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A Numerical Method for a Nonlinear Singularly Perturbed Interior Layer Problem Using an Approximate Layer Location

Jason Quinn
Dublin Institute of Technology, jason.quinn@dit.ie

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J Quinn

School of Mathematical Sciences, Dublin Institute of Technology, Ireland.

Abstract

A class of nonlinear singularly perturbed interior layer problems is examined in this paper. Solutions exhibit an interior layer at an a priori unknown location. A numerical method is presented that uses a piecewise uniform mesh refined around numerical approximations to successive terms of the asymptotic expansion of the interior layer location. The first term in the expansion is used exactly in the construction of the approximation which restricts the range of problem data considered. Denote the perturbation parameter as ε and the number of mesh intervals to be used as N. The method is shown to converge point-wise to the true solution with a first order convergence rate (overlooking a logarithmic factor) for sufficiently small $\varepsilon \leq N^{-1}$. A numerical experiment is presented to demonstrate the convergence rate established.

Keywords: Singularly Perturbed, Interior Layer, Nonlinear, Shishkin Mesh
2000 MSC: 65L11, 65L12, 65L20

1. Introduction

Singularly perturbed nonlinear interior layer problems typically exhibit interior layers at a priori unknown locations (e.g. [12]). This scenario poses a difficulty when designing numerical methods for such problems as it is not initially known where to refine a mesh to resolve any interior layers. In this paper, we establish a numerical method for a class of nonlinear interior layer problems using an asymptotic expansion of the interior layer location as a sufficient approximation to the true location for sufficiently small values of the perturbation parameter ε. Successive terms in the asymptotic expansion may not be explicitly known and hence must be approximated numerically.

In [4], Nefedov et. al examine a problem class containing nonlinear problems of the form

$$(\varepsilon u'' + uu' - bu)(x) = q(x), \quad x \in (-1, 1), \quad u(-1) < 0 < u(1). \tag{1.1}$$

Under certain conditions on the problem data, the solution exhibits an interior layer centred around an a priori unknown point $d_\varepsilon \in (-1, 1)$. In [4], an asymptotic expansion of d_ε is constructed. The objective in this paper is to establish a numerical method for a problem class containing (1.1) based on the asymptotic analysis given in [4].

The numerical method we present consists of constructing numerical approximations to the first three terms in the asymptotic expansion of the interior layer location. We then solve nonlinear finite difference schemes on a piece-wise uniform Shishkin mesh to the left and right of the numerical approximation to the layer location at which the fine mesh is centred and numerical solutions are set to zero. The numerical method is shown to admit a point-wise error estimate for approximations U^N of (1.1) of the form

$$|(u - U^N)(x_i)| \leqslant CN^{-1}\ln(N), \quad \text{for } \varepsilon \leqslant CN^{-1}, \tag{1.2}$$

where x_i are mesh points on the proposed mesh and C represents a generic positive constant independent of ε and N throughout the paper.

In [9], Shishkin presents a numerical method for a class of problems containing (1.1). In the classical case where $\varepsilon \geqslant N^{-2/5}$, a finite difference scheme is solved globally on a uniform mesh and a point-wise error estimate is presented, establishing a sufficient convergence of the method for increasing N. In the singularly perturbed case where $\varepsilon < N^{-2/5}$, numerical approximations $\bar{U}^N$ of (1.1) are set to zero at a numerical approximation d_0^{*N} to the leading term of the asymptotic expansion of the interior layer location. Finite difference schemes are solved to the left and right of this point at which a Shiskin mesh is centred. The error estimate for the singularly perturbed case is presented as follows:

$$|(u - \bar{U}^N)(x_i)| \leqslant CN^{-1/5}(\ln(N))^{1/2}, \quad |x_i - d_0^{*N}| \geqslant m,$$
$$|u(x_i + (d_\varepsilon - d_0^{*N})) - \bar{U}^N(x_i)| \leqslant CN^{-1/5}(\ln(N))^{1/2}, \quad |x_i - d_0^{*N}| \leqslant 2m,$$

where m is described as a sufficiently small constant independent of ε. We note, for mesh points in a neighborhood of d_0^{*N}, the error estimate is not point-wise. The main difference between the numerical analytic methods used here and in [9] is that we consider higher order terms in the asymptotic expansion of the interior layer location. However, here we consider a significant restriction on the problem data in (1.1) described as follows. The left and right reduced solutions associated with problem (1.1) when $\varepsilon = 0$ are the functions u_L and u_R satisfying the first order nonlinear problems:

$$u_L {u_L}' - bu_L = q, \qquad x \in (-1, 1], \qquad u_L(-1) = u(-1); \tag{1.3a}$$

$$u_R {u_R}' - bu_R = q, \qquad x \in [-1, 1), \qquad u_R(1) = u(1). \tag{1.3b}$$

In [4], Nefedov et. al define the layer location d_ε as the point where the solution of problem (1.1) satisfies $u(d_\varepsilon) = \frac{1}{2}(u_L + u_R)(d_\varepsilon)$. The leading term d_0 in the asymptotic expansion of d_ε is defined as the solution of the equation

$$(u_L + u_R)(d_0) = 0. \tag{1.3c}$$

The error estimate (1.2) holds for any approximation d_0^N of the leading term d_0 satisfying

$$|d_0 - d_0^N| \leqslant C\varepsilon N^{-1}. \tag{1.4}$$

Given that we are considering the problem (1.1) for any arbitrary small value of the perturbation parameter ε, the required accuracy of d_0^N motivates the following restriction on the problem data in (1.1):

Assumption: The problem data in (1.1) *is such that* (1.3) *is explicitly solvable.*

Hence, under this assumption, we choose $d_0^N = d_0$.

For problem (1.1), it can be shown that there exists a unique point τ_ε s.t. $u(\tau_\varepsilon) = 0$. In [7], a theoretical numerical method is presented based on the assumption of the existence of a *close* approximation T_ε of τ_ε. The numerical method sets the numerical approximation $\hat{U}^N$ of (1.1) to zero at the point T_ε and solves two discrete problems to the left and right of T_ε. The error estimate presented in [7] is of the form

$$|(u - \hat{U})(x_i)| \leqslant CN^{-1}\ln(N) + C\varepsilon^{-1}|T_\varepsilon - \tau_\varepsilon|.$$

It follows that to obtain a sufficiently accuarate point-wise approximation to (1.1) using the proposed method, one needs an approximation T_ε to τ_ε *closer* than $C\varepsilon$.

The problem class containing (1.1) studied in [4] is a time-periodic problem for which (1.1) corresponds to the steady-state problem for specific problem data. Both the time-periodic problems and steady state problems considered in [4] were previously considered by Vasil'eva in [10] and [11]. The algorithm used to find successive terms in the asymptotic expansion of the layer location in the three articles ([4], [10] and [11]) is identical and corresponds to imposing a C^1-matching condition on the left and right decompositions of the solution at the layer location.

We consider the approximation bound of the asymptotic expansion given in [4] as the most employable in the numerical analysis in this paper and hence focus on the results therein.

The paper is structured as follows. In §2, we state the problem under study and describe the assumptions on the problem data.

In §3, we establish existence and uniqueness of the continuous solution. In §3.1, the terms in the asymptotic expansion of the interior layer location are presented and their numerical approximations are specified. In §3.2, we analyse the continuous problem to the left and right of the numerical approximation to the layer location by decomposing the solution on either side into a sum of a smooth regular component and a singular layer component.

In §4, we state the discrete problem and analyse the corresponding left and right discrete problems on either side of the numerical approximation to the layer location in a similar manner to the continuous problem.

In §5 we perform a numerical experiment. In Appendix A, for the sake of completeness, we perform the algorithm described in [4] to find successive terms in the asymptotic expansion of the layer location for the problem under study and establish properties of these terms required in the continuous and discrete analysis.

Finally, in Appendix B, we establish an algorithm to approximate the second term in the asymptotic expansion of the layer location.

2. Statement of problem and assumptions on problem data

Consider the following problem class: find a function $y_\varepsilon \in C^2[\Omega_\varepsilon]$ such that

$$(\varepsilon y_\varepsilon'' + (F(y_\varepsilon))' - by_\varepsilon)(x) = q(x), \quad x \in \Omega_\varepsilon := (-1,1), \tag{2.1a}$$

$$y_\varepsilon(-1) = A < 0, \quad y_\varepsilon(1) = B > 0, \tag{2.1b}$$

$$b, q \in C^2, \quad b(x) \geqslant 0, \quad b \not\equiv 0, \quad x \in \Omega_\varepsilon, \quad F \in C^3[\Psi^-, \Psi^+]; \tag{2.1c}$$

$$\Psi^- := A - 3\sqrt{\max_{\bar{\Omega}_\varepsilon}\{0, q(x)\}}, \quad \Psi^+ := B + 3\sqrt{|\min_{\bar{\Omega}_\varepsilon}\{0, q(x)\}|} \;. \tag{2.1d}$$

We consider this problem for the boundary conditions A, B and the function F satisfying

$$\begin{array}{c} A^2 - 4(|A|\|b\| + |\min_{\bar{\Omega}_\varepsilon}\{0, q(x)\}|) > \gamma_L^2, \quad \gamma_L > 0, \\ B^2 - 4(B\|b\| + \max_{\bar{\Omega}_\varepsilon}\{0, q(x)\}|) > \gamma_R^2, \quad \gamma_R > 0, \end{array} \tag{2.1e}$$

$$f(s) := F'(s), \quad f(0) = 0, \quad |f(s)| \geqslant |s|, \quad sf(s) > 0, \quad s \in [\Psi^-, \Psi^+]. \tag{2.1f}$$

By assuming (2.1f), we may analyse (2.1a) in the same manner as (1.1). The left and right reduced solutions, r_L and r_R respectively, of ((2.1); $\varepsilon = 0$) satisfy

$$((F(r_{L\setminus R}))' - br_{L\setminus R})(x) = q(x), \quad r_L(-1) = A, \quad r_R(1) = B. \tag{2.2}$$

We now place additional restrictions on f below so that the problem (2.1) is contained within the problem class considered in [4] and hence the results therein may be applied to the problem (2.1). Assume that

$$f(-s) = -f(s) \quad \forall \;\; s \in [\Psi^-, \Psi^+], \tag{2.3a}$$

$$\exists \quad d_0 \;\in\; (-1,1) \quad \text{s.t.} \quad (r_L + r_R)(d_0) = 0, \tag{2.3b}$$

$$b(d_0) > 0, \quad r_R(d_0) > \tfrac{|q(d_0)|}{b(d_0)}. \tag{2.3c}$$

These assumptions are sufficient to ensure that the solution of (2.1) exhibits an interior layer at a point $d_\varepsilon \in (-1,1)$, which has an asymptotic expansion with leading term d_0 defined in (2.3b).

Remark 1. *In [4], it is required that $I(x) := \int_{r_L(x)}^{r_R(x)} f(s)\,ds$ has a root d_0. The assumption that f is odd in (2.3a) provides a relatively simple expression* *to obtain this root i.e. the root is the same as the solution to* $r_R(x) = -r_L(x)$*. Moreover, it is required that* $I'(d_0) < 0$*. Using* (2.2) *along with the assumption that* f *is odd allows us to simplify this requirement to that given in* (2.3c)*.*

Furthermore, the assumption that f *is odd is convenient later in that it allows us to simplify the expressions for successive terms in the asymptotic expansion of the layer location.*

The numerical approximation to the interior layer location established later includes the term d_0 explicitly. We now assume the following:

Assumption 1. *The problem data in* (2.1) *is such that the solution* d_0 *of* ((2.2), (2.3b)) *can be solved explicitly.*

Remark 2. *A subclass of problem* (2.1) *for which* d_0 *is solvable explicitly is the problem with* $f(s) = G'(s)s$*,* $G(s) \in C^2$ *is odd,* $g(s) := G'(s) \geqslant 1$ *for* $s \in [\Psi^-, \Psi^+]$ *and* $b(x) = H'(x)$ *where* H *has an explicitly known inverse* H^{-1}*. In which case*

$$d_0 = H^{-1}(\tfrac{1}{2}(H(-1) + H(1) - G(A) - G(B))).$$

Note that the error bound established for the numerical method given in §4 holds when d_0 *is approximated by* d_0^N *s.t.* $|d_0 - d_0^N| \leqslant C\varepsilon N^{-1}$*. However, we do not analyse an algorithm to obtain such an approximation of* d_0 *in this paper.*

3. Analysis of the continuous problem

We first establish existence and uniqueness for the solution of (2.1). We use the method of upper and lower solutions (see [1, §1.5]) in the same manner as for the problem studied in [8].

Lemma 1. *Assuming* (2.1c), (2.1d) *and* (2.1f)*, there exists a unique solution* $y_\varepsilon \in C^2[\bar{\Omega}_\varepsilon]$ *of the problem* (2.1) *satisfying*

$$\Psi^- \leqslant y_\varepsilon(x) \leqslant \Psi^+, \qquad \varepsilon|y'_\varepsilon(x)| \leqslant C, \qquad x \in \bar{\Omega}_\varepsilon,$$

where $\Psi^\pm$ *are defined in* (2.1d)*.*

Proof. Define the functions

$$\underline{y}(x) := A - \sqrt{\max_{\bar{\Omega}_\varepsilon}\{0, q(x)\}}(x+2); \qquad \overline{y}(x) := B + \sqrt{|\min_{\bar{\Omega}_\varepsilon}\{0, q(x)\}|}(2-x). \tag{3.1}$$

Clearly $\underline{y} < 0$ and $\underline{y}' \leqslant 0$. Using (2.1f), we have $\varepsilon \underline{y}'' + f(\underline{y})\underline{y}' - b\underline{y} - q \geqslant \underline{y}\ \underline{y}' - |\min_{\bar{\Omega}_\varepsilon}\{0, q(x)\} \geqslant 0$. Hence, $\underline{y}$ is a lower solution of (2.1). We can similarly establish that the function $\overline{y}(x)$ is an upper solution of (2.1).

Suppose y_1 and y_2 are two solutions of (2.1). Then $\underline{y} \leqslant y_1, y_2 \leqslant \overline{y}$ and $\chi := y_1 - y_2$ satisfies the linear problem $\varepsilon\chi'' + (\chi \int_0^1 f(ty_1 + (1-t)y_2)\, dt)' - b\chi = 0$, $\chi(\pm 1) = 0$. Clearly 0 is both a lower and an upper solution, hence the solution of (2.1) is unique.

Since $y_\varepsilon \in C^2$, by the mean value theorem there exists $z \in (-1, -1+\varepsilon)$ s.t. $y_\varepsilon'(z) = \varepsilon^{-1}(y_\varepsilon(-1+\varepsilon) - y_\varepsilon(-1))$. It follows that $\varepsilon|y_\varepsilon'(z)| \leqslant 2\max|\Psi^\pm|$. Integrate both sides of (2.1a) from -1 to z, to obtain the bound on $\varepsilon|y_\varepsilon'(x)|$ in the Lemma statement. □

The following Lemma establishes bounds on the reduced solutions.

Lemma 2. *Assuming (2.1c)-(2.1f), the reduced solutions of ((2.1); $\varepsilon = 0$) defined by (2.2) uniquely exist and satisfy*

$$\Psi^- \leqslant r_L(x) < -\gamma_L < 0, \qquad 0 < \gamma_R < r_R(x) \leqslant \Psi^+, \qquad x \in \bar{\Omega}_\varepsilon.$$

where $\Psi^\pm$ are defined in (2.1d).

Proof. Note the definition of upper and lower solutions for initial value problems in [5]. Using a suitable transform, we can easily extend the definition to the case of terminal value problems. Using (2.1e)-(2.1f), it can be checked that the function $\underline{y}$ in (3.1) and the function g^- defined as

$$g^-(x) := -\sqrt{A^2 - 2(|A|\|b\| + |\min_{\bar{\Omega}_\varepsilon}\{0, q(x)\}|)(x+1)} < -\gamma_L, \tag{3.2a}$$

are lower and upper solutions of ((2.2); r_L) respectively. In the same manner, we can show that the functions $\overline{y}$ in (3.1) and g^+ defined as

$$g^+(x) := \sqrt{B^2 + 2(B\|b\| + \max_{\bar{\Omega}_\varepsilon}\{0, q(x)\})(x-1)} > \gamma_R, \tag{3.2b}$$

are upper and lower solutions of ((2.2); r_R) respectively. We can establish uniqueness in the usual manner. □

3.1. Interior layer location

Define the function $\phi(x) := \frac{1}{2}(r_L + r_R)(x)$ and the point d_ε s.t. $y_\varepsilon(d_\varepsilon) = \phi(d_\varepsilon)$. From [4], the point d_0 in (2.3b) is the first term in the asymptotic expansion of $d_\varepsilon = d_0 + \varepsilon d_1 + \cdots$. Denote the truncated expansion as $d_\varepsilon^{(k)} := d_0 + \varepsilon d_1 + \cdots + \varepsilon^k d_k$.

The algorithm to evaluate the d_i's for $i = 0, 1, 2$ in the expansion of d_ε is performed in Appendix A. It is found that $d_\varepsilon^{(2)} := d_0 + \varepsilon d_1 + \varepsilon^2 d_2$ where

$$d_1 = -\tfrac{q}{br_R f(r_R)} - \tfrac{f(r_R)}{2br_R}(v_{1L} + v_{1R})\Big|_{x=d_0}, \tag{3.3a}$$

$$|d_2| \leqslant C, \tag{3.3b}$$

$$|y_\varepsilon(d_\varepsilon^{(2)})| \leqslant C\varepsilon, \tag{3.3c}$$

where q, b and f are the problem data in (2.1a), $r_{L\backslash R}$ and d_0 are the solutions of (2.2)-(2.3b) and the functions $v_{1L\backslash R}$ satisfy the linear problems:

$$f(r_L){v_{1L}}' + ((f(r_L))' - b)v_{1L} = -r_L'', \quad v_{1L}(-1) = 0; \tag{3.4a}$$

$$f(r_R){v_{1R}}' + ((f(r_R))' - b)v_{1R} = -r_R'', \quad v_{1R}(1) = 0. \tag{3.4b}$$

Note, using (2.2) and the assumptions on the problem data in (2.1), we can show that

$$\|v_{1L\backslash R}\| \leqslant C. \tag{3.5}$$

Using Assumption 1 and the Numerical Algorithm in Appendix B and (3.3b), we can generate approximations d_0^N and d_1^N s.t.

$$D_\varepsilon^N := d_0^N + \varepsilon d_1^N; \qquad d_0^N = d_0; \qquad |d_1 - d_1^N| \leqslant CN^{-1}, \tag{3.6a}$$

$$|d_\varepsilon^{(2)} - D_\varepsilon^N| \leqslant C\varepsilon(\varepsilon^{-1}|d_0 - d_0^N| + N^{-1} + \varepsilon) \leqslant C\varepsilon(N^{-1} + \varepsilon). \tag{3.6b}$$

Hence, using Lemma 1, we can show that

$$|y_\varepsilon(D_\varepsilon^N)| \leqslant |y_\varepsilon(d_\varepsilon^{(2)})| + \|y_\varepsilon'\||d_\varepsilon^{(2)} - D_\varepsilon^N| \leqslant C(\varepsilon + N^{-1}). \tag{3.7}$$

We now analyse (2.1) to the left and right of the point D_ε^N to establish bounds on the solution and it's derivatives required for the error analysis of the proposed numerical method.

3.2. Analysis of left and right problems

Noting (3.7), we consider the problem (2.1) as left and right boundary value problems defined either side of D_ε^N as follows:

$$y_\varepsilon(x) = \{y_L(x),\ x < D_\varepsilon^N; \qquad y_\varepsilon(D_\varepsilon^N),\ x = D_\varepsilon^N; \qquad y_R(x),\ x > D_\varepsilon^N\}; \tag{3.8a}$$

$$\begin{array}{ccc} \varepsilon y_L'' + (F(y_L))' - by_L = q, & & \varepsilon y_R'' + (F(y_R))' - by_R = q, \\ x \in (-1, D_\varepsilon^N), & ; & x \in (D_\varepsilon^N, 1), \\ y_L(-1) = A,\ \ y_L(D_\varepsilon^N) = y_\varepsilon(D_\varepsilon^N), & & y_R(D_\varepsilon^N) = y_\varepsilon(D_\varepsilon^N),\ \ y_R(1) = B. \end{array} \tag{3.8b}$$

Lemma 3. *Assuming (2.1c)-(2.1f), if y_ε is the solution of (2.1), then for sufficiently small ε and sufficiently large N, the solutions y_L and y_R of (3.8) uniquely exist and satisfy*

$$\Psi^- \leqslant y_L(x) \leqslant C(\varepsilon + N^{-1}), \quad x \in [-1, D_\varepsilon^N],$$
$$-C(\varepsilon + N^{-1}) \leqslant y_R(x) \leqslant \Psi^+, \quad x \in [D_\varepsilon^N, 1],$$

where $\Psi^\pm$ are defined in (2.1d).

Proof. As with (2.1), construction of upper and lower solutions of (3.8) suffices to prove existence. We can show that the functions $\underline{y}$ and $\overline{y}$ in (3.1) are lower and upper solutions for ((3.8); y_L) and ((3.8); $\underline{y_R}$) respectively. Consider the functions $g^\mp$ in (3.2) and the solutions $\hat{y}^\mp$ to the second order terminal/initial value problems:

$$\varepsilon \hat{y}^{\mp\,\prime\prime} + f(\hat{y}^\mp)\hat{y}^{\mp\,\prime} = f(g^\mp)g^{\mp\prime}, \tag{3.9a}$$

$$\hat{y}^\mp(D_\varepsilon^N) = \pm C(\varepsilon + N^{-1}), \qquad \varepsilon \hat{y}^{\mp\,\prime}(D_\varepsilon^N) = F(g^\mp) - F(\hat{y}^\mp(D_\varepsilon^N)). \tag{3.9b}$$

Integrating both sides from any x in the respective problem domain to the point D_ε^N, we can show that $\hat{y}^\mp$ satisfy

$$\varepsilon \hat{y}^{\mp\,\prime} + F(\hat{y}^\mp) - F(g^\mp) = 0 \quad \text{for } x \text{ s.t.} \quad \mp(x - D_\varepsilon^N) > 0, \tag{3.10a}$$

$$\hat{y}^\mp(D_\varepsilon^N) = \pm C(\varepsilon + N^{-1}). \tag{3.10b}$$

Substituting $\tilde{y}^\mp = \pm C(\varepsilon + N^{-1})$ into (3.10) and using (2.1f), for sufficiently small ε and large N, we have

$$F(\tilde{y}^\mp) - F(g^\mp) = \textstyle\int_{g^\mp}^{\tilde{y}^\mp} f(s)\ ds = \int_{g^\mp}^{0} f(s)\ ds + O((\varepsilon + N^{-1})) < 0.$$

Hence $\tilde{y}^{\mp}$ is an upper/lower solution for the tvp/ivp (3.10). Since $g^{\mp\prime} > 0$, we can see that $g^{\mp}$ are lower and upper solutions for (3.10; $\hat{y}^{-}$) and (3.10; $\hat{y}^{+}$) respectively. Hence we have $A \leqslant g^{-}(x) \leqslant \tilde{y}^{-} \leqslant C(\varepsilon + N^{-1})$ and $-C(\varepsilon + N^{-1}) \leqslant \tilde{y}^{+} \leqslant g^{+}(x) \leqslant B$.

Using (2.1f), (3.2a) and (3.9) we can show that

$$(\varepsilon \hat{y}^{-\prime} + F(\hat{y}^{-}))' - b\hat{y}^{-} - q(x) \leqslant f(g^{-})g^{-\prime} + \|b\||A| - q(x) \leqslant g^{-}g^{-\prime} + \|b\||A| - q(x) \leqslant 0.$$

Hence $\hat{y}^{-}$ is an upper solution for (3.8; y_L). We may analogously show that $\hat{y}^{+}$ is a lower solution for (3.8; y_R) and the bounds in the Lemma statement follow. Uniqueness may be established in the same manner as in the proof of Lemma 1. □

We decompose each solution $y_{L\backslash R}$ of (3.8) into a regular component $v_{L\backslash R}$ and a layer component $w_{L\backslash R}$. The regular components satisfy

$$(\varepsilon v''_{L\backslash R} + (F(v_{L\backslash R}))' - bv_{L\backslash R})(x) = q(x), \quad x \in (0, D_\varepsilon^N)\backslash(D_\varepsilon^N, 1), \tag{3.11a}$$

$$v_L(-1) = A, \quad v_{L\backslash R}(D_\varepsilon^N) = (r_{L\backslash R} + \varepsilon v_{1L\backslash R})(D_\varepsilon^N), \qquad v_R(1) = B, \tag{3.11b}$$

where the $r_{L\backslash R}$ and $v_{1L\backslash R}$ are the solutions to (2.2) and (3.4) respectively. Note the following identity for any $k \in C^1[a,b]$:

$$k(a) - k(b) = (a-b)\int_0^1 k'(ta + (1-t)b)\ dt\ . \tag{3.12}$$

We can use this to show that the left and right layer components $w_{L\backslash R} := y_{L\backslash R} - v_{L\backslash R}$ satisfy the problems

$$(\varepsilon w'_{L\backslash R} + \xi_{L\backslash R} w_{L\backslash R})' - bw_{L\backslash R} = 0, \quad x \in (0, D_\varepsilon^N)\backslash(D_\varepsilon^N, 1), \tag{3.13a}$$

$$w_{L\backslash R}(D_\varepsilon^N) = (y_\varepsilon - v_{L\backslash R})(D_\varepsilon^N), \qquad w_{L\backslash R}(\mp 1) = 0, \tag{3.13b}$$

$$\xi_{L\backslash R}(x) := \int_0^1 f(ty_{L\backslash R}(x) + (1-t)v_{L\backslash R}(x))\ dt\ . \tag{3.13c}$$

Lemma 4. *Assuming (2.1c)-(2.1f), for $k = 1,2$, the solutions $v_{L\backslash R}$ and $w_{L\backslash R}$ of (3.11) and (3.13) respectively satisfy the bounds*

$$\Psi^{-} \leqslant v_L(x) \leqslant -\gamma_L,\ x \in [-1, D_\varepsilon^N], \quad \gamma_R \leqslant v_R(x) \leqslant \Psi^{+},\ x \in [D_\varepsilon^N, 1], \tag{3.14a}$$

$$|w_{L\backslash R}(x)| \leqslant \max\{|\Psi^{\pm}|\} e^{-\frac{|\gamma_{L\backslash R}|}{2\varepsilon}|D_\varepsilon^N - x|}, \quad x \in [-1, D_\varepsilon^N]\backslash x \in [D_\varepsilon^N, 1], \tag{3.14b}$$

$$\|v_{L\backslash R}^{(k)}\| \leqslant C(1 + \varepsilon^{2-k}), \tag{3.14c}$$

$$|w_{L\backslash R}^{(k)}| \leqslant C\varepsilon^{-k} e^{-\frac{|\gamma_{L\backslash R}|}{2\varepsilon}|D_\varepsilon^N - x|}, \quad x \in [-1, D_\varepsilon^N]\backslash x \in [D_\varepsilon^N, 1], \tag{3.14d}$$

where $\Psi^{\pm}$ *and* $\gamma_{L\backslash R}$ *are defined in* (2.1d)-(2.1e) *and* $z^{(k)} := \frac{d^k z}{dx^k}$ *where* z *is any of the functions* $v_{L\backslash R}$ *or* $w_{L\backslash R}$.

Proof. Using Lemma 2 and (3.5) with (3.11), we have $|v_{L\backslash R}(D_\varepsilon^N)| \geqslant |\gamma_{L\backslash R}|$ for sufficiently small ε. Repeat the method used in Lemma 3 but replacing the terminal/initial condition in (3.10) to $\hat{y}^{\mp}(D_\varepsilon^N) = \mp\gamma_{L\backslash R}$ to establish the bounds in (3.14a).

Using the identity in (3.12), we can show that the functions $v_{2L\backslash R}$ defined by $\varepsilon^2 v_{2L\backslash R} := v_{L\backslash R} - r_{L\backslash R} - \varepsilon v_{1L\backslash R}$ satisfy the problems

$$\varepsilon v_2'' + f(v)v_2' + B(x)v_2 = Q(x) := -v_1'' - f'(r)v_1v_1' - (r+\varepsilon v_1)'\mathcal{I}v_1^2, \quad (3.15a)$$

$$x \in (-1, D_\varepsilon^N)\backslash(D_\varepsilon^N, 1), \quad v_{2L\backslash R}(\mp 1) = v_{2L\backslash R}(D_\varepsilon^N) = 0, \quad (3.15b)$$

$$P(x) := ((r+\varepsilon v_1)'(f'(r) + (v - r + \varepsilon v_1)\mathcal{I}) - b)(x) \quad (3.15c)$$

$$\mathcal{I}(x) := \int_0^1 t \int_0^1 f''(r + st(v-r))\ ds\ dt. \quad (3.15d)$$

Note, we have omitted the left and right subscripts on the functions v, r, v_1, v_2, P and Q in (3.15). Using (2.1f), Lemma 2 and (3.4), we can show that $\overline{v} := \frac{\|Q\|}{\max\{\|P\|,1\}}(e^{\frac{2\max\{\|P\|,1\}}{|\gamma_L|}(x+1)} - 1)$ and $-\overline{v}$ are upper and lower solutions for ((3.15); v_{2L}) respectively. We bound v_{2R} in the same manner.

We can establish that $\varepsilon|v_2{}'_{L\backslash R}| \leqslant C$ using the same method to bound the derivative of y_ε in Lemma 1. It follows from (2.1c) and (3.15) that $\varepsilon|v_2{}''_{L\backslash R}| \leqslant C/\varepsilon$. Hence, writing $v_{L\backslash R}$ as $v_{L\backslash R} = r_{L\backslash R} + \varepsilon v_{1L\backslash R} + \varepsilon^2 v_{2L\backslash R}$, and noting the smoothness of the problem data in (2.1c), we can establish the bounds in (3.14c).

We now prove the bound on the left layer component w_L. Since $w_L(D_\varepsilon^N) = (y_\varepsilon - v_L)(D_\varepsilon^N) \geqslant \gamma_L - C(\varepsilon + N^{-1}) > 0$ for sufficiently small ε and large N, we can show that 0 is a lower solution of ((3.13); w_L).

Consider the function $\overline{w}(x) := w_L(d_\varepsilon)\exp(\int_x^{D_\varepsilon^N} \xi_L(t)\ dt) > 0$, which is composition of C^2 functions. Substitute $\underline{w}(x)$ into ((3.13); w_L) to verify that it is an upper solution. Hence $0 \leqslant w_L(x) \leqslant \overline{w}(x)$ on $x \in [-1, D_\varepsilon^N]$.

We now bound $\overline{w}$ by $Ce^{-C/\varepsilon|D_\varepsilon^N - x|}$ by showing that $\xi_L \leqslant -C < 0$. Recall from Lemma 3 that $\Psi^- \leqslant y_L(x) \leqslant C(\varepsilon + N^{-1})$. At points x s.t. $y_L(x) \leqslant 0$, using (2.1f) and $v_L(x) < -\gamma_L$ we can show that

$$\begin{aligned} \xi_L(x) &= \int_0^1 f(ty_L(x) + (1-t)v_L(x))\ dt \\ &\leqslant \int_0^1 ty_L(x) + (1-t)v_L(x)\ dt \ = \tfrac{1}{2}(y_L + v_L)(x) \leqslant -\tfrac{1}{2}\gamma_L. \end{aligned}$$

At points x s.t. $0 < y_L(x) \leqslant C(\varepsilon + N^{-1})$, using the identity (3.12) we can show that

$$\xi_L(x) \leqslant \int_0^1 f((1-t)v_L(x))\, dt \ + \tfrac{1}{2}|y_L(x)|\|f'\| \leqslant -\tfrac{1}{2}\gamma_L.$$

Hence $w_L(x) \leqslant \overline{\overline{w}}(x) := w_L(D_\varepsilon^N)e^{-\frac{|\gamma_L|}{2\varepsilon}(D_\varepsilon^N - x)}$. We can the solution of ((3.13); w_R) in the same manner.

Since $w_L \in C^2$, by the mean value theorem there exists $z \in (-1, -1+\varepsilon)$ s.t. $\varepsilon w_L'(z) = w_L(-1+\varepsilon) - w_L(-1) = w_L(-1+\varepsilon) \leqslant \overline{\overline{w}}(-1+\varepsilon) \leqslant C\overline{\overline{w}}(-1)$ in an ε-neighbourhood. Integrating both sides of (2.1a) from -1 to z, we obtain $|\varepsilon w_L'(-1)| = |\varepsilon w_L'(z) + (\xi_L w_L)(z) - \int_{-1}^z (bw_L)(t)| \leqslant C\overline{\overline{w}}(-1)$.

Integrating both sides of (2.1a) from -1 to any $x \leqslant D_\varepsilon^N$ yields $\varepsilon|w_L'(x)| \leqslant C\overline{\overline{w}}(-1) + C\overline{\overline{w}}(x) \leqslant C\overline{\overline{w}}(x)$. Use ((3.13); w_L) and Lemma 1 to establish the bound $\varepsilon|w_L''(x)| \leqslant C/\varepsilon\overline{\overline{w}}(x)$. Complete the proof by bounding w_R' and w_R'' using the same method. □

4. The discrete problem and error analysis

We approximate the solution of (2.1) by the discrete mesh function Y_ε defined around the point D_ε^N in (3.6a) as follows:

$$Y_\varepsilon(x_i) = \{Y_L(x_i),\ x_i < D_\varepsilon^N; \qquad 0,\ x_i = D_\varepsilon^N; \qquad Y_R(x_i),\ x_i > D_\varepsilon^N\}; \tag{4.1a}$$

$$\begin{array}{ccc} D^-[\varepsilon D^+ Y_L + F(Y_L)] - bY_L = q, & & D^+[\varepsilon D^- Y_R + F(Y_R)] - bY_R = q, \\ x_i \in (-1, D_\varepsilon^N), & ; & x_i \in (D_\varepsilon^N, 1), \\ Y_L(-1) = A, \quad Y_L(D_\varepsilon^N) = 0, & & Y_R(D_\varepsilon^N) = 0, \quad Y_R(1) = B, \end{array} \tag{4.1b}$$

where x_i are points on the piecewise-uniform mesh $\Omega_\varepsilon^N := \bar{\Omega}_\varepsilon^N \setminus \{x_0, x_N\}$ centred at the point D_ε^N (in (3.6a)) defined as

$$\sigma_L := \min\{\tfrac{1}{2}(D_\varepsilon^N + 1), \tfrac{2\varepsilon}{\gamma_L}\ln(N)\}, \quad \sigma_R := \min\{\tfrac{1}{2}(1 - D_\varepsilon^N), \tfrac{2\varepsilon}{\gamma_R}\ln(N)\}, \tag{4.2a}$$

$$H_L := \tfrac{4}{N}(D_\varepsilon^N - \sigma_L + 1), \quad h_{L\setminus R} := \tfrac{4}{N}\sigma_{L\setminus R}, \quad H_R := \tfrac{4}{N}(1 - D_\varepsilon^N - \sigma_R), \tag{4.2b}$$

$$\bar{\Omega}_\varepsilon^N := \left\{ x_i \;\middle|\; \begin{array}{cc} x_i = -1 + H_L i, & 0 \leqslant i \leqslant \frac{N}{4}, \\ x_i = D_\varepsilon^N - \sigma_L + h_L(i - \frac{N}{4}), & \frac{N}{4} < i \leqslant \frac{N}{2}, \\ x_i = D_\varepsilon^N + h_R(i - \frac{N}{2}), & \frac{N}{2} < i \leqslant \frac{3N}{4}, \\ x_i = D_\varepsilon^N + \sigma_R + H_R(i - \frac{3N}{4}), & \frac{3N}{4} < i \leqslant N, \end{array} \right\}, \tag{4.2c}$$

$$\bar{\Omega}_{L\setminus R}^N := \{x_i \in \bar{\Omega}_\varepsilon^N | i \leqslant \tfrac{N}{2} \setminus i \geqslant \tfrac{N}{2}\}; \quad \Omega_{L\setminus R}^N := \bar{\Omega}_{L\setminus R}^N \setminus \{x_0, x_{\frac{N}{2}}, x_N\}. \tag{4.2d}$$

Immediately, from (3.7), we have

$$|(y_\varepsilon - Y_\varepsilon)(D_\varepsilon^N)| \leqslant C(\varepsilon + N^{-1}) \leqslant CN^{-1}.$$

Throughout this section, we use the notation $h_i := x_i - x_{i-1}$. Note from (4.2) that $h_i = h$ for all $i \in [N/4 + 1, 3N/4]$ and

$$h/\varepsilon \leqslant CN^{-1}\ln(N) \tag{4.3}$$

which is sufficiently small for sufficiently large N.

For the discrete problem (4.1), we can establish the method of discrete upper and lower solutions in the same manner as for the discrete problem in [8].

Lemma 5. *Assuming* (2.1c)-(2.1f)*, if* $Y_{L\setminus R}$ *are the solutions of* (4.1)*, then*

$$\Psi^- \leqslant Y_L(x_i) \leqslant 0, \quad x_i \in \bar{\Omega}_L^N; \qquad 0 \leqslant Y_R(x_i) \leqslant \Psi^+, \quad x_i \in \bar{\Omega}_R^N.$$

Proof. Using (2.1f), we can show that the linear functions $\underline{y}$ and $\overline{y}$ in (3.1) are discrete lower and upper solutions for ((4.1); Y_L) and ((4.1); Y_R) respectively.

The notion of discrete upper and lower solutions for first order difference equations $D^\pm Z_i = f(x_i, Z_i)$ can be defined analogously as for the corresponding continuous problem (see [5]) using an extension of the proof in [6, Theorem 4].

Consider the functions $g^{\mp}$ in (3.2) and the solutions $\hat{Y}^{\mp}$ to the discrete terminal/initial value problems:

$$\varepsilon D^{\pm}\hat{Y}^{\mp} + F(\hat{Y}^{\mp}) - F(g^{\mp}) = 0 \text{ for } x_i \text{ s.t. } \mp(x_i - D^N_\varepsilon) > 0, \quad \hat{Y}^{\mp}(D^N_\varepsilon) = 0. \tag{4.4}$$

Substituting $\tilde{Y}^{\mp} = 0$ into the left hand side of (3.10) and using (2.1f), we have $\varepsilon D^{\pm}\tilde{Y}^{\mp} + F(\tilde{Y}^{\mp}) - F(g^{\mp}) = \int_{g^{\mp}}^{0} f(s)\,ds < 0$. Hence $\tilde{Y}^{\mp}$ is an upper/lower solution for the tvp/ivp (3.10). Since $g^{\mp\prime} > 0$, we can see that $g^{\mp}$ are lower and upper solutions for (4.4; $\hat{Y}^-$) and (4.4; $\hat{Y}^+$) respectively. Hence $A \leqslant g^-(x) \leqslant \tilde{Y}^- \leqslant 0$ and $0 \leqslant \tilde{Y}^+ \leqslant g^+(x) \leqslant B$.

Using the identity $D^-[(g^-)(x_i)^2] = (g^-(x_i) + g^-(x_{i-1}))D^-[g^-(x_i)]$ with (2.1f) and (3.2a) we can show that

$$\begin{aligned} &\varepsilon D^- D^+ \hat{Y}^- + D^- F(\hat{Y}^-) - b\hat{Y}^- - q(x) \\ \leqslant &(g^-(x_i) + g^-(x_{i-1}))D^-[g^-(x_i)] + \|b\||A| - q(x) \leqslant 0. \end{aligned}$$

Hence $\hat{Y}^-$ is an upper solution for (4.1; Y_L). We analogously show that $\hat{Y}^+$ is a lower solution for (4.1; Y_R) and the bounds in the Lemma statement follow. □

We decompose each discrete solution $Y_{L\backslash R}$ of (4.1) into the sum of a regular components $V_{L\backslash R}$ and a layer component $W_{L\backslash R}$. The left and right regular components are defined as the solutions of the following discrete problems:

$$D^{\mp}[\varepsilon D^{\pm} V_{L\backslash R} + F(V_{L\backslash R})] - bV_{L\backslash R} = q, \quad x_i \in \Omega^N_{L\backslash R}, \tag{4.5a}$$

$$V_{L\backslash R}(\mp 1) = A\backslash B, \qquad V_{L\backslash R}(D^N_\varepsilon) = v_{L\backslash R}(D^N_\varepsilon). \tag{4.5b}$$

The left and right discrete layer components are defined as the solutions of the following discrete problems

$$D^{\mp}[\varepsilon D^{\pm} W_{L\backslash R} + \xi^N_{L\backslash R} W_{L\backslash R}] - bW_{L\backslash R} = 0, \quad x_i \in \Omega^N_{L\backslash R}, \tag{4.6a}$$

$$W_{L\backslash R}(D^N_\varepsilon) = -V_{L\backslash R}(D^N_\varepsilon), \qquad W_{L\backslash R}(\mp 1) = 0, \tag{4.6b}$$

$$\xi^N_{L\backslash R}(x_i) := \int_0^1 f(tY_{L\backslash R}(x_i) + (1-t)V_{L\backslash R}(x_i))\,dt\ . \tag{4.6c}$$

We now determine bounds on $V_{L\backslash R}$, $W_{L\backslash R}$ and on the error $|V_{L\backslash R} - v_{L\backslash R}|$.

Lemma 6. *Assuming* (2.1c)-(2.1f)*, If* $v_{L\backslash R}$ *and* $w_{L\backslash R}$ *are the solutions of* (3.11) *and* (3.13) *and* $V_{L\backslash R}$ *and* $W_{L\backslash R}$ *are the solutions of* (4.5) *and* (4.6) *respectively then we have the bounds*

$$\Psi^- \leqslant V_L(x_i) \leqslant -\gamma_L, \quad \gamma_R \leqslant V_R(x_i) \leqslant \Psi^+, \quad x_i \in \bar{\Omega}^N_{L\backslash R},$$
$$|(V_{L\backslash R} - v_{L\backslash R})(x_i)| \leqslant CN^{-1}, \quad x_i \in \bar{\Omega}^N_{L\backslash R},$$
$$|W_{L\backslash R}(x_i)| \leqslant \max\{|\Psi^\pm|\} e^{-\frac{\gamma_{L\backslash R}}{2\varepsilon}|D^N_\varepsilon - x_i|}, \quad x_i \in \bar{\Omega}^N_{L\backslash R},$$
$$|W_{L\backslash R}(x_i)| \leqslant \max\{|\Psi^\pm|\} N^{-1} \quad \textit{for} \quad i \leqslant \tfrac{N}{4} \backslash i \geqslant \tfrac{3N}{4} \quad \textit{if} \quad \sigma_{L\backslash R} = \tfrac{2\gamma_{L\backslash R}}{\varepsilon} \ln(N).$$

Proof. To bound $V_{L\backslash R}$, repeat the method used in Lemma 5 but replacing the terminal/initial condition in (4.4) by $\hat{Y}^\mp(D^N_\varepsilon) = \mp\gamma_{L\backslash R}$ to establish the bounds on $V_{L\backslash R}$ given in the Lemma statement. Using (3.11) and (4.5), we can show that the error $E_V := V_L - v_L$ satisfies the following discrete problem:

$$D^-[\varepsilon D^+ E_V + \xi^N_V E_V] - bE_V = \tau_V, \quad x_i \in \Omega^N_L, \tag{4.7a}$$
$$E_V(-1) = E_V(D^N_\varepsilon) = 0, \tag{4.7b}$$
$$\xi^N_V(x_i) := \int_0^1 f(tV_L(x_i) + (1-t)v_L(x_i))\, dt\ , \tag{4.7c}$$
$$\tau_V(x_i) := \varepsilon[\tfrac{d^2}{dx^2} - D^-D^+]v_L(x_i) + [\tfrac{d}{dx} - D^-]F(v_L(x_i)). \tag{4.7d}$$

We first estimate the truncation error τ_V. Note the following for any $k \in C^1$:

$$\tau^\pm[k(x_i)] := (\tfrac{d}{dx} - D^\pm)k(x_i) = \pm\frac{1}{h_{i+1\backslash h_i}} \int_{x_i}^{x_{i\pm1}} k'(x_i) - k'(t)\, dt\ , \tag{4.8a}$$
$$\tau_V(x_i) = \varepsilon D^- \tau^+[v_L(x_i)] + \tau^-[(F(v_L) + \varepsilon v_L')(x_i)]. \tag{4.8b}$$

Using (2.1c), the equation ((3.11); v_L) and (3.14c), we have

$$\|(F(v_L))''\| \leqslant \|F''\|\|v_L'\|^2 + \|F'\|\|v_L''\| \leqslant C; \ \max_{t\in[x_{i-1},x_i]} \varepsilon|v_L''(x_i) - v_L''(t)| \leqslant Ch_i.$$

Hence, using (4.8), we can show that

$$|\tau_V(x_i) - \varepsilon D^-\tau^+[v_L(x_i)]| \leqslant \hat{C}N^{-1}; \qquad \tau^+[v_L(x_i)] \leqslant CN^{-1}. \tag{4.9}$$

Consider the mesh functions $E^\pm$ satisfying the discrete terminal value problems:

$$\varepsilon D^+ E^\pm + \xi^N_V E^\pm = -T^\pm, \quad x_i < D^N_\varepsilon, \quad E^\pm(D^N_\varepsilon) = 0, \tag{4.10a}$$
$$T^\pm := \varepsilon(\pm\|\tau^+[v_L(x_i)]\| - \tau^+[v_L(x_i)]) \pm \hat{C}N^{-1}(1 + x_i). \tag{4.10b}$$

Clearly $\pm T^{\pm} \geqslant 0$ and from (4.9) we have $\|T^{\pm}\| \leqslant \tilde{C}N^{-1}$. Since $V_L, v_L < -\gamma_L$, using (2.1f), we can show that $\xi_V^N \leqslant -\gamma_L$. With this, we can easily show that 0 and $\tilde{C}N^{-1}/\gamma_L$ are lower and upper solutions of ((4.10); E^+) respectively and that $-\tilde{C}N^{-1}/\gamma_L$ and 0 are lower and upper solutions of ((4.10); E^-) respectively. Hence

$$\begin{aligned} &D^-[\varepsilon D^+E^+ + \xi_V^N E^+] - bE^+ - \tau_V \\ =& - D^-[T^+] - bE^+ - \tau_V \leqslant -\hat{C}N^{-1} + |\varepsilon D^-\tau^+[v_L(x_i)] - \tau_V| \leqslant 0, \end{aligned}$$

and E^+ is an upper solution for (4.7). We can show that E^- is a lower solution in the same manner. We may bound $V_R - v_R$ analogously.

We now prove the bound on the left discrete layer component W_L. Since $W_L(D_\varepsilon^N) = -V_L(D_\varepsilon^N) \geqslant \gamma_L > 0$ we can show that 0 is a lower solution of ((4.6); W_L). Consider the function $\overline{W}(x_i)$ defined for $0 \leqslant i \leqslant \frac{N}{2}$ as

$$\overline{W}(x_i) = W_L(d_\varepsilon) \prod_{j=i}^{N/2} \Big(1 - \tfrac{h_{j+1}}{\varepsilon}\xi_L^N(x_j)\Big)^{-1} > 0, \quad \overline{W}(x_{\frac{N}{2}}) = W_L(d_\varepsilon).$$

Substitute $\underline{W}$ into ((4.6); W_L) to verify that it is an upper solution. Hence $0 \leqslant W_L(x_i) \leqslant \overline{W}(x_i)$ for $0 \leqslant i \leqslant N/2$. We now bound $\overline{W}$ by $Ce^{-C/\varepsilon|D_\varepsilon^N - x|}$ by showing that $\xi_L^N \leqslant -C < 0$. Recalling from Lemma 5 that $\Psi^- \leqslant Y_L \leqslant 0$ and using $V_L \leqslant -\gamma_L$ and (2.1f), we can show that

$$\xi_L^N(x_i) = \int_0^1 f(tY_L(x_i) + (1-t)V_L(x_i))\,dt \leqslant \tfrac{1}{2}(Y_L + V_L)(x) \leqslant -\tfrac{1}{2}\gamma_L\ . \tag{4.11}$$

Note the following inequalities:

$$e^{(1-t/2)t} \leqslant 1 + t \leqslant e^t, \quad t > 0; \qquad e^t \leqslant 1 + 2t, \quad t \in [0, 0.5]; \tag{4.12a}$$
$$e^{-(1+t)t} \leqslant 1 - t \leqslant e^{-t} \leqslant 1 - t/2, \quad t \in [0, 0.5]. \tag{4.12b}$$

Using these with (4.11) and (4.3) we can establish the exponential bound on the left discrete layer component in the Lemma statement, and hence it's negligibility outside the fine mesh in the case where $\sigma_L = C\varepsilon\ln(N)$ (see (4.2)). We may bound the solution of ((4.6); W_R) in a similar manner. $\square$

A bound on the error $Y_\varepsilon - y_\varepsilon$ is given in the following lemma.

Theorem 7. *If y_ε is the solution of (2.1) and Y_ε is the solution of (4.1)-(4.2), then*

$$\|\overline{Y}_\varepsilon^N - y_\varepsilon\| \leqslant CN^{-1}\ln(N)$$

where $\overline{Y}_\varepsilon^N$ is the linear interpolant of Y_ε and $\|k\| = \max_{t\in[-1,1]} |k(t)|$.

Proof. We first bound $Y_L - y_L$. If $\sigma_L = 2/\gamma_L\varepsilon\ln(N)$ then for $i \leqslant \frac{N}{4}$, using Lemma 4, Lemma 6 and (4.2) we have

$$|(Y_L - y_L)(x_i)| \leqslant |W_L(x_i)| + |w_L(x_i)| + |(V_L - v_L)(x_i)| \leqslant CN^{-1}.$$

Consider all mesh points in $(x_J, D_\varepsilon^N)\cap\bar{\Omega}_\varepsilon^N$ where $J = x_{\frac{N}{4}}$ if $\sigma_L = 2/\gamma_L\varepsilon\ln(N)$ or $J = 0$ otherwise for which by (4.3), we have $h_i/\varepsilon \leqslant CN^{-1}\ln N$. The error $E_Y := Y_L - y_L$ over (x_J, D_ε^N) satisfies the following discrete problem:

$$D^-[\varepsilon D^+ E_Y + \xi_Y^N E_Y] - bE_Y = \tau_Y, \quad x_i \in \Omega_L^N \cap (x_J, D_\varepsilon^N), \tag{4.13a}$$

$$|E_Y(x_J)| \leqslant CN^{-1}, \quad |E_Y(D_\varepsilon^N)| = CN^{-1}, \tag{4.13b}$$

$$\xi_Y^N(x_i) := \int_0^1 f(tY_L(x_i) + (1-t)y_L(x_i))\,dt\ , \tag{4.13c}$$

$$\tau_Y(x_i) := \varepsilon[\tfrac{d^2}{dx^2} - D^-D^+]y_L(x_i) + [\tfrac{d}{dx} - D^-]F(y_L(x_i)). \tag{4.13d}$$

Using (2.1c), the equation ((3.8); y_L) and Lemma 4, we have

$$\max_{t\in[x_{i-1},x_i]\subset[x_J,D_\varepsilon^N]} |y_L^{(k)}(t)| \leqslant C(1 + \varepsilon^{-k}e^{-\frac{\gamma_L}{2\varepsilon}(D_\varepsilon^N - x_i)})$$

for $k = 0, 1, 2$. Hence, using (4.3), (4.8) and (4.12) for sufficiently large N, we can show that

$$\begin{aligned}|\tau_Y(x_i)| &\leqslant Ch(|y_L'| + |y_L'|^2 + |y_L''|) \leqslant Ch(1 + \varepsilon^{-2}e^{-\frac{\gamma_L}{2\varepsilon}(D_\varepsilon^N - x_i)})\\ &\leqslant Ch(1 + \varepsilon^{-2}\left(1 + \tfrac{\gamma_L}{2\varepsilon}h\right)^{i-N/2}) \leqslant \hat{C}h(1 + \varepsilon^{-1}D^-\left(1 + \tfrac{\gamma_L}{2\varepsilon}h\right)^{i+1-N/2})\\ &:= \hat{C}h(1 + \varepsilon^{-1}D^-\Delta(x_i)).\end{aligned}$$

Using (3.7) and Lemma 4 and considering the cases $x < D_\varepsilon^N - C\varepsilon$ and $D_\varepsilon^N - C\varepsilon \leqslant x \leqslant D_\varepsilon^N$ separately for sufficiently small ε and large N, we can show that

$$\begin{aligned}y_L(x) = v_L(x) + w_L(x) &\leqslant v_L(x) - v_L(D_\varepsilon^N)e^{-\frac{\gamma_L}{2\varepsilon}(D_\varepsilon^N - x)} + |y(D_\varepsilon^N)|\\ &\leqslant -\frac{\gamma_L}{2}(1 - e^{-\frac{\gamma_L}{2\varepsilon}(D_\varepsilon^N - x)}) + CN^{-1} =: \beta(x) + CN^{-1}.\end{aligned}$$

Using this and rewriting ξ_Y^N we have

$$\begin{aligned}\xi_Y^N(x_i) = &\int_0^1 f(tY_L(x_i)+(1-t)y_L(x_i))\ dt\\ = &\int_0^1 f(tY_L(x_i))\ dt + y_L(x_i)\int_0^1 (1-t)\int_0^1 f'((tY_L+s(1-t)y_L)(x_i))dsdt.\end{aligned}$$

Hence, using (2.1f) and $Y_L \leqslant 0$, we have

$$\xi_Y^N(x_i) \leqslant \begin{cases} y_L(x_i) \leqslant \beta(x) + CN^{-1}, & y_L(x_i) \leqslant 0,\\ CN^{-1}, & y_L(x_i) > 0.\end{cases}$$

Hence, for sufficiently small ε and large N, we can show that

$$\varepsilon D^+\beta(x_i) + \xi_Y^N(x_i)\beta(x_i) \geqslant \tfrac{\gamma_L^2}{8}. \tag{4.15}$$

Consider the mesh functions $E^\pm$ satisfying the discrete terminal value problems:

$$\varepsilon D^+E^\pm + \xi_Y^N E^\pm = -T^\pm := \mp \hat{C}h(1 + x_i + \varepsilon^{-1}\Delta(x_i)), \tag{4.16a}$$

$$E^\pm(D_\varepsilon^N) = \pm\bar{C}N^{-1}. \tag{4.16b}$$

Clearly $\pm T^\pm \geqslant 0$, it follows that 0 is a lower/upper solution for ((4.16); $E^\pm$). Using (4.3), we have $\|T^\pm\| \leqslant \tilde{C}N^{-1}\ln N$. Consider the mesh functions $\bar{E}^\pm := \mp\frac{8}{\gamma_L^2}(\tilde{C}N^{-1}\ln N + \bar{C}\|f\|N^{-1})\beta(x) \pm \bar{C}N^{-1}$. Using (4.15) and (4.16), we can show that

$$\begin{aligned}&\varepsilon D^+\bar{E}^+ + \xi_Y^N\bar{E}^+ + T^+\\ \leqslant&(-\tfrac{8}{\gamma_L^2}[\varepsilon D^+\beta(x_i) + \xi_Y^N(x_i)\beta(x_i)] + 1)(\tilde{C}N^{-1}\ln N + \bar{C}\|f\|N^{-1}) \leqslant 0.\end{aligned}$$

Thus $\bar{E}^+$ is an upper solution of ((4.16); E^+). We may analagously check that $\bar{E}^+$ is a lower solution of ((4.16); E^-). Hence

$$\begin{aligned}&D^-[\varepsilon D^+E^+ + \xi_Y^N E^+] - bE^+ - \tau_Y\\ =&-D^-[T^+] - bE^+ - \tau_Y \leqslant -\hat{C}h(1+\varepsilon^{-1}D^-\Delta(x_i)) + |\tau_Y(x_i)| \leqslant 0.\end{aligned}$$

Hence E^+ is an upper solution for (4.13). We can show that E^- is a lower solution in the same manner. We may bound $Y_R - y_R$ in the same manner.

The global bound follows as in [6, pg. 381]. □

5. Numerical examples

5.1. Example 1

Consider the following example problem case of (2.1):

$$\varepsilon y'' + yy' - b(x)y = 0, \quad x \in (-1,1), \quad b(x) = 1/8(x+1)e^{1/4(x+1)^2}, \tag{5.1a}$$
$$y(-1) = A = -3.25, \quad y(1) = B = 3.5 + 1/4(e^1 - 2e^{1/16}). \tag{5.1b}$$

For this problem data, we can show that

$$r_L(x) = A + \tfrac{1}{4}(e^{\frac{1}{4}(x+1)^2} - 1),\ r_R(x) = B + \tfrac{1}{4}(e^{\frac{1}{4}(x+1)^2} - e^1), d_0 = -0.5, \tag{5.2a}$$
$$v_{1L}(x) = -\int_{-1}^{x} \tfrac{b'(t)}{r_L(t)} dt \ , \quad v_{1R}(x) = \int_{x}^{1} \tfrac{b'(t)}{r_R(t)} dt \ , \quad d_1 = -\tfrac{(v_{1L}+v_{1R})(d_0)}{2b(d_0)}, \tag{5.2b}$$
$$\sqrt{A^2 - 4|A|\|b\|} > 1.25 = \gamma_L, \quad \sqrt{B^2 - 4B\|b\|} > 1.75 = \gamma_R, \tag{5.2c}$$

where $r_{L\setminus R}$ and d_0 are defined in (2.2)-(2.3) and $v_{1L\setminus R}$ and d_1 are defined in (3.3)-(3.4) and $\gamma_{L\setminus R}$ are as defined in (2.1e). For this particular example, to approximate d_1, we use a standard numerical integration method to obtain an estimate d_1^N s.t. $|d_1 - d_1^N| \leqslant CN^{-1}$. The finite difference scheme in (4.1) is nonlinear. Note that

$$D^-[\tfrac{1}{2}Y_L(x_i)^2] = \tfrac{1}{2}(Y_L(x_i) + Y_L(x_{i-1}))D^-Y_L(x_i), \tag{5.3a}$$
$$D^-[\tfrac{1}{2}Y_L(x_i)^2] = \tfrac{1}{2}(Y_L(x_i) + Y_L(x_{i-1}))D^-Y_L(x_i). \tag{5.3b}$$

We choose to approximate the discrete solution of (4.1) using the adaptive time-step algorithm described in [2, pg. 196] with linearised difference schemes motivated by the identities (5.3) as follows.

Numerical Scheme 1. *Approximate the discrete solutions Y_L and Y_R of (4.1) each with a sequence of mesh functions L_j^N and R_j^N where L_j^N satisfies the linear difference scheme:*

$$\varepsilon D^- D^+ L_j^N + \tfrac{1}{2}(L_{j-1}^N(x_i) + L_{j-1}^N(x_{i-1}))D^- L_j^N - bL_j^N = \tfrac{1}{k_j}(L_j^N - L_{j-1}^N)(x_i),$$
$$x_i \in \Omega_L^N, \quad L_j^N(-1) = A, \quad L_j^N(D_\varepsilon^N) = 0, \tag{5.4}$$

where x_i are meshpoints on the mesh Ω_ε^N in (4.2) *with mesh parameters in* (5.2) *and where R_j^N satisfies an analogous difference scheme. Motivated by the bounds in Lemma 4, we choose the initial conditions*

$$L_0^N(x_i) = r_L(x_i)+|r_L(D_\varepsilon^N)|e^{-\frac{\gamma_L}{2\varepsilon}(D_\varepsilon^N-x)},\ R_0^N(x_i) = r_R(x_i)-r_R(D_\varepsilon^N)e^{-\frac{\gamma_R}{2\varepsilon}(x-D_\varepsilon^N)}.$$

See [2, pg. 196] for a description of how the iterative step size k_j is chosen. Construct an approximation of Y_ε^N in (4.1) *with a specified tolderance ϵ_{tol} as follows:*

$$U_\varepsilon^N(x_i) = \{L_{J_L}^N(x_i),\quad x_i < D_\varepsilon^N;\quad 0,\quad x_i = D_\varepsilon^N;\quad L_{J_R}^N(x_i),\quad x_i > D_\varepsilon^N\}, \tag{5.5a}$$

$$\textit{where}\quad \max_{x_i} \frac{1}{k_{J_{L\setminus R}}}|(L^N\setminus R^N{}_{J_{L\setminus R}} - L^N\setminus R^N{}_{J_{L\setminus R}-1})(x_i)| \leqslant \epsilon_{tol}. \tag{5.5b}$$

Fig. 1 (a) displays a sample numerical solution generated from Numerical Scheme 1. Fig. 1 (b) displays sample numerical solutions when $D_\varepsilon^N = d_0$ is used instead of $D_\varepsilon^N = d_0 + \varepsilon d_1^N$ in the mesh (4.2), along with cases when $D_\varepsilon^N = d_0+\varepsilon d_1^N$ is used. Note the slight displacement of the layer for increasing N in Fig. 1 (b), which is negligible however compared to the displacement when $D_\varepsilon^N = d_0$ is used.

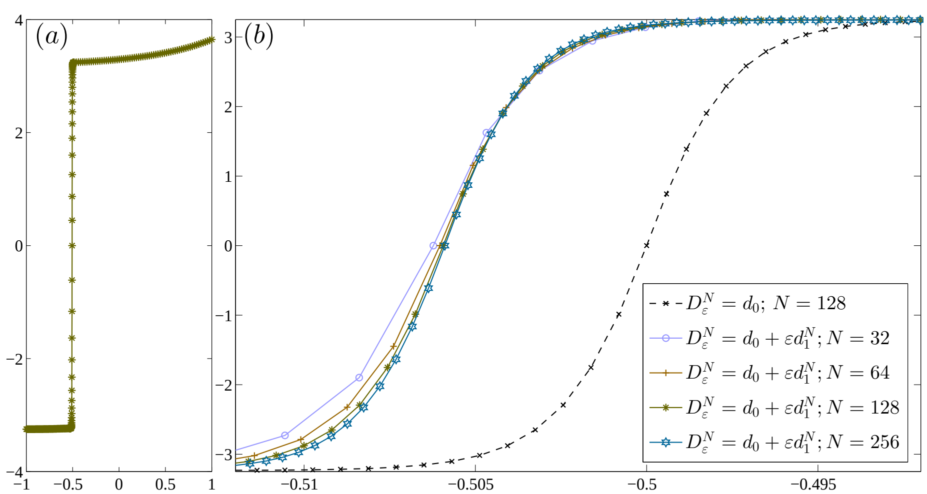

Figure 1: (a) Numerical solution over $[-1, 1]$ constructed with Numerical Scheme 1 with $\varepsilon = 2^{-8}$ and $N = 128$; (b) Numerical solutions over $[-0.512, -0.492]$ constructed with Numerical Scheme 1 with $\varepsilon = 2^{-8}$ and a selection of choices for the point D_ε^N in the mesh ((4.2)-(5.2)).

We estimate the orders of convergence using the double mesh principle ([2]). For sample values of N and ε, we construct numerical solutions U_ε^N as in (5.5). We construct the computed orders of global convergence p_ε^N and the computed orders of uniform global convergence p^N as follows:

$$m_\varepsilon^N := \max_{\Omega_\varepsilon^N \cup \Omega_\varepsilon^{2N}} |\overline{U}_\varepsilon^N - \overline{U}_\varepsilon^{2N}|; p_\varepsilon^N = \log_2\left(\frac{m_\varepsilon^N}{m_\varepsilon^{2N}}\right); p^N = \log_2\left(\frac{\max_\varepsilon\{m_\varepsilon^N\}}{\max_\varepsilon\{m_\varepsilon^{2N}\}}\right), \quad (5.6)$$

where $\overline{U}_\varepsilon^K$, $K = N, 2N$ are the linear interpolants of $\overline{U}_\varepsilon^K$ onto the mesh $\Omega_\varepsilon^N \cup \Omega_\varepsilon^{2N}$. Table 1 displays the computed orders of global and uniform global convergence defined in (5.6) which demonstrate the first order convergence rate (with a logarithmic factor) established in Theorem 7.

Table 1: Computed orders of global and uniform global convergence defined in (5.6) generated from numerical solutions of Numerical Scheme 1 for example problem ((5.1)-(5.2)).

	p_ε^N							
ε	**N=32**	**N=64**	**N=128**	**N=256**	**N=512**	**N=1024**	**N=2048**	**N=4096**
$\mathbf{2^{-3}}$	1.02	1.01	1.00	0.96	0.76	1.01	1.00	1.00
$\mathbf{2^{-4}}$	1.07	1.04	1.03	1.02	1.01	0.97	0.81	0.83
$\mathbf{2^{-5}}$	0.97	0.77	0.85	1.04	1.02	1.01	1.01	1.00
$\mathbf{2^{-6}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{2^{-7}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{2^{-8}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{2^{-9}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{2^{-10}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{2^{-20}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{2^{-30}}$	0.97	0.87	0.81	0.88	0.87	0.87	0.88	0.88
$\mathbf{p^N}$	**0.97**	**0.77**	**0.85**	**0.94**	**0.87**	**0.87**	**0.88**	**0.88**

We now demonstrate a distinction between numerical solutions of (4.1) generated over the mesh (4.2) centred at the appropriate point $D_\varepsilon^N = d_0 + \varepsilon d_1^N$ and at the point d_0 for all values of ε and N. For the same sample values of N and ε, we construct numerical solutions W_ε^N as in (5.5) on the mesh $\Omega_\varepsilon^{N,0}$ ((4.2)-(5.2)) centred at d_0 as opposed to $d_0 + \varepsilon d_1^N$. For each sample value of ε, we construct a fine mesh numerical solution U_ε^{16384} in (5.5) on the mesh $\Omega_\varepsilon^{16384,\varepsilon}$ ((4.2)-(5.2)) centred at $d_0 + \varepsilon d_1^N$ with $N = 2^{14}$. Table 2 displays the quantities

$$E_\varepsilon^N := \max_{\Omega_\varepsilon^{N,0} \cup \Omega_\varepsilon^{16384,\varepsilon}} |\overline{W}_\varepsilon^N - \overline{U}_\varepsilon^{16384}|, \quad (5.7)$$

where $\overline{W}_\varepsilon^N$ and $\overline{U}_\varepsilon^{16384}$ are the linear interpolants of the mesh functions W_ε^N and U_ε^{16384} onto the mesh $\Omega_\varepsilon^{N,0} \cup \Omega_\varepsilon^{16384,\varepsilon}$.

Table 2: Computed errors (5.7) between numerical solutions generated from Numerical Scheme 1 on mesh ((4.2), (5.2)) centred at the leading term d_0 of the asymptotic expansion of the layer location and a fine mesh numerical solution generated on the mesh ((4.2), (5.2)) centred at $d_0 + \varepsilon d_1^N$ where d_1^N is an approximation to the second term in the asymptotic expansion of the layer location.

	E_ε^N						
ε	**N=32**	**N=64**	**N=128**	**N=256**	**N=512**	**N=1024**	**N=2048**
$\mathbf{2^{-3}}$	5.310	5.317	5.351	5.380	5.402	5.417	5.427
$\mathbf{2^{-4}}$	5.303	5.309	5.341	5.368	5.390	5.403	5.412
$\mathbf{2^{-5}}$	5.300	5.304	5.335	5.362	5.383	5.396	5.404
$\mathbf{2^{-10}}$	5.296	5.299	5.329	5.355	5.375	5.389	5.396
$\mathbf{2^{-15}}$	5.296	5.299	5.329	5.355	5.375	5.388	5.396
$\mathbf{2^{-20}}$	5.296	5.299	5.329	5.355	5.375	5.388	5.396
$\mathbf{2^{-25}}$	5.296	5.299	5.329	5.355	5.375	5.388	5.396
$\mathbf{2^{-30}}$	5.296	5.299	5.329	5.355	5.375	5.388	5.396

Note, some sample values of ε used to construct Tables 1-2 are hidden.

5.2. Example 2

In this example, we consider the following case of the time-periodic problem analysed in [4]:

$$\varepsilon(u_{xx} - u_t) + uu_x - b(t)u = 0, \quad x \in (-1, 1), \tag{5.8a}$$

$$u(-1, t) = A(t) = -6 + \tfrac{1}{4}\sin(4\pi t), \quad u(1, t) = B(t) = 5.25 - \tfrac{1}{2}\sin(4\pi t), \tag{5.8b}$$

$$b(t) = 1 + \tfrac{1}{2}\cos(4\pi t), \quad u(x, t) = u(x, t + T). \tag{5.8c}$$

For this problem data, we can show that

$$r_L(x, t) = A(t) + b(t)(x + 1), \qquad r_R(x, t) = B(t) + b(t)(x - 1), \tag{5.9a}$$

$$d_0(t) = -\tfrac{1}{2b(t)}(A + B)(t) = \frac{3 + \sin(4\pi t)}{8 + 4\cos(4\pi t)}, \tag{5.9b}$$

$$v_{1L}(x, t) = \tfrac{d}{dt}[\tfrac{A(t)}{b(t)}]\ln(1 + \tfrac{b(t)}{A(t)}(x + 1)) + \tfrac{d}{dt}[\ln(b(t))](x + 1), \tag{5.9c}$$

$$v_{1R}(x, t) = \tfrac{d}{dt}[\tfrac{B(t)}{b(t)}]\ln(1 + \tfrac{b(t)}{B(t)}(x - 1)) + \tfrac{d}{dt}[\ln(b(t))](x - 1), \tag{5.9d}$$

$$d_1(t) = -\frac{1}{2b(t)}(2d_0'(t) + (v_{1L} + v_{1R})(d_0(t), t)), \tag{5.9e}$$

where $r_{L\backslash R}$ are the reduced solutions, $v_{1L\backslash R}$ are the $O(\varepsilon)$ regular components and $d_0(t)$ s.t. $(r_L + r_R)(d_0(t), t) = 0$ and $d_1(t)$ are the leading and first terms respectively of the asmyptotic expansion of the layer location $d_\varepsilon(t)$ of the solution of (5.8). We consider the example problem (5.8) for $0 \leqslant t \leqslant 1.5$ and construct a numerical approximation for the solution u. We estimate $u(x, 0)$ with the solution of the problem

$$\varepsilon g'' + gg' - b(0)g = 0, \quad x \in (-1, 1), \tag{5.10a}$$

$$g(-1) = A(0) = -6, \quad g(1) = B(0) = 5.25, \quad b(0) = 1.5, \tag{5.10b}$$

$$g \text{ has interior layer at } d_\varepsilon(0) \text{ sufficiently close to } d_0(0) + \varepsilon d_1(0). \tag{5.10c}$$

We construct a numerical approximation G_ε^N of (5.10) using Numerical Scheme (1) on the mesh ((4.2),$\gamma_{L\backslash R} = 1$) centred at the point $D_\varepsilon(t) := d_0(t) + \varepsilon d_1(t)$ when $t = 0$. We consider the problem (5.8) as left and right problems around the curve $D_\varepsilon(t)$ for all time and set numerical approximations to zero along the curve $D_\varepsilon(t)$. The left and right problem domains are non-rectangular hence we consider these problems under the change of variables $\xi_{L\backslash R}(x, t) := m_{L\backslash R}(t)(x \pm 1) \mp 1$ where $m_{L\backslash R}(t) = (D_\varepsilon(0) \pm 1)/(D_\varepsilon(t) \pm 1)$ s.t. $\xi_L(-1, t) = -1$, $\xi_{L\backslash R}(D_\varepsilon(t), t) = D_\varepsilon(0)$, $\xi_R(1, t) = 1$. The left and right transformed problems can be written as follows:

$$\varepsilon(m(t)^2 y_{\xi\xi} - y_t) + (m(t)y - \varepsilon \tfrac{m'(t)}{m(t)}(\xi \pm 1))y_\xi - b(t)y = 0, \tag{5.11a}$$

$$y(-1, t) = A(t), \quad y_{L\backslash R}(D_\varepsilon(0), t) = 0, \quad y(+1, t) = B(t), \tag{5.11b}$$

$$y(x, 0) = g(x), \tag{5.11c}$$

where here, we have ommitted the left and right subscripts of m, ξ and y. We approximate the solutions y_L and y_R of the problems (5.11) with the linear finite difference schemes

$$\varepsilon(m_L(t_j)^2\delta_x^2 Y_L - D_t^- Y_L) + \bar{L}(x_i,t_j,t_{j-1})D_x^- Y_L(x_i,t_j) - b(t_j)Y_L(x_i,t_j) = 0, \tag{5.12a}$$

$$\bar{L}(x_i,t_j,t_{j-1}) := (m_L(t_j)Y_L(x_i,t_{j-1}) - \varepsilon\tfrac{m_L'(t_j)}{m_L(t_j)}(x_i+1)), \tag{5.12b}$$

$$0 < i < N/2, \quad Y_L(-1,t_j) = A(t_j), \quad Y_L(D_\varepsilon(0),t_j) = 0, \tag{5.12c}$$

$$\varepsilon(m_R(t)^2\delta_x^2 Y_R - D_t^- Y_R) + \bar{R}(x_i,t_j,t_{j-1})D_x^+ Y_R(x_i,t_j) - b(t)Y_R(x_i,t_j) = 0, \tag{5.12d}$$

$$\bar{R}(x_i,t_j,t_{j-1}) := (m_R(t_j)Y_R(x_i,t_{j-1}) - \varepsilon\tfrac{m_R'(t_j)}{m_R(t_j)}(x_i-1)), \tag{5.12e}$$

$$N/2 < i < N, \quad Y_R(D_\varepsilon(0),t_j) = 0, \quad Y_R(1,t_j) = B(t_j), \tag{5.12f}$$

$$Y(x_i,0) = G_\varepsilon^N(x_i), \tag{5.12g}$$

where (x_i,t_j) are nodes on the grid $\Omega \times \{t_j | t_j = j\frac{1.5}{N}\}$ where Ω is the same mesh used to construct the approximation G_ε^N of (5.10), that is ((4.2),$D_\varepsilon^N = d_0(0) + \varepsilon d_1(0)$, $\gamma_{L\backslash R} = 1$). Figure 2 displays a numerical solution generated from the difference scheme (5.12) and transformed to the variables (x,t).

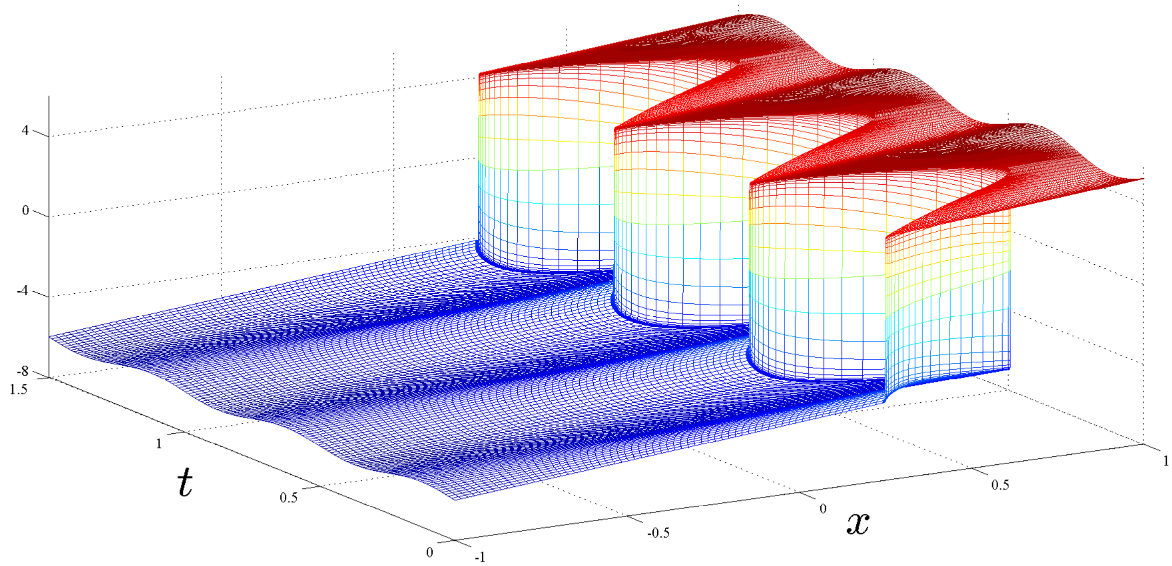

Figure 2: Numerical approximation transformed to the x and t variables of the solution of the example case ((5.8),(5.9)) of time-periodic problems studied in [4] generated from the difference scheme (5.12) using an approximation of (5.10) constructed from the Numerical Scheme (1) as an in initial condition.

Appendix A. Algorithm to construct the asymptotic expansion of the layer location

Here, we perform the algorithm as described in [4] to find the first three terms in the asymptotic expansion of d_ε and simplify the expressions for these terms using the assumptions on the problem data ((2.1f), (2.3)).

Decompose the solution of (2.1) into left and right boundary value problems whose solutions $y^\pm$ are to be further decomposed into the asymptotic expansions of a regular component $U^\pm$ in the variable x and a layer component $Q^\pm$ in the stretched variable $\xi = (x - d_\varepsilon)/\varepsilon$:

$$y_\varepsilon(x) = \{y^-(x), \quad x < d_\varepsilon; \quad \phi(d_\varepsilon), \quad x = d_\varepsilon; \quad y^+(x), \quad x > d_\varepsilon\};$$

$$\begin{matrix} \varepsilon y^{-\prime\prime} + (F(y^-))' - by^- = q, \\ y^-(-1) = A, \quad y^-(d_\varepsilon) = \phi(d_\varepsilon), \end{matrix} ; \qquad \begin{matrix} \varepsilon y^{+\prime\prime} + (F(y^+))' - by^+ = q, \\ y^+(d_\varepsilon) = \phi(d_\varepsilon), \quad y^+(1) = B, \end{matrix} ;$$

$$y^\pm(x) = U^\pm(x) + Q^\pm(\xi) = \sum_{i=0}^{\infty} \varepsilon^i \left(U_i^\pm(x) + Q_i^\pm(\xi)\right).$$

Note, the left and right layer components are solutions to problems defined over the domains $\xi < 0$ and $\xi > 0$ respectively. It is assumed in [4] that $\frac{dQ_i^\pm}{d\xi}(\pm\infty) = 0$. The choice of the appropriate boundary conditions to assign to $Q_i^\pm(0)$ is determined by using a standard Taylor series expansion to expand $y^\pm(x)$ and $\phi(x)$ around $x = d_\varepsilon$ (or $\xi = 0$) as follows:

$$\begin{aligned} y^\pm(d_\varepsilon) &= \sum_{i=0}^{\infty} \varepsilon^i \left(U_i^\pm(d_\varepsilon) + Q_i^\pm(0) \right) \\ &= [U_0^\pm(d_0) + Q_0^\pm(0)] + \varepsilon[d_1 U_0^{\pm\prime}(d_0) + U_1^\pm(d_0) + Q_1^\pm(0)] + \\ &\quad \varepsilon^2[d_2 U_0^{\pm\prime}(d_0) + \tfrac{1}{2} d_1^2 U_0^{\pm\prime\prime}(d_0) + d_1 U_1^{\pm\prime}(d_0) + U_2^\pm(d_0) + Q_2^\pm(0)] + \cdots ; \\ \phi(d_\varepsilon) &= [\phi(d_0)] + \varepsilon[d_1 \phi'(d_0)] + \varepsilon^2[d_2 \phi'(d_0) + \tfrac{1}{2!} d_1^2 \phi''(d_0)] + \cdots . \end{aligned}$$

Hence obtain the appropriate value of $Q_i^\pm(0)$ by equating each term in the expansions by powers of ε. Moreover, the d_i are determined by using the Taylor series expansion to expand $\varepsilon y^{\pm\prime}(x)$ around $x = d_\varepsilon$ (or $\xi = 0$) as follows:

$$\begin{aligned} \varepsilon y^{\pm\prime}(d_\varepsilon) &= \varepsilon \sum_{i=0}^{\infty} \varepsilon^i \left({U^\pm}'_i(x) \mid_{x=d_\varepsilon} + \varepsilon^{-1} {Q^\pm}'_i(\xi) \mid_{\xi=0} \right) \\ &= [{Q^\pm}'_0(0)] + \varepsilon[U_0^{\pm\prime}(d_0) + {Q^\pm}'_1(0)] + \varepsilon^2[(d_1 U_0^{\pm\prime\prime} + U_1^{\pm\prime})(d_0) + {Q^\pm}'_2(0)] \cdots . \end{aligned}$$

The left and right C^1-matching condition for the $O(1)$ terms reads ${Q^-}'_0(0) = {Q^+}'_0(0)$, the same for the $O(\varepsilon)$ terms reads $U_0^{-\prime}(d_0) + {Q^-}'_1(0) = U_0^{+\prime}(d_0) + {Q^+}'_1(0)$ and so on. We now find explicit values for d_0 and d_1 below. The $O(1)$ left and right regular and layer components satisfy:

$$\begin{gathered} U_0^\mp(x) \equiv r_{L\setminus R}(x); \\ {Q^\mp}''_0(\xi) + [F(r_{L\setminus R}(d_0) + Q_0^\mp(\xi))]' = 0, \\ Q_0^\mp(\mp\infty) = 0, \quad Q_0^\mp(0) = (\phi - r_{L\setminus R})(d_0), \end{gathered} \tag{A.1}$$

where $r_{L\setminus R}$ are the reduced solutions defined in (2.2). Note, the assumption in (2.3) predetermines that $\phi(d_0) = 0$. Using the method of upper and lower solutions with (A.1), we can show that solutions exist which satisfy

$$|Q_0^\pm(\pm\xi)| \leqslant |U_0^\pm(d_0)| \left(1 - \tanh\left(\frac{|U_0^\pm(d_0)|}{2}\xi\right)\right), \quad \xi \geqslant 0. \tag{A.2}$$

Note that in the fundamental subclass of problems (2.1), where $f(s) = s$, the inequality in (A.2) can be replaced by an equality.

Using (2.1f) and Lemma 2 and integrating both equations in (A.1) from 0 to $\pm\infty$, we can show that $(Q^{-\backslash +}{}_0)'(0) = \int_{\phi(d_0)}^{r_{L\backslash R}(d_0)} f(t)\, dt$. The C^1 matching condition reads

$$Q^{-\prime}{}_0(0) = Q^{+\prime}{}_0(0) \quad \Leftrightarrow \quad \int_{r_L(d_0)}^{r_R(d_0)} f(t)\, dt = 0 \quad \overset{f \text{ odd}}{\Leftrightarrow} \quad r_R(d_0) = -r_L(d_0). \tag{A.3}$$

The assumption on the problem data in (2.3) ensures that (A.3) has a solution in $(-1, 1)$ which satisfies $(r_L + r_R)(d_0)$. Define the function $\hat{Q}$ such that $Q_0^+(\xi) =: -\hat{Q}(-\xi)$, since f is odd and $r_L(d_0) = -r_R(d_0)$, we can use the equations for $Q^{-\backslash +}{}_0$ to show that $\hat{Q}(\xi) \equiv Q_0^-(\xi)$, $\forall\ \xi \leqslant 0$. Hence $Q_0^+(\xi) \equiv -Q_0^-(-\xi)$, $\forall\ \xi \geqslant 0$ and it follows that

$$\textstyle\int_{-\infty}^{0} Q_0^-(\xi)\, d\xi + \int_0^\infty Q_0^+(\xi)\, d\xi = 0 \tag{A.4}$$

which will be used momentarily to simplify the expression for d_1.

The $O(\varepsilon)$ left and right regular and layer components satisfy

$$\begin{array}{c} f(U_0^\pm)U_1^{\pm\prime} + (f'(U_0^\pm)U_0^{\pm\prime} - b)U_1^\pm = -U_0^{\pm\prime\prime}, \\ U_1^\pm(\pm 1) = 0, \end{array} ; \tag{A.5}$$

$$\begin{array}{c} Q^{\pm\prime\prime}{}_1(\xi) + [f(U_0^\pm(d_0) + Q_0^\pm(\xi))Q^\pm{}_1(\xi)]' \\ = b(d_0)Q_0^\pm(\xi) + \frac{d}{d\xi}[((d_1+\xi)U_0^{-\prime}(d_0) + U_1^\pm(d_0))(f(U_0^\pm(d_0)) - f(U_0^\pm(d_0) + Q_0^\pm(\xi)))], \\ Q_1^\pm(\pm\infty) = 0, \quad Q_1^\pm(0) = d_1(\phi' - U_0^{\pm\prime})(d_0) - U_1^\pm(d_0). \end{array} \tag{A.6}$$

From [4, pg. 95], solutions to (A.6) exist satisfying

$$|Q_1^\pm(\pm\xi)| \leqslant Ce^{-C\xi}, \quad \xi \geqslant 0. \tag{A.7}$$

Integrating both equations in (A.6) from 0 to $\pm\infty$, we can show that

$$Q_1^{\pm\prime}(0) = b(d_0)\ \textstyle\int_{\pm\infty}^{0} Q^\pm{}_0(t)\, dt + (d_1 U_0^{\pm\prime}(d_0) + U_1^\pm(d_0))f(U_0^\pm(d_0))\ .$$

Using (2.1f), (A.1) and (A.4), it follows from the C^1-matching condition that

$$r_L'(d_0) + Q^{-\prime}{}_1(0) = r_R'(d_0) + Q^{-\prime}{}_1(0) \quad \Leftrightarrow \quad d_1 = \left.\tfrac{r_R' - r_L' + U_1^+ f(r_R) - U_1^- f(r_L)}{r_L' f(r_L) - r_R' f(r_R)}\right|_{x=d_0}.$$

Using (2.1f), (2.2) and (A.3), we can simplify the expression for d_1 to

$$d_1 = -\frac{q}{br_R f(r_R)} - \frac{f(r_R)}{2br_R}(U_1^- + U_1^+)\bigg|_{x=d_0}. \tag{A.8}$$

Remark 3. *For the fundamental subclass of problems of* (2.1) *where* $f(s) = s$*, if these problems are homogeneous* ($q(x) \equiv 0$) *then from* (2.2) *we have* $r'_{L\backslash R}(x) = b(x)$ *and hence from* (A.5) *we can solve for* ${U_1}^{\pm}$ *explicitly as* ${U_1}^{\pm}(x) = -\int_{\pm 1}^{x} ({U_0}''/U_0)(t)dt$*. Hence, for* b *constant, observe from* (A.8) *that* $d_1 \equiv 0$*.*

Finally, before reviewing the main result in [4], we verify that d_2 is a bounded quantity for the problem (2.1). The $O(\varepsilon^2)$ left and right regular and layer components satisfy:

$$\begin{array}{c} f(U_0^{\pm}){U_2^{\pm}}' + (f'(U_0^{\pm}){U_0^{\pm}}' - b)U_2^{\pm} = -{U_1^{\pm}}'' - (\tfrac{1}{2}{U_1^{\pm}}^2 f'(U_0^{\pm}))', \\ U_2^{\pm}(\pm 1) = 0, \end{array} ;$$

$$\begin{aligned} &{Q^{\pm}}''_2(\xi) + [f(U_0^{\pm} + Q_0^{\pm}(\xi)){Q^{\pm}}_2(\xi)]' \\ &= (d_1+\xi)b' Q_0^{\pm}(\xi) + bQ_1^{\pm}(\xi) \\ &+ \frac{d}{d\xi}\Big[(d_2{U_0^-}' + \tfrac{1}{2}(d_1+\xi)^2{U_0^-}'' + (d_1+\xi){U_1^{\pm}}' + U_2^{\pm})(f(U_0^{\pm}) - f(U_0^{\pm} + Q_0^{\pm}(\xi)))\Big] \\ &+ \frac{d}{d\xi}\Big[\tfrac{1}{2}((d_1+\xi){U_0^-}' + U_1^{\pm})^2 F''(U_0^{\pm}) \\ &- \tfrac{1}{2}((d_1+\xi){U_0^-}' + U_1^{\pm} + Q_1^{\pm}(\xi))^2 F''(U_0^{\pm} + Q_0^{\pm}(\xi))\Big], \\ &Q_2^{\pm}(\pm\infty) = 0, \quad Q_2^{\pm}(0) = d_2(\phi' - {U_0^{\pm}}')(d_0) + \tfrac{1}{2}d_1^2(\phi'' - {U_0^{\pm}}'')(d_0) - (d_1{U_1^{\pm}}' - U_2^{\pm})(d_0). \end{aligned} \tag{A.9}$$

Note, for brevity, all occurrences of the functions b, $U_i^{\pm}$ and their derivatives in (A.9) represent those functions evaluated at $x = d_0$. Also, using the differentiability assumptions of b, q and F stated in (2.1), we can show that $\|U_j^{\pm(2-j)}\| \leqslant C$ for $j = 0, 1, 2$. Integrating both equations in (A.9) from 0 to $\pm\infty$, one can write down the C^1-matching condition and use (A.2), (A.4) and (A.7) to find that

$$\begin{aligned} |d_2| \leqslant & C\max\{|d_1|^i, \|f\|, \|f'\|, \|U_j^{\pm(2-j)}\|, \|{U_0^{\pm}}'\|\} \\ &+ 2\|b\| \max|\ \textstyle\int_{\pm\infty}^{0} Q_1^{\pm}(t)\ dt\ | + 2\|b'\|\ \textstyle\int_0^{\infty} tQ_0^{+}(t)\ dt\ \leqslant C \end{aligned}$$

for $i = 1, 2$ and $j = 0, 1, 2$.

At the point $x = d_\varepsilon^{(2)}$, when $n = 1$, the asymptotic error bound in [4, Theorem 2.1] reads

$$\left| y_\varepsilon(d_\varepsilon^{(2)}) - \left[\left(U_0^-(d_\varepsilon^{(2)}) + Q_0^-(0)\right) + \varepsilon\left(U_1^-(d_\varepsilon^{(2)}) + Q_1^-(0)\right)\right]\right| \leqslant C\varepsilon.$$

Using the boundary conditions $Q_0^-(0)$ and $Q_1^-(0)$, we can easily rewrite this error bound to show that $|y_\varepsilon(d_\varepsilon^{(2)})| \leqslant C\varepsilon$.

Appendix B. Numerical Algorithm to construct an approximation for the second term in the asymptotic expansion of the layer location

Step 1: If not known explicitly, approximate the reduced solutions $r_{L\setminus R}$ in (2.2) with mesh functions L^N and R^N satisfying the linearised difference schemes:

$$\bar{\Gamma}^N := \{x_i | x_i = -1 + (2/N)i, \quad 0 \leqslant i \leqslant N\}, \quad \Gamma^N := \bar{\Gamma}^N \setminus \{x_0, x_N\} \tag{B.1a}$$

$$(f(L^N(x_{i-1}))D^- L^N - bL^N)(x_i) = q(x_i), \quad x_i \in \bar{\Gamma}^N \setminus \{x_0\}, \quad L^N(x_0) = A; \tag{B.1b}$$

$$(f(R^N(x_{i+1}))D^+ R^N - bR^N)(x_i) = q(x_i), \quad x_i \in \bar{\Gamma}^N \setminus \{x_N\}, \quad R^N(x_N) = B; \tag{B.1c}$$

Step 2: Approximate $v_{1L\setminus R}$ in (3.4) with mesh functions V^- and V^+ satisfying the linear difference schemes:

$$(f(L^N)D^- + (f'(L^N)D^- L^N - b))V^- = -\delta^2 L^N, \tag{B.2a}$$

$$x_i \in \bar{\Gamma}^N \setminus \{x_0\}, \quad V^-(x_0) = 0, \tag{B.2b}$$

$$(f(R^N)D^+ + (f'(R^N)D^+ R^N - b))V^+ = -\delta^2 R^N, \tag{B.2c}$$

$$x_i \in \bar{\Gamma}^N \setminus \{x_N\}, \quad V^+(x_N) = 0. \tag{B.2d}$$

Step 3: Construct d_1^N as follows

$$d_1^N = -\tfrac{q}{bR^N f(R^N)} - \tfrac{f(R^N)}{2bR^N}(V^- + V^+)\Big|_{x_i = d_0^-}, \tag{B.3}$$

where d_0^- is the greatest mesh point $x_i \in \Gamma^N$ satisfying $x_i \leqslant d_0^N = d_0$.

We now proceed to prove that d_1^N in (B.3) is a sufficient approximation to d_1 in (3.3a). Consider the general family of discrete problems in which (B.1c) is contained: find a mesh function Z on the arbitrary mesh $\{y_i\}_0^N$ s.t.

$$D^+[F(Z(y_i))] = k(y_i, Z(y_i)), \quad y_i \in \{y_i\}_0^{N-1}, \quad Z(y_N) = B > 0. \tag{B.4}$$

Theorem 8. *If there exists mesh functions $\underline{Z}$ and $\overline{Z}$ s.t. $0 \leqslant \underline{Z} \leqslant \overline{Z}$, $\underline{Z}(y_N) \leqslant B \leqslant \overline{Z}(y_N)$ and $D^+[F(\underline{Z})] - k(y_i, \underline{Z}) \geqslant 0 \geqslant D^+[F(\underline{Z})] - k(y_i, \underline{Z})$, then there exists a mesh function Z s.t. $\underline{Z} \leqslant Z \leqslant \overline{Z}$ which satisfies (B.4).*

Proof. We establish existence at the last internal mesh point from the boundary, that is y_{N-1}. Define the map $T:\mathbb{R}\to\mathbb{R}: z\mapsto F(B)-(F(z)+g(y_{N-1},z)(y_N-y_{N-1}))$. Using (3.12), we have

$$\begin{aligned}T(\underline{Z}(x_{N-1})) =& F(B)-(F(\underline{Z}(y_{N-1}))+g(y_{N-1},\underline{Z}(y_{N-1}))(y_N-y_{N-1}))\\ \geqslant& F(B)-F(\underline{Z}(y_N))\\ =&(B-\underline{Z}(y_N))\ \int_0^1 f(\underline{Z}(y_N)+t(B-\underline{Z}(y_N)))\ dt\ \ \geqslant 0.\end{aligned}$$

Similarly, $T(\overline{Z}(y_{N-1}))\leqslant 0$. Hence, there exists a value $Z(y_{N-1})\in[\underline{Z}(y_N),\overline{Z}(y_N)]$ s.t. $T(Z(y_{N-1}))=0$. Complete the proof inductively. □

Lemma 9. *If L^N and R^N are the discrete solutions of* (B.1) *and r_L and r_R are the solutions of* (2.2) *then*

$$\begin{aligned}\max\{|(L^N-r_L)(x_i)|,|(R^N-r_R)(x_i)|\}\leqslant CN^{-1},\quad x_i\in\bar\Gamma^N,\\ \max\{|(D^-L^N-r_L')(x_i)|,|(D^+R^N-r_R')(x_i)|\}\leqslant CN^{-1},\quad x_i\in\Gamma^N,\\ \max\{|(\delta^2L^N-r_L'')(x_i)|,|(\delta^2R^N-r_R')(x_i)|\}\leqslant CN^{-1},\quad x_i\in\Gamma^N.\end{aligned}$$

Proof. We can show that $\overline{y}$ and g^+ defined in (3.1) and (3.2b) respectively are upper and lower solutions of ((B.1c)) according to Theorem 8. Define the mesh function $\tilde{L}^N$ over the meshpoints $t_{N-i}:=-x_i$ as $\tilde{L}^N(t_{N-i}):=-L^N(x_i)$. Using (B.1b), we can show that $\tilde{L}^N$ satisfies the following discrete terminal value problem

$$f(-\tilde{L}^N(t_{N-i+1}))D^+\tilde{L}^N(t_{N-i})+b(-t_{N-i})\tilde{L}^N(t_{N-i})=q(-t_{N-i}),\tag{B.5a}$$
$$\tilde{L}^N(t_N)=-A>0,\tag{B.5b}$$

which is a subclass of (B.4). We can show that $\overline{\tilde{L}}:=-A+\sqrt{|\min\{0,-q\}|}(2-t_{N-i})$ and $\underline{\tilde{L}}$ s.t. $\underline{\tilde{L}}^2=A^2+2(|A|\|b\|+\max\{0,-q\})(t_{N-i}-1)$ are upper and lower solutions of (B.5).

The error $E_L:=L^N-r_L$ satisfies the following linear problem

$$f(L^N(x_{i-1}))D^-E_L+K(E_L-h_iD^-L^N)-bE_L=f(L^N(x_{i-1}))\tau^-[r_L],\tag{B.6a}$$
$$E_L(-1)=0,\quad \tau^-[r_L]:=(D^--\tfrac{d}{dx})r_L(x_i)\tag{B.6b}$$
$$K:=(\ \int_0^1 f'((1-t)r_L(x_i)+tL^N(x_{i-1}))\ dt\ )r_L'(x_i).\tag{B.6c}$$

Using (2.2), (B.1b) and standard truncation error estimates, we can complete the proof. □

Lemma 10. *If V^- and V^+ are the discrete solutions of* (B.2) *and v_{1L} and v_{1R} are the solutions of* (3.4) *then*

$$\max\{|(V^{\mp} - v_{1L\setminus R})(x_i)|\} \leqslant CN^{-1}, \quad x_i \in \bar{\Gamma}^N.$$

Furthermore, d_1 and d_1^N as defined in (3.3a) *and* (B.3) *satisfy*

$$|d_1 - d_1^N| \leqslant CN^{-1}.$$

Proof. Use the same method of proof as in that of [8, Lemma 2.2] to establish the error estimate.

Subtract the expressions for d_1 and d_1^N in (3.3a) and (B.3) and use the error estimates for $|R^N - r_R|$ and $|(V^{\mp} - v_{1L\setminus R})(x_i)|$ to complete the proof. □

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