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Nonlinear Behaviour of Sea Surface Waves Based on Low-Gradient Phase-Only Scattering Effects

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Abstract—Nonlinear sea waves generated by the wind, including freak waves, are considered to be phenomena that can be modelled using the nonlinear (cubic) Schrödinger equation, for example. However, there is a problem with this approach which is that sea surface waves, driven by wind speeds of varying strength, must be considered to be composed of two distinct types, namely, linear waves and nonlinear waves. In this paper, we consider a different approach to modelling ‘nonlinear’ waves that is based on a solution to the linear wave equation under a low-gradient, phase-only condition. This approach is entirely compatible with the fluid equations of motion (the Navier-Stokes equations) and is thereby not based on a phenomenological model such as the nonlinear Schrödinger equation.

I. INTRODUCTION

There are a number of aspects about the dynamical behaviour of the sea surface that are obvious. For example, the state of the sea can change radically from ‘calm’ to ‘rough’ and the height and wavelength of sea waves can vary significantly. In nearly all cases, ‘sea states’ are determined by the interaction of the wind (in particular, the wind force, i.e. the rate of change of wind velocity) with the sea surface and as a general rule, greater ‘wind energy’ results in greater ‘wave power’.

A. Linear Sea Wave Models

There are two principal measurable properties of sea surface waves: their height and period of oscillation. Real ocean waves do not generally occur at a single frequency but have a frequency distribution for which a range of linear models have been developed. For example, if $s(t)$ denotes the spectral density as a function of the wave period t in seconds, then, for a linear wave pattern, it can be shown that [1] and [2]

$$s(t) = \alpha t^3 \exp(-\beta t^4)$$

where α and β are given by

$$\alpha = 8.10 \times 10^{-3} \frac{g^2}{(2\pi)^4}$$

and

$$\beta = 0.74 \left(\frac{g}{2\pi v} \right)$$

respectively, v is the wind velocity measured 19.5m above still water and g is the acceleration due to gravity.

Models of this type are non-realistic for a number of reasons: (i) They do not take into account that wave states are

non-stationary; (ii) there is no modelling of the connectivity between the wind velocity and wave energy; (iii) it is assumed that the wind velocity is constant and no statistical variability in wind velocity is taken into account; (iv) they assume that local wind conditions and swell are correlated and are thereby not capable of explaining the ‘split spectra’ phenomenon, for example, in which the spectrum of the wavefield consists of two distinct peaks.

Linear wave spectrum models assume that the distance over which the waves develop and the duration for which the wind blows are sufficient for the waves to achieve their maximum energy for the given wind speed. It is assumed that waves can be represented by sinusoidal forms. This relies on the following: (i) Waves vary in a regular way around an average wave height; (ii) there are no energy losses due to friction or turbulence, for example; (iii) the wave height is much smaller than the wavelength.

Linear models are used to predict wave height, at least on a statistical basis. These assume that wave height conforms to a Rayleigh distribution given by

$$P(h) = \frac{h}{\sigma^2} \exp\left(-\frac{h^2}{2\sigma^2}\right)$$

where h is the wave crest height and σ is the most probable wave height. The ‘Significant Wave Height’ (SWH) is then defined as the average of one third of the maximum wave height which, based on this Rayleigh distribution, is given by

$$\text{SWH} = 2.2\sigma$$

In high storm condition with significant wave heights $\sim 15\text{m}$, this statistical model suggests that it is rare to obtain waves higher than 15m and that the probability of obtaining waves with heights of more than twice the SWH is of the order of 10^{-5} . This result is a direct consequence of assuming linear models for deep ocean surface wavefields and can not account for the existence of ‘freak waves’ that have been observed and measured with increasing regularity throughout the worlds oceans.

B. Nonlinear Sea Wave Models

Freak waves, e.g. [4], [5] [6] and [7], have been known about for many years but it is only relatively recently that experimental data has been obtained on their occurrence and research has been undertaken into their cause. A well known

example of experimental evidence for freak waves is given in Figure 1 which is a signal of the wave height (in metres) as a function of time (in second) recorded on New year's Day, 1995, using a radar pulse-echo system setup on the *Draupner* oil rig on the North Sea off Norway [8], [9]. In this case, a freak wave of approximately 26m was measured.

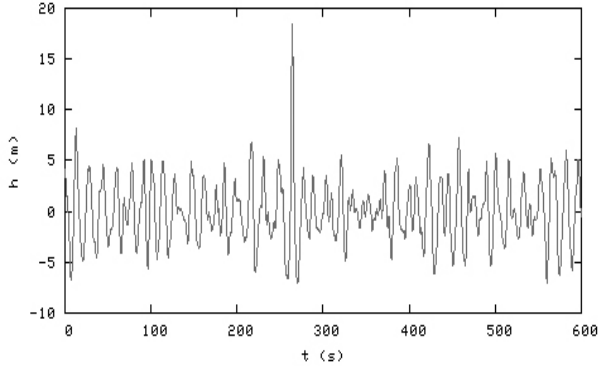


Fig. 1. Wave height (in metres) as a function of time (in seconds) recorded on New year's Day, 1995, using a radar pulse-echo system setup on the *Draupner* oil rig in the North Sea off Norway.

The example given in Figure 1 is typical of freak waves generated in deep water that evolve in stormy conditions with high wind energies. A freak wave is not the same as a Tsunami that are mass displacement generated waves that propagate at high speed and are more or less unnoticeable in deep water, rising in wave height as they approach the shoreline. A freak wave is a spatially and temporally localized event that most frequently occurs far out at sea. Freak waves of up to 35m in height are much more common than probability theory would predict using a Rayleigh distribution for wave heights. They appear to occur in all of the world's oceans many times every year during a storm. This has called for a reexamination of the reasons for their existence, as well as reconsideration of the implications for ocean-going ship design [7] and wave energy conversion technology, e.g. [10], [11], [12] and [13].

There appear to be three principal categories of freak waves: (i) **Walls of water** travelling up to 10km over the ocean surface before become extinct; (ii) **Three sisters** which are groups of three waves; **Single, giant storm waves** that build up to more than four times the average height of storm waves and then collapse over a relatively small time scale (in seconds). These wave types are only three of a range of freak wave phenomena that have yet to be fully classified. It is clear that, whatever the range and diversity of freak waves, their existence can not be explained using linear wave models. One of the most common models used to explain these effects is the Nonlinear Schrödinger equation which is the focus of this paper. However, there are other physical reasons for the generation of freak waves which include the following:

Diffraction Effects. Like optical and acoustic waves, sea surface waves can be diffracted producing a diffraction pattern that is related to the angle of incidence and the shape of

the coast and/or seabed. In some cases, the diffraction pattern produces a focus where a collection of relatively small waves coherently combine in phase to produce a freak wave. The basic physics of this effect is the same as in optics except in terms of scale where the wavelength of the wavefield is relatively large and the frequency spectrum is very low.

Current Focusing. If storm force waves are driven together from opposite directions as opposing 'current' then the wavelength of the waves are shortened causing an increase in wave height. Oncoming wave trains are then 'compressed' together into a freak wave.

It is known that freak waves occur in deep water when diffraction effects can not be the cause and current focusing is weak. In this case, the freak wave is taken to be the result of a nonlinear effect in which the energy of many randomly generated waves is combined into a single wave front which continues to grow until collapsing under its own weight.

II. MODELS BASED ON NONLINEAR SCHRÖDINGER EQUATION

One of the underlying models for explaining freak waves is the Non-Linear Schrödinger (NLS) equation. For example, for a one-dimensional model, the 'cubic' NLS is given by (for normalised units)

$$i\partial_t\Psi(x,t) + \partial_x^2\Psi(x,t) + 2\Psi(x,t)|\Psi(x,t)|^2 = 0$$

where Ψ is the wave function. This equation has two types of 'soliton solutions' associated with a group of wave functions. The first is the 'soliton' solution given by [14]

$$\Psi(x,t) = \frac{\exp(it)}{\cosh(x)}$$

This solution describes an envelope that does not change its form with time. The second class of solutions are of the form [15], [16]

$$\Psi(x,t) = \exp(2it) \frac{\cosh(\Omega t - 2i\theta) - \cos(\theta) \cos(px)}{\cosh(\Omega t) - \cos(\theta) \cos(px)}$$

where

$$p = 2 \sin \theta \quad \text{and} \quad \Omega = 2 \sin(2\theta)$$

The amplitude is periodic in time with frequency Ω and for real θ , the solution tends to an unperturbed plane wave as $|x| \rightarrow \infty$. For $|x| \rightarrow 0$, the solution describes a wave that begins as a modulated plane wave and evolves into one or several peaks that 'extract energy' from the surrounding peaks. The peak values for $\Psi(x,t)$ are twice the amplitude of the unperturbed value of the wave function and for imaginary θ , $\Psi(x,t)$ becomes a space period wave that tends to an unperturbed plane wave as $|t| \rightarrow \infty$, [17]. The maximum value of the peak amplitude is approximately three times the unperturbed value (depending on θ). For the limiting case, when $\theta \rightarrow 0$, we can consider the algebraic solution [18]

$$\Psi(x,t) = \exp(2it) \left(1 - \frac{4(4+4it)}{1+4x^2+16t^2} \right)$$

It is this aspect of the available analytical solutions that has been considered responsible for the generation of deep water freak. The freak waves observed in the numerical simulations based on the NLS equation can be approximately modelled by this algebraic solution [19].

III. SCATTERING MODEL FOR A FREAK WAVE

One of the principal issues associated with using the (cubic) NLS equation to model freak waves is that it is based on a phenomenology, like the Schrödinger equation itself (linear or otherwise). In this section, we illustrate how to obtain analogous (nonlinear) behaviour based on a novel approach to solving the Helmholtz scattering problem which, in turn, is based on the classical linear wave equation for variable wavespeed. We consider a solution to this equation based on a phase only condition for the sum of the incident and scattered wavefield and aim to develop a solution that provides a model for the time evolution of a freak wave.

A. Inhomogeneous Wave Equation

Consider the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})} \frac{\partial^2}{\partial t^2} \right) \Psi(\mathbf{r}, t) = -s(\mathbf{r}, t)$$

where the three-dimensional wave function Ψ is ‘driven’ by a source function $s(\mathbf{r}, t)$ and where $c(\mathbf{r})$ is the variable wavespeed function. Let

$$\Psi(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \psi(\mathbf{r}, \omega) \exp(i\omega t) d\omega$$

and

$$s(\mathbf{r}, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} S(\mathbf{r}, \omega) \exp(i\omega t) d\omega$$

so that, with

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2} [1 + \gamma(\mathbf{r})]$$

we can write

$$(\nabla^2 + k^2) \psi(\mathbf{r}, \omega) = -k^2 \gamma(\mathbf{r}) \psi(\mathbf{r}, \omega) - S(\mathbf{r}, \omega) \quad (1)$$

where $k = \omega/c_0$ and γ is the ‘scattering function’. This result can be derived from the (linearised) Navier-Stokes equations where Ψ represents the scalar velocity field and γ represents variations in the material density for the case when the viscosity is negligible and the compressibility is a constant.

Let

$$\psi(\mathbf{r}, \omega) = \psi_i(\mathbf{r}, \omega) + \psi_s(\mathbf{r}, \omega)$$

where ψ_s is the scattered field whose solution we require and ψ_i is the incident wave which is the solution of

$$(\nabla^2 + k^2) \psi_i(\mathbf{r}, \omega) = -S(\mathbf{r}, \omega)$$

given by

$$\psi_i(\mathbf{r}, \omega) = \frac{\exp(ikr)}{4\pi r} \otimes_{\mathbf{r}} S(\mathbf{r}, \omega)$$

where $\otimes_{\mathbf{r}}$ denotes the convolution integral over \mathbf{r} . Equation (1) is then reduced to

$$(\nabla^2 + k^2) (\psi_i + \psi_s) = -k^2 \gamma (\psi_i + \psi_s) - S$$

is then reduced to

$$(\nabla^2 + k^2) \psi_s = -k^2 \gamma \psi \quad (2)$$

B. Low Gradient Wave-function Condition

Consider the case where the scattered wave function has a very low gradient such that

$$|\nabla \psi| \ll k$$

and equation (2) is reduced to the form

$$\psi_i^* \psi_s + |\psi_s|^2 = -\gamma |\psi|^2$$

This result is consistent with the application of a low spatial frequency condition.

C. Phase Only Solution

If we consider the function $\psi(\mathbf{r}, \omega)$ to be a phase only wavefield where, for unit amplitude,

$$\psi(\mathbf{r}, \omega) = \exp[i\theta_\psi(\mathbf{r}, \omega)]$$

then

$$\psi_s = -\frac{\psi_i}{|\psi_i|^2} (\gamma + |\psi_s|^2)$$

which has first and second order solutions given by

$$\psi_s^{(1)} = -\frac{\gamma \psi_i}{|\psi_i|^2}$$

and

$$\psi_s^{(2)} = -\frac{\gamma \psi_i}{|\psi_i|^2} \left(1 + \frac{\gamma}{|\psi_i|^2} \right)$$

respectively.

D. Model for a Separable Source Function in the Low Frequency Limit

Let $S(\mathbf{r}, \omega) = R(\mathbf{r})F(\omega)$ so that in the low frequency limit

$$\psi_i(\mathbf{r}, \omega) \sim \frac{1}{4\pi r} \otimes_{\mathbf{r}} R(\mathbf{r})F(\omega), \quad \omega \in [-\Omega, \Omega], \quad \Omega \rightarrow 0$$

In this case, the solution for $\Psi_s^{(1)}$ becomes

$$\Psi_s^{(1)}(\mathbf{r}, t) = \frac{2\gamma(\mathbf{r})}{r^{-1} \otimes_{\mathbf{r}} R(\mathbf{r})} \int_{-\Omega}^{\Omega} \frac{F(\omega)}{|F(\omega)|^2 + \epsilon} \exp(i\omega t) d\omega$$

where ϵ is a regularising constant.

Figure 2 shows a numerical example of the temporal characteristics associated with this solution for $\epsilon = 10^{-9}$ computed over a 1000 element array. The spectrum $F(\omega)$ is computed from a zero mean Gaussian distributed noise field. The figure compares this result with the signal

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) \exp(i\omega t) d\omega$$

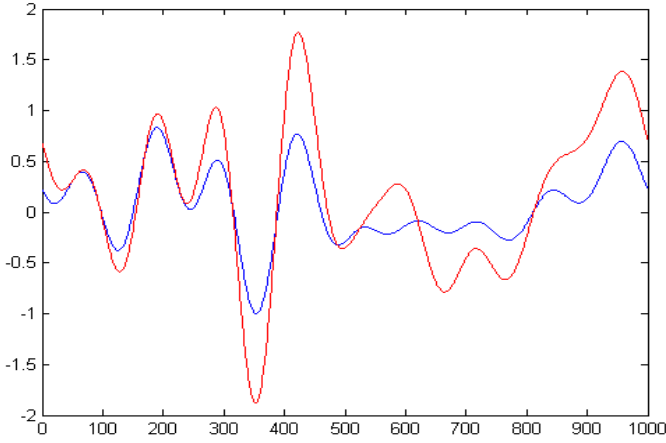


Fig. 2. Example simulation of the time dependent component of the wave function $\Psi = \Psi_i + \Psi_s^{(1)}$ (Red) and the normalised source function $f(t)$ (Blue) for $\Omega = 10$.

used to ‘source’ the solution.

This example illustrates the ability for a low spatial frequency and phase only scattering model to generate large amplitude waves even when the scattering function is a constant. This is due to the phase only solution being characterised by the inverse filter $F/|F|^2$. In contrast, if the phase only condition is relaxed, then the first and second order scattered fields become

$$\psi_s^{(1)} = -\gamma\psi_i$$

and

$$\psi_s^{(2)} = -\gamma(1 - \gamma)^2\psi_i - \gamma^2\psi_i$$

respectively, and, for constant γ , the temporal characteristics of both solutions are characterised by $F(\omega)$.

IV. CONCLUSION

In contrast to using the cubic NLS equation for modelling nonlinear sea surface wave such as freak waves, in this paper we have considered a low spatial frequency scattering model to illustrate that there are other ‘routes’ to explaining and modelling freak wave occurrences. Under the phase only condition, this approach reveals that the time-dependent characteristics of the scattered wave are determined by the inverse of the spectrum of the source function. Unlike phenomenological models such as the (cubic) NLS equation, the scattering model considered here (Section III) is based on the Navier-Stokes equations. In order to illustrate the solution method considered and some of its characteristics, we have considered a wave equation for the scalar velocity propagating in a non-viscous fluid where the relaxation time is zero. Thus, a further development will be to consider a model for wave propagation and scattering in a homogeneous viscous fluid.

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