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Georgi Grahovski
Dublin Institute of Technology

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The Generalised Zakharov-Shabat System and the Gauge Group Action

Georgi G. Grahovski[1,2]

[1] *School of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland*

[2] *Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko chausee, Sofia 1784, Bulgaria*

E-mail: `georgi.grahovski@dit.ie` `grah@inrne.bas.bg`

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Abstract

The generalized Zakharov–Shabat systems with complex-valued *non-regular* Cartan elements and the systems studied by Caudrey, Beals and Coifman (CBC systems) and their gauge equivalent are studied. This study includes: the properties of fundamental analytical solutions (FAS) for the gauge-equivalent to CBC systems and the minimal set of scattering data; the description of the class of nonlinear evolutionary equations, solvable by the inverse scattering method, and the recursion operator, related to such systems; the hierarchies of Hamiltonian structures. The results are illustrated on the example of the multi-component nonlinear Schrödinger (MNLS) equations and the corresponding gauge-equivalent multi-component Heisenberg ferromagnetic (MHF) type models, related to $so(5,\mathbb{C})$ algebra.

1 Introduction

The multi-component Zakharov-Shabat (ZS) system leads to such important systems as the multi-component non-linear Schrödinger equation (NLS), the N-wave type equations, etc. All of these systems are integrable via the inverse scattering method. In the class of nonlinear evolution equations (NLEE) related to the Zakharov–Shabat (ZS) system [54, 52], the Lax operator belonging to $sl(2,\mathbb{C})$ algebra is studied. This class of NLEE contains physically important equations as the nonlinear Schrödinger equation (NLS), the sine-Gordon, Korteweg–de-Vriez (KdV) and the modified Korteweg–de-Vriez (mKdV) equations. In the recent years, the gauge equivalent systems to various versions of the generalised Zakharov-Shabat have been systematically studied [13, 21, 22, 23, 24, 27, 28, 29, 30, 31, 48, 49, 50, 47]. Recently, it was also shown [10] that the spectral problem for the Degasperis-Procesi equation can be cast into Zakharov-Shabat form with an $sl(3,\mathbb{C})$ Lax pair with additional $\mathbb{Z}_3$ and $\mathbb{Z}_2$ symmetries.

Here, we consider the $n\times n$ system [7, 9, 15]:

$$L\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + Q(x,t) - \lambda J \right) \Psi(x,t,\lambda), \qquad (1.1)$$

where the potential $Q(x,t)$ takes values in the semi-simple Lie algebra $\mathfrak{g}$ [45, 20, 53, 16]:

$$Q(x,t) = \sum_{\alpha\in\Delta_+} (q_\alpha(x,t)E_\alpha + q_{-\alpha}(x,t)E_{-\alpha}) \in \mathfrak{g}_J \qquad J = \sum_{j=1}^{r} a_j H_j \in \mathfrak{h}.$$

For the case of complex J, we refer to this system as the Caudrey-Beals-Coifman (CBC) system. Here, J is a *non-regular* element in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathfrak{g}_J$ is the image of ad_J, $\{E_\alpha, H_i\}$ form the Cartan–Weyl Basis in $\mathfrak{g}$, Δ_+ is the set of positive roots of the algebra, $r = \mathrm{rank}\,\mathfrak{g} = \dim\mathfrak{h}$. For more details, see section 2 below. The *non-regularity* of the Cartan elements means that $\mathfrak{g}_J$ is *not* spanned by all root vectors E_α of $\mathfrak{g}$, i.e. $\alpha(J) \neq 0$ for any root α of $\mathfrak{g}$.

The given NLEE, as well as the other members of its hierarchy, possess a Lax representation of the form (according to (1.1)): $[L(\lambda), M_P(\lambda)] = 0$, where

$$M_P \Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + \sum_{k=-S}^{P-1} V_k(x,t) - \lambda^P f_P I \right) \Psi(x,t,\lambda) = 0, \qquad I \in \mathfrak{h}, \tag{1.2}$$

which must hold identically with respect to λ. A standard procedure generalising the one proposed by Ablowitz, Kaup, Newell and Segur (AKNS) [1] allows us to evaluate $V_k(x,t)$ in terms of $q(x,t)$ and its x-derivatives. Here and below, we consider only the class of potentials $q(x,t)$ vanishing fast enough for $|x| \to \infty$. Then, one may also check that the asymptotic value of the potential in $M_P(\lambda)$, namely $f^{(P)}(\lambda) = f_P \lambda^P I$, may be understood as the dispersion law of the corresponding NLEE.

Another important step in the development of the Inverse Scattering Method (ISM) is the introduction of the reduction group by A. V. Mikhailov [44], and further developed in [15, 16, 53, 45]. This allows one to prove that some of the well known models in field theory [44], and also a number of new interesting NLEE [44, 15, 45, 19], are integrable by the ISM and possess special symmetry properties. As a result, its potential $q(x,t)$ has a very special form and J *can no-longer be chosen real*.

This problem of constructing the spectral theory for (1.1) in the most general case, when J has an arbitrary complex eigenvalues, was initialized by Beals, Coifman and Caudrey [3, 4, 5, 9], and continued by Zhou [55] in the case when the algebra $\mathfrak{g}$ is $sl(n)$, $q(x,t)$ vanishing fast enough for $|x| \to \infty$, and no *a priori* symmetry conditions are imposed on $q(x,t)$. This was done later for any semi-simple Lie algebras by Gerdjikov and Yanovski [33].

The applications of the differential geometric and Lie algebraic methods to soliton type equations lead to the discovery of a close relationship between the multi-component (matrix) NLS equations and the symmetric and homogeneous spaces [16]. In [16] it was shown that the integrable MNLS systems have a Lax representation of the form (1.1), where J is a constant element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$ and $Q(x,t) \equiv [J, \widetilde{Q}(x,t)] \in \mathfrak{g}/\mathfrak{h}$. In other words, $Q(x,t)$ belongs to the co-adjoint orbit $\mathcal{M}_J$ of $\mathfrak{g}$ passing through J. Later on, this approach was extended to other types of multi-component integrable models, like the derivative NLS, Korteweg-de-Vriez and modified Korteweg-de-Vriez, N-wave, Davey-Stewartson, Kadomtsev-Petviashvili equations [2, 14].

The choice of J determines the dimension of $\mathcal{M}_J$ which can be viewed as the phase space of the relevant nonlinear evolution equations (NLEE). It is is equal to the number of roots of $\mathfrak{g}$ such that $\alpha(J) \neq 0$. Taking into account that if α is a root, then $-\alpha$ is also a root of $\mathfrak{g}$, then $\dim \mathcal{M}_J$ is always even [43].

The degeneracy of J means that the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ of elements commuting with J (i.e., the kernel of the operator ad_J) is non-commutative. This makes more difficult the derivation of the fundamental analytic solutions (FAS) of the Lax operator (1.1) and the construction of the corresponding (generating) recursion operator Λ. The explicit construction of the recursion operator related to (1.1), using the gauge covariant approach [20, 18], is outlined in[26].

As we mentioned above the Lax operator for the MNLS equations formally has the form (1.1) but now J is no longer a regular element of $\mathfrak{h}$. This means that the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ of elements commuting with J (i.e., the kernel of the operator ad_J) is a non-commutative one. The dispersion law of the MNLS eqn. is quadratic in λ: $f_{\mathrm{MNLS}} = 2\lambda^2 J$. The general form of the MNLS equations and their M-operators is:

$$i\frac{dq}{dt} + 2\mathrm{ad}\,_J^{-1}\frac{d^2q}{dx^2} + [q, \pi_0[q, \mathrm{ad}\,_J^{-1}q]] - 2i(\mathbb{1} - \pi_0)[q, \mathrm{ad}\,_J^{-1}q_x] = 0, \tag{1.3}$$

$$M(\lambda)\psi \equiv \left(i\frac{d}{dt} - V_0^{\mathrm{d}} + 2i\mathrm{ad}_J^{-1}q_x(x,t) + 2\lambda q(x,t) - 2\lambda^2 J \right)\psi(x,t,\lambda) = 0, \tag{1.4}$$

where $V_0^d = \pi_0\left([q, \mathrm{ad}\,_J^{-1}q_x]\right)$ and π_0 is the projector onto $\mathfrak{g}_J$; (see also Section 4.1 below).

The zero-curvature condition $[L(\lambda), M_P(\lambda)] = 0$, is invariant under the action of the group of gauge transformations [56]. Therefore, the gauge equivalent systems are again completely integrable, possess a hierarchy of Hamiltonian structures, etc, [12, 52, 33, 56].

The structure of this paper is as follows: In Section 2 we summarize some basic facts about the reduction group and Lie algebraic details. The construction of the fundamental analytic solutions (FAS) is sketched in Section 3 which is done separately for the case of real Cartan elements (Section 3.1) and for complex ones (Section 3.2). The gauge equivalent MHF's to the MNLS systems are described in Section 4. In Section 5 we present an example of a MNLS type system, related to the $so(5)$ Lie algebra and its corresponding MHF one.

The present article presents an extension of our results [35].

2 Preliminaries

2.1 Simple Lie Algebras

Here, we fix up the notations and the normalization conditions for the Cartan-Weyl generators of $\mathfrak{g}$ [38]. We introduce $h_k \in \mathfrak{h}$, $k = 1, \ldots, r$ and E_α, $\alpha \in \Delta$, where $\{h_k\}$ are the Cartan elements dual to the orthonormal basis $\{e_k\}$ in the root space $\mathbb{E}^r$. Along with h_k, we introduce also

$$H_\alpha = \frac{2}{(\alpha,\alpha)} \sum_{k=1}^{r} (\alpha, e_k) h_k, \qquad \alpha \in \Delta, \tag{2.1}$$

where (α, e_k) is the scalar product in the root space $\mathbb{E}^r$ between the root α and e_k. The commutation relations are given by [8]:

$$[h_k, E_\alpha] = (\alpha, e_k)E_\alpha, \quad [E_\alpha, E_{-\alpha}] = H_\alpha, \quad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\}. \end{cases}$$

We will denote by $\vec{a} = \sum_{k=1}^{r} a_k e_k$ the r-dimensional vector dual to $J \in \mathfrak{h}$; obviously $J = \sum_{k=1}^{r} a_k h_k$. If J is a regular real element in $\mathfrak{h}$, then without restrictions we may use it to introduce an ordering in Δ. Namely, we will say that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, \vec{a}) > 0$ ($(\alpha, \vec{a}) < 0$ respectively). The normalization of the basis is determined by:

$$E_{-\alpha} = E_\alpha^T, \quad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha,\alpha)}, \quad N_{-\alpha,-\beta} = -N_{\alpha,\beta}, \quad N_{\alpha,\beta} = \pm(p+1), \tag{2.2}$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \ldots, p$ $\alpha + (p+1)\beta \notin \Delta$ and $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{g}$ [36, 38]. The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections A_α^*; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as $A_\alpha^* \vec{y} = \vec{y} - \frac{2(\alpha,\vec{y})}{(\alpha,\alpha)}\alpha$, $\alpha \in \Delta$. All Weyl reflections A_α^* form a finite group $W_\mathfrak{g}$ known as the Weyl group. One may introduce, in a natural way, an action of the Weyl group on the Cartan-Weyl basis, namely [11, 39]:

$$A_\alpha^*(H_\beta) \equiv A_\alpha H_\beta A_\alpha^{-1} = H_{A_\alpha^*\beta}, \qquad A_\alpha^*(E_\beta) \equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{A_\alpha^*\beta}, \quad n_{\alpha,\beta} = \pm 1.$$

It is also well known that the matrices A_α are given (up to a factor from the Cartan subgroup) by $A_\alpha = \mathrm{e}^{E_\alpha}\mathrm{e}^{-E_{-\alpha}}\mathrm{e}^{E_\alpha} H_A$, where H_A is a conveniently chosen element from the Cartan subgroup such that $H_A^2 = \mathbb{1}$.

As we already mentioned in the Introduction, the MNLS equations correspond to the Lax operator (1.1) with non-regular (constant) Cartan elements $J \in \mathfrak{h}$. If J is a regular element of the Cartan subalgebra of $\mathfrak{g}$, then ad_J has as many different eigenvalues as is the number of the roots of the algebra and they are given by $a_j = \alpha_j(J)$, $\alpha_j \in \Delta$. Such J's can be used to introduce ordering in the root system by assuming that $\alpha > 0$ if $\alpha(J) > 0$. In what follows, we will assume that all roots for which $\alpha(J) > 0$ are positive.

Obviously, we can consider the eigensubspaces of ad_J as a grading of the algebra $\mathfrak{g}$. In what follows, we will consider symmetric spaces related to maximally degenerated J, i.e. ad_J has only four non-vanishing eigenvalues: $\pm a$ and $\pm 2a$. Then $\mathfrak{g}$ is split into a direct sum of the subalgebra $\mathfrak{g}_0$ and the linear subspaces $\mathfrak{g}_\pm$:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad \mathfrak{g}_\pm = \bigoplus_{k=1}^{2} \mathfrak{g}_{\pm k} \qquad \mathfrak{g}_\pm = \mathrm{l.c.}\,\{X_{\pm j} \,|\, [J, X_\pm] = \pm k a X_\pm\}\,, \qquad k = 1, 2.$$

The subalgebra $\mathfrak{g}_0$ contains the Cartan subalgebra $\mathfrak{h}$ and also all root vectors $E_{\pm\alpha} \in \mathfrak{g}$ corresponding to the roots α such that $\alpha(J) = (\vec{a}, \alpha) = 0$. The root system Δ is split into subsets of roots $\Delta = \theta_0 \cup \theta_+ \cup (-\theta_+)$, where:

$$\theta_0 = \{\alpha \in \Delta \,|\, \alpha(J) = 0\}\,, \quad \theta_+ = \{\alpha \in \Delta \,|\, \alpha(J) > 0\}\,. \tag{2.3}$$

2.2 The Reduction Group

The principal idea underlying Mikhailov's reduction group [44] is to impose algebraic restrictions on the Lax operators L and M, which will be automatically compatible with the corresponding equations of motion. Due to the purely Lie-algebraic nature of the Lax representation, this is most naturally done by embedding the reduction group as a subgroup of $\mathrm{Aut}\,\mathfrak{g}$ – the group of automorphisms of $\mathfrak{g}$. Obviously, to each reduction imposed on L and M, there will correspond a reduction of the space of fundamental solutions $\mathbf{S}_\Psi \equiv \{\Psi(x,t,\lambda)\}$ of (1.1).

Some of the simplest $\mathbb{Z}_2$-reductions of Zakharov–Shabat systems have been known for a long time (see [44]) and are related to outer automorphisms of $\mathfrak{g}$ and $\mathfrak{G}$, namely:

$$C_1(\Psi(x,t,\lambda)) = A_1 \Psi^\dagger(x,t,\kappa(\lambda)) A_1^{-1} = \widetilde{\Psi}^{-1}(x,t,\lambda), \qquad \kappa(\lambda) = \pm\lambda^*, \tag{2.4}$$

$$C_2(\Psi(x,t,\lambda)) = A_3 \Psi^*(x,t,\kappa(\lambda)) A_3^{-1} = \widetilde{\Psi}(x,t,\lambda), \tag{2.5}$$

where A_1 and A_3 are elements of the group of authomorphisms $\mathrm{Aut}\,\mathfrak{g}$ of the algebra $\mathfrak{g}$. Since our aim is to preserve the form of the Lax pair, we limit ourselves to automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. The reduction group, G_R, is a finite group which preserves the Lax representation, i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations:

i) $G_R \subset \mathrm{Aut}\mathfrak{g}$

ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane.

To each $g_k \in G_R$, we relate a reduction condition for the Lax pair as follows [44]:

$$C_k(U(\Gamma_k(\lambda))) = \eta_k U(\lambda), \tag{2.6}$$

where $U(x,\lambda) = q(x) - \lambda J$, $C_k \in \mathrm{Aut}\ \mathfrak{g}$ and $\Gamma_k(\lambda)$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group, then for each g_k, there exists an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

It is well known that $\mathrm{Aut}\,\mathfrak{g} \equiv V \otimes \mathrm{Aut}\,_0\mathfrak{g}$, where V is the group of outer automorphisms (the symmetry group of the Dynkin diagram) and $\mathrm{Aut}\,_0\mathfrak{g}$ is the group of inner automorphisms. Since we start with $I, J \in \mathfrak{h}$, it is natural to consider only those inner automorphisms that preserve the Cartan subalgebra $\mathfrak{h}$. Then $\mathrm{Aut}\,_0\mathfrak{g} \simeq \mathrm{Ad}\,_H \otimes W$ where $\mathrm{Ad}\,_H$ is the group of similarity transformations with elements from the Cartan subgroup and W is the Weyl group of $\mathfrak{g}$.

Generically, each element $g_k \in G$ maps λ into a fraction-linear function of λ. Such action however is appropriate for a more general class of Lax operators which are fraction linear functions of λ.

3 The Caudrey–Beals–Coifman systems

3.1 Fundamental analytical solutions and scattering data for real J.

The direct scattering problem for the Lax operator (1.1) is based on the Jost solutions:

$$\lim_{x\to\infty} \psi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \tag{3.1}$$

and the scattering matrix

$$T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda). \tag{3.2}$$

The fundamental analytic solutions (FAS) $\chi^{\pm}(x,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\mathrm{Im}\,\lambda \gtrless 0$ and are related to the Jost solutions by [20, 32]

$$\chi^{\pm}(x,\lambda) = \phi(x,\lambda)S_J^{\pm}(\lambda) = \psi^{\pm}(x,\lambda)T_J^{\mp}(\lambda)D_J^{\pm}(\lambda), \tag{3.3}$$

where $T_J^{\pm}(\lambda)$, $S_J^{\pm}(\lambda)$ and $D_J^{\pm}(\lambda)$ are the factors of the Gauss decomposition of the scattering matrix:

$$\begin{gathered}
T(\lambda) = T_J^{-}(\lambda)D_J^{+}(\lambda)\hat{S}_J^{+}(\lambda) = T_J^{+}(\lambda)D_J^{-}(\lambda)\hat{S}_J^{-}(\lambda) \qquad (3.4)\\
T_J^{\pm}(\lambda) = \exp\left(\sum_{\alpha>0} t^{\pm}_{\pm\alpha,J}(\lambda)E_\alpha\right), \qquad S_J^{\pm}(\lambda) = \exp\left(\sum_{\alpha>0} s^{\pm}_{\pm\alpha,J}(\lambda)E_\alpha\right),\\
D_J^{+}(\lambda) = I\exp\left(\sum_{j=1}^{r} \frac{2d_J^{+}(\lambda)}{(\alpha_j,\alpha_j)}H_j\right), \qquad D_J^{-}(\lambda) = I\exp\left(\sum_{j=1}^{r} \frac{2d_J^{-}(\lambda)}{(\alpha_j,\alpha_j)}H_j^{-}\right).
\end{gathered}$$

Here, $H_j = H_{\alpha_j}$, $H_j^- = w_0(H_j)$, $\hat{S} \equiv S^{-1}$, I is an element from the universal center of the corresponding Lie group $\mathfrak{G}$ and the superscript $+$ (or $-$) in the Gauss factors means upper- (or lower-) block-triangularity for $T_J^{\pm}(\lambda)$, $S_J^{\pm}(\lambda)$ and shows that $D_J^{+}(\lambda)$ (or $D_J^{-}(\lambda)$) are analytic functions with respect to λ for $\mathrm{Im}\,\lambda > 0$ (or $\mathrm{Im}\,\lambda < 0$, respectively).

On the real axis $\chi^{+}(x,\lambda)$ and $\chi^{-}(x,\lambda)$ are linearly related by:

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G_{J,0}(\lambda), \qquad G_{J,0}(\lambda) = S_J^{+}(\lambda)\hat{S}_J^{-}(\lambda), \tag{3.5}$$

and the sewing function $G_{J,0}(\lambda)$ may be considered as a minimal system of scattering data provided that the Lax operator (1.1) has no discrete eigenvalues [20].

3.2 The CBC Construction for Semisimple Lie Algebras

Here, we will sketch the construction of the FAS for the case of complex-valued regular Cartan elements J: $\alpha(\psi) \neq 0$, following the general ideas of Beals and Coifmal [3, 6] for the $sl(n)$ algebras and [33] for the orthogonal and symplectic algebras. These ideas consist of the following:

1. For potentials $q(x)$ with small norm $||q(x)||_{L^1} < 1$, one can divide the complex λ–plane into sectors and then construct an unique FAS $m_\nu(x,\lambda)$ which is analytic in each of these sectors Ω_ν;

2. For these FAS in each sector, there is a certain Gauss decomposition problem for the scattering matrix $T(\lambda)$ which has a unique solution in the case of absence of discrete eigenvalues.

The main difference between the cases of real-valued and complex-valued J lies in the fact that for complex J the Jost solutions and the scattering data exist only for the potentials on compact support.

We define the regions (sectors) Ω_ν as consisting of those λ's for which $\mathrm{Im}\,(\lambda\alpha(J)) \neq 0$ for any $\alpha \in \Delta$. Thus, the boundaries of the Ω_ν's consist of the set of straight lines:

$$l_\alpha \equiv \{\lambda : \mathrm{Im}\,\lambda\alpha(J) = 0, \qquad \alpha \in \Delta\}, \tag{3.6}$$

and to each root α, we can associate a certain line l_α; different roots may define coinciding lines.

Note that with the change from λ to $\lambda e^{i\eta}$ and J to $Je^{-i\eta}$ (this leads the product $\lambda\alpha(J)$ invariant), we can always choose l_1 to be along the positive real λ axis.

To introduce an ordering in each sector Ω_ν, we choose the vector $\vec{a}_\nu(\lambda) \in \mathbb{E}^r$ to be dual to the element $\mathrm{Im}\,\lambda J \in \mathfrak{h}$. Then, in each sector we split Δ into

$$\Delta = \Delta_\nu^+ \cup \Delta_\nu^-, \qquad \Delta_\nu^\pm = \{\alpha \in \Delta : \mathrm{Im}\,\lambda\alpha(J) \gtrless 0,\ \lambda \in \Omega_\nu\}. \tag{3.7}$$

If $\lambda \in \Omega_\nu$ then $-\lambda \in \Omega_{M+\nu}$ (if the lines l_α split the complex λ-plane into $2M$ sectors). We also need the subset of roots:

$$\delta_\nu = \{\alpha \in \Delta\ :\ \mathrm{Im}\,\lambda\alpha(J) = 0,\ \lambda \in l_\nu\} \tag{3.8}$$

which will be a root system of some subalgebra $\mathfrak{g}_\nu \subset \mathfrak{g}$. Then, we can write that

$$\mathfrak{g} = \bigoplus_{\nu=1}^{M} \mathfrak{g}_\nu \qquad \Delta = \bigcup_{\nu=1}^{M} \delta_\nu \qquad \delta_\nu = \delta_\nu^+ \cup \delta_\nu^-, \qquad \delta_\nu^\pm = \delta_\nu \cap \Delta_\nu^\pm.$$

Thus, we can describe in more detail the sets $\Delta_\nu^\pm$:

$$\Delta_k^+ = \delta_1^+ \cup \delta_2^+ \cup \cdots \cup \delta_k^+ \cup \delta_{k+1}^- \cup \cdots \cup \delta_M^-, \quad \Delta_{k+M}^+ = \Delta_k^-, \quad k = 1, \ldots, M. \tag{3.9}$$

Note that each ordering in Δ can be obtained from the "canonical" one by an action of a properly chosen element of the weyl group $\mathfrak{W}(\mathfrak{g})$.

Now, in each sector Ω_ν, we introduce the FAS $\chi_\nu(x,\lambda)$ and $m_\nu(x,\lambda) = \chi_\nu(x,\lambda)e^{i\lambda Jx}$ satisfying the equivalent equation:

$$i\frac{dm_\nu}{dx} + q(x)m_\nu(x,\lambda) - \lambda[J, m_\nu(x,\lambda)] = 0, \qquad \lambda \in \Omega_\nu. \tag{3.10}$$

If $q(x)$ is a potential on compact support, then the FAS $m_\nu(x,\lambda)$ are related to the Jost solutions by

$$m_\nu(x,\lambda) = \phi(x,\lambda)S^+_{J,\nu}(\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \psi(x,\lambda)T^-_{J,\nu}(x,\lambda)D^+_{J,\nu}(\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx}, \tag{3.11}$$
$$m_{\nu-1}(x,\lambda) = \phi(x,\lambda)S^-_{J,\nu}(\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx} = \psi(x,\lambda)T^+_{J,\nu}(x,\lambda)D^-_{J,\nu}(\lambda)\mathrm{e}^{\mathrm{i}\lambda Jx}, \qquad \lambda\in l_\nu.$$

From the definitions of $m_\nu(x,\lambda)$ and the scattering matrix $T(\lambda)$, we have

$$T(\lambda) = T^-_{J,\nu}(\lambda)D^+_{J,\nu}(\lambda)\hat{S}^+_{J,\nu}(\lambda) = T^+_{J,\nu}(\lambda)D^-_{J,\nu}(\lambda)\hat{S}^-_{J,\nu}(\lambda), \quad \lambda\in l_\nu \tag{3.12}$$

where, in the first equality, we take $\lambda=\mu\mathrm{e}^{\mathrm{i}0}$ and for the second– $\lambda=\mu\mathrm{e}^{-\mathrm{i}0}$ with $\mu\in l_\nu$. The corresponding expressions for the Gauss factors have the form:

$$S^+_{J,\nu}(\lambda) = \exp\left(\sum_{\alpha\in\Delta^+_\nu} s^+_{\nu,\alpha}(\lambda)E_\alpha\right), \qquad S^-_{J,\nu}(\lambda) = \exp\left(\sum_{\alpha\in\Delta^+_{\nu-1}} s^-_{\nu,\alpha}(\lambda)E_{-\alpha}\right),$$
$$T^+_{J,\nu}(\lambda) = \exp\left(\sum_{\alpha\in\Delta^+_{\nu-1}} t^+_{\nu,\alpha}(\lambda)E_\alpha\right), \qquad T^-_{J,\nu}(\lambda) = \exp\left(\sum_{\alpha\in\Delta^+_\nu} t^-_{\nu,\alpha}(\lambda)E_{-\alpha}\right),$$
$$D^+_{J,\nu}(\lambda) = \exp(\mathbf{d}^+_\nu(\lambda)\cdot\mathbf{H}_\nu), \qquad D^-_{J,\nu}(\lambda) = \exp(\mathbf{d}^-_\nu(\lambda)\cdot\mathbf{H}_{\nu-1}). \tag{3.13}$$

Here $\mathbf{d}^\pm_\nu(\lambda) = (d^\pm_{\nu,1},\ldots,d^\pm_{\nu,r})$ is a vector in the root space and

$$\mathbf{H}_\eta = \left(\frac{2H_{\eta,1}}{(\alpha_{\eta,1},\alpha_{\eta,1})},\ldots,\frac{2H_{\eta,r}}{(\alpha_{\eta,r},\alpha_{\eta,r})}\right), \qquad (\mathbf{d}^\pm_\nu(\lambda),\mathbf{H}_\eta) = \sum_{k=1}^r \frac{2d^\pm_{\nu,k}(\lambda)H_{\eta,k}}{(\alpha_{\eta,k},\alpha_{\eta,k})}, \tag{3.14}$$

where $\alpha_{\eta,k}$ is the k-th simple root of $\mathfrak{g}$ with respect to the ordering Δ^+_η and $H_{\eta,k}$ are their dual elements in the Cartan subalgebra $\mathfrak{h}$.

4 The Gauge Group Action

Before proceeding with the study of the gauge-equivalent systems, the following remark is in order:

We can use the gauge transformation commuting with J to simplify Q; in particular, we can remove all components of Q in $\mathfrak{g}_0$; effectively, this means that our $Q(x,t) = Q_+(x,t)+Q_-(x,t)\in\mathfrak{g}_+\cup\mathfrak{g}_-$ can be viewed as a local coordinate in the co-adjoint orbit $\mathcal{M}_J\simeq\mathfrak{g}\backslash\mathfrak{g}_0$:

$$Q_+(x,t) = \sum_{\alpha\in\theta_+} q_\alpha(x,t)E_\alpha, \qquad Q_-(x,t) = \sum_{\alpha\in\theta_-} p_\alpha(x,t)E_{-\alpha}. \tag{4.1}$$

4.1 The class of the gauge equivalent NLEEs

The notion of gauge equivalence allows one to associate to any Lax pair of the type (1.1), (1.2) an equivalent one [33], solvable by the inverse scattering method for the gauge equivalent linear problem:

$$\widetilde{L}\widetilde{\psi}(x,t,\lambda) \equiv \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} - \lambda S(x,t)\right)\widetilde{\psi}(x,t,\lambda) = 0,$$
$$\widetilde{M}\widetilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dt} - 2i\lambda\,\mathrm{ad}^{-1}_{\mathcal{S}}\mathcal{S}_x - 2\lambda^2\mathcal{S}\right)\widetilde{\psi}(x,t,\lambda) = 0, \tag{4.2}$$

where $\widetilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda)$, $S = \mathrm{Ad}_g \cdot J \equiv g^{-1}(x,t)Jg(x,t)$, and $g(x,t) = m_\nu(x,t,0)$ is FAS at $\lambda = 0$. The functions $m_\nu(x,t,\lambda)$ are analytic with respect to λ in each sector Ω_ν and do not lose their analyticity for $\lambda = 0$ (in the case of potential on compact support). From the integral representation for the FAS $m_\nu(x,t,\lambda)$ at $\lambda = 0$, it follows that

$$m_1(x,t,0) = \cdots = m_\nu(x,t,0) = \cdots = m_{2M}(x,t,0).$$

Therefore, the gauge group action is well defined. The zero-curvature condition $[\widetilde{L}, \widetilde{M}] = 0$ gives:

$$i\frac{d\mathcal{S}}{dt} + 2\frac{d}{dx}\left(\mathrm{ad}_{\mathcal{S}}^{-1}\frac{d\mathcal{S}}{dx}\right) = 0. \tag{4.3}$$

Both Lax operators $L(\lambda)$ and $\widetilde{L}(\lambda)$ have equivalent spectral properties and spectral data and therefore, the classes of NLEE's related to them are equivalent.

Following [1], one can consider more general $\widetilde{M}$-operators of the form:

$$\widetilde{M}(\lambda)\widetilde{\Psi} \equiv \mathrm{i}\frac{d\widetilde{\Psi}}{dt} + \left(\sum_{k=1}^{N}\widetilde{V}_k(x,t)\lambda^k\right)\widetilde{\Psi}(x,t,\lambda) = 0, \qquad f(\lambda) = \lim_{x\to\pm\infty}\widetilde{V}(x,t,\lambda), \tag{4.4}$$

where $\widetilde{V}(x,t,\lambda) = \sum_{k=1}^{N}\widetilde{V}_k(x,t)\lambda^k$. The Lax representation $[\widetilde{L}(\lambda), \widetilde{M}(\lambda)] = 0$ leads to recurrent relations between $\widetilde{V}_k(x,t) = \widetilde{V}_k^{\mathrm{f}} + \widetilde{V}_k^{\mathrm{d}}$

$$\begin{aligned}
&\widetilde{V}_{k+1}^{\mathrm{f}}(x,t) \equiv \pi_{\mathcal{S}}(\widetilde{V}_{k+1}) = \widetilde{\Lambda}_\pm \widetilde{V}_k^{\mathrm{f}}(x,t) + \mathrm{i}\,\mathrm{ad}_{\mathcal{S}}^{-1}[C_k, \mathrm{ad}_{\mathcal{S}}^{-1}\mathcal{S}_x(x,t)], \\
&\widetilde{V}_k^{\mathrm{d}}(x,t) \equiv (\mathbb{1} - \pi_{\mathcal{S}})(\widetilde{V}_k) = \widetilde{C}_k + \int_{\pm\infty}^{x} dy\,[\mathrm{ad}_{\mathcal{S}}^{-1}\mathcal{S}_x(y,t), \widetilde{V}_k^{\mathrm{f}}(y,t)], \qquad k = 1,\dots,N;
\end{aligned} \tag{4.5}$$

where $\pi_{\mathcal{S}} = \mathrm{ad}_{\mathcal{S}}^{-1} \circ \mathrm{ad}_{\mathcal{S}}$ and $\widetilde{C}_k = (\mathbb{1} - \pi_{\mathcal{S}})\widetilde{C}_k$ are block-diagonal integration constants, for details see, e.g. [1, 16]. These relations are resolved by the recursion operators (4.6):

$$\widetilde{\Lambda} = \mathrm{Ad}_g \cdot \Lambda = \frac{1}{2}\left(\widetilde{\Lambda}_+ + \widetilde{\Lambda}_-\right), \qquad \widetilde{\Lambda}_\pm = \mathrm{Ad}_g \cdot \Lambda_\pm \tag{4.6}$$

where

$$\Lambda_\pm Z = \mathrm{ad}_J^{-1}(\mathbb{1} - \pi_0)\left(i\frac{dZ}{dx} + [q(x), Z(x)] + i\left[q(x), \int_{\pm\infty}^{x} dy\,\pi_0[q(y), Z(y)]\right]\right), \tag{4.7}$$

and we assume that $Z \equiv \pi_0 Z \in \mathcal{M}_{\mathcal{S}}$, where $\pi_0 = \mathrm{ad}_J^{-1} \circ \mathrm{ad}_J$ is the projector onto the off-diagonal part. As a result, we obtain that the class of (generically nonlocal) multi-component Heisenberg feromagnet (MHF) type models, solvable by the ISM, have the form:

$$i\mathrm{ad}_{\mathcal{S}}^{-2}\frac{d\mathcal{S}}{dt} = \sum_{k=0}^{N}\widetilde{\Lambda}_\pm^{N-k}\left[\widetilde{C}_k, \mathrm{ad}_{\mathcal{S}}^{-2}\mathcal{S}(x,t)\right], \qquad f(\lambda) = \begin{pmatrix} f^+(\lambda) & 0 \\ 0 & f^-(\lambda)\end{pmatrix}, \tag{4.8}$$

where $f(\lambda) = \sum_{k=0}^{N}\widetilde{C}_k\lambda^{N-k}$ determines their dispersion law. The NLEE (4.3) become local if $f(\lambda) = f_0(\lambda)\mathcal{S}$, where $f_0(\lambda)$ is a scalar function. In particular, if $f(\lambda) = -2\lambda^2\mathcal{S}$ we get the MHF eqn. (4.3).

4.2 The Minimal Set of Scattering Data for $L(\lambda)$ and $\widetilde{L}(\lambda)$

We skip the details about CBC construction which can be found in [33] and go to the minimal set of scattering data for the case of complex J which are defined by the sets $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows:

$$
\mathcal{F}_1 = \bigcup_{\nu=1}^{2M} \mathcal{F}_{1,\nu}, \qquad \mathcal{F}_2 = \bigcup_{\nu=1}^{2M} \mathcal{F}_{2,\nu},
$$

$$
\mathcal{F}_{J,1,\nu} = \{\rho^{\pm}_{J,B,\nu,\alpha}(\lambda),\ \alpha \in \delta^+_\nu,\ \lambda \in l_\nu\} \qquad \mathcal{F}_{J,2,\nu} = \{\tau^{\pm}_{J,B,\nu,\alpha}(\lambda),\ \alpha \in \delta^+_\nu,\ \lambda \in l_\nu\}, \tag{4.9}
$$

where

$$
\rho^{\pm}_{J,B,\nu,\alpha}(\lambda) = \langle S^{\pm}_{J,\nu}(\lambda) B \hat{S}^{\pm}_{J,\nu}(\lambda), E_{\mp\alpha}\rangle, \qquad \tau^{\pm}_{J,B,\nu,\alpha}(\lambda) = \langle T^{\pm}_{J,\nu}(\lambda) B \hat{T}^{\pm}_{J,\nu}(\lambda), E_{\mp\alpha}\rangle, \tag{4.10}
$$

with $\alpha \in \delta^+_\nu$, $\lambda \in l_\nu$ and B is a properly chosen regular element of the Cartan subalgebra $\mathfrak{h}$. Without loss of generality, we can take in (4.10) $B = H_\alpha$. Note that the functions $\rho^{\pm}_{J,B,\nu,\alpha}(\lambda)$ and $\tau^{\pm}_{J,B,\nu,\alpha}(\lambda)$ are continuous functions of λ for $\lambda \in l_\nu$.

If we choose J in such way that $2M = |\Delta|$– the number of the roots of $\mathfrak{g}$, then to each pair of roots $\{\alpha, -\alpha\}$ one can relate a separate pair of rays $\{l_\alpha, l_{\alpha+M}\}$, and $l_\alpha \neq l_\beta$ if $\alpha \neq \pm\beta$. In this case, each of the subalgebras $\mathfrak{g}_\alpha$ will be isomorphic to $sl(2)$.

In order to determine the scattering data for the gauge equivalent equations, we need to start with the FAS for these systems:

$$
\widetilde{m}^{\pm}_\nu(x,\lambda) = g^{-1}(x,t) m^{\pm}_\nu(x,\lambda) g_-, \tag{4.11}
$$

where $g_- = \lim_{x\to-\infty} g(x,t)$ and due to (1.2) and $g_- = \hat{T}(0)$. In order to ensure that the functions $\widetilde{\xi}^{\pm}(x,\lambda)$ are analytic with respect to λ, the scattering matrix $T(0)$ at $\lambda = 0$ must belong to $\mathfrak{H} \otimes \mathfrak{G}_0$, where $\mathfrak{H}$ is the corresponding Cartan subgroup and $\mathfrak{G}_0$ is the subgroup, that corresponds to the subalgebra $\mathfrak{g}_0$. Then, equation (4.11) provides the fundamental analytic solutions of $\widetilde{L}$. We can calculate their asymptotics for $x \to \pm\infty$ and thus establish the relations between the scattering matrices of the two systems:

$$
\lim_{x\to-\infty} \widetilde{\xi}^+(x,\lambda) = e^{-\mathrm{i}\lambda J x} T(0) S^+_J(\lambda) \hat{T}(0) \qquad \lim_{x\to\infty} \widetilde{\xi}^+(x,\lambda) = e^{-\mathrm{i}\lambda J x} T^-_J(\lambda) D^+_J(\lambda) \hat{T}(0) \tag{4.12}
$$

with the result: $\widetilde{T}(\lambda) = T(\lambda)\hat{T}(0)$. The factors in the corresponding Gauss decompositions are related by:

$$
\widetilde{S}^{\pm}_J(\lambda) = T(0) S^{\pm}_J(\lambda) \hat{T}(0), \qquad \widetilde{T}^{\pm}_J(\lambda) = T^{\pm}_J(\lambda) \qquad \widetilde{D}^{\pm}_J(\lambda) = D^{\pm}_J(\lambda)\hat{T}(0).
$$

On the real axis, again, the FAS $\widetilde{\xi}^+(x,\lambda)$ and $\widetilde{\xi}^-(x,\lambda)$ are related by $\widetilde{\xi}^+(x,\lambda) = \widetilde{\xi}^-(x,\lambda)\widetilde{G}_{J,0}(\lambda)$ with the normalization condition $\widetilde{\xi}(x,\lambda = 0) = \mathbb{1}$ and $\widetilde{G}_{J,0}(\lambda) = \widetilde{S}^+_J(\lambda)\hat{\widetilde{S}}^-_J(\lambda)$ again can be considered as a minimal set of scattering data.

The minimal set of scattering data for the gauge-equivalent CBC systems are defined by the sets $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ as follows:

$$
\widetilde{\mathcal{F}}_1 = \bigcup_{\nu=1}^{2M} \widetilde{\mathcal{F}}_{1,\nu}, \qquad \widetilde{\mathcal{F}}_2 = \bigcup_{\nu=1}^{2M} \widetilde{\mathcal{F}}_{2,\nu},
$$

$$
\widetilde{\mathcal{F}}_{1,\nu} = \{\widetilde{\rho}^{\pm}_{B,\nu,\alpha}(\lambda),\ \alpha \in \delta^+_\nu,\ \lambda \in l_\nu\} \qquad \widetilde{\mathcal{F}}_{2,\nu} = \{\widetilde{\tau}^{\pm}_{B,\nu,\alpha}(\lambda),\ \alpha \in \delta^+_\nu,\ \lambda \in l_\nu\}, \tag{4.13}
$$

where

$$
\widetilde{\rho}^{\pm}_{J,B,\nu,\alpha}(\lambda) = \langle T_J(0) S^{\pm}_{J,\nu}(\lambda) B \hat{S}^{\pm}_{J,\nu}(\lambda) \hat{T}_J(0), E_{\mp\alpha}\rangle, \quad \widetilde{\tau}^{\pm}_{J,B,\nu,\alpha}(\lambda) = \langle T^{\pm}_{J,\nu}(\lambda) B \hat{T}^{\pm}_{J,\nu}(\lambda), E_{\mp\alpha}\rangle \tag{4.14}
$$

with $\alpha \in \delta^+_\nu$, $\lambda \in l_\nu$ and B is again a properly chosen regular element of the Cartan subalgebra $\mathfrak{h}$. Without loss of generality, we can take in (4.14) $B = H_\alpha$ (as in (4.10)). The functions $\widetilde{\rho}^{\pm}_{B,\nu,\alpha}(\lambda)$ and $\widetilde{\tau}^{\pm}_{B,\nu,\alpha}(\lambda)$ are continuous functions of λ for $\lambda \in l_\nu$, and have the same analyticity properties as the functions $\rho^{\pm}_{B,\nu,\alpha}(\lambda)$ and $\tau^{\pm}_{B,\nu,\alpha}(\lambda)$.

4.3 Integrals of Motion and Hierarchies of Hamiltonian Structures

If $q(x,t)$ evolves according to the MNLS (5.4), then

$$i\frac{dS_J^{\pm}}{dt} - 2\lambda^2[J, S_J^{\pm}(t,\lambda)] = 0, \qquad i\frac{dT_J^{\pm}}{dt} - 2\lambda^2[J, T_J^{\pm}(t,\lambda)] = 0, \qquad \frac{dD_J^{\pm}}{dt} = 0. \tag{4.15}$$

This means that the MNLS eq. (5.4) has four series of integrals of motion. This is due to the special (degenerate) choice of the dispersion law $f_{\rm MNLS} = -2\lambda^2 J$. We have to remember, however, that only two of these four series are in involution, which in turn is related to the non-commutativity of the subalgebra $\mathfrak{g}_J$.

Both classes of NLEE's are infinite dimensional, completely integrable Hamiltonian systems and possess hierarchies of Hamiltonian structures.

The phase space $\mathcal{M}_{\rm MNLS}$ is the linear space of all off-diagonal matrices $q(x,t)$ tending fast enough to zero for $x \to \pm\infty$. The hierarchy of pair-wise compatible symplectic structures on $\mathcal{M}_{\rm MNLS}$ is provided by the 2-forms:

$$\Omega^{(k)}_{\rm MNLS} = {\rm i}\int_{-\infty}^{\infty} dx {\rm tr}\, \left(\delta q(x,t) \wedge \Lambda^k[J, \delta q(x,t)]\right), \tag{4.16}$$

where $\Lambda = (\Lambda_+ + \Lambda_-)/2$ is the generating (recursion) operator for (1.1) defined in (4.7). The symplectic forms $\Omega^{(k)}_{\rm MNLS}$ can be expressed in terms of the scattering data for $L(\lambda)$:

$$\begin{aligned} \Omega^{(k)}_{\rm MNLS} &= \frac{c_k}{2\pi}\sum_{\nu=1}^{M}\int_{\lambda\in l_\nu\cup l_{M+\nu}} d\lambda\lambda^k\left(\Omega^+_{0,\nu}(\lambda) - \Omega^-_{0,\nu}(\lambda)\right), \\ \Omega^{\pm}_{0,\nu}(\lambda) &= \left\langle \hat{D}^{\pm}_{J,\nu}(\lambda)\hat{T}^{\mp}_{J,\nu}(\lambda)\delta T^{\mp}_{J,\nu}(\lambda) D^{\pm}_{J,\nu}(\lambda) \wedge \hat{S}^{\pm}_{J,\nu}(\lambda)\delta S^{\pm}_{J,\nu}(\lambda)\right\rangle. \end{aligned} \tag{4.17}$$

Note that the kernels of $\Omega^{(k)}_{\rm MNLS}$ differ only by the factor λ^k so all of them can be cast into canonical form simultaneously.

The phase space $\mathcal{M}_{\rm MHF}$ of the gauge equivalent to the MNLS systems is the manifold of all $\mathcal{S}(x,t)$, satisfying appropriate boundary conditions. The family of compatible 2-forms is:

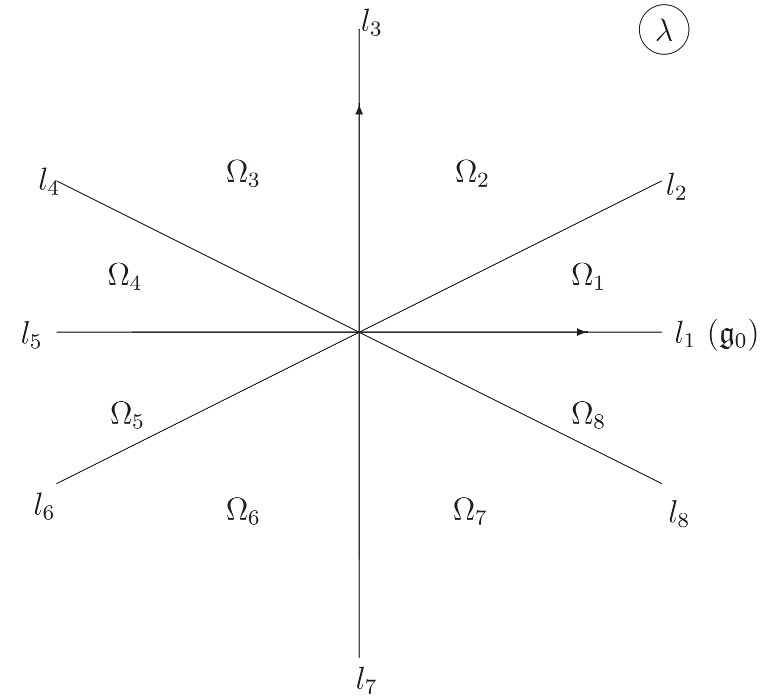
$$\widetilde{\Omega}^{(k)}_{\rm MHF} = \frac{i}{4}\int_{-\infty}^{\infty} dx {\rm tr}\, \left(\delta S^{(0)} \wedge \widetilde{\Lambda}^k[S^{(0)}, \delta S^{(0)}(x,t)]\right). \tag{4.18}$$

Again, like for the "canonical" MNLS models, the symplectic forms $\Omega^{(k)}_{\rm MHF}$ for their gauge-equivalent MHF's can be expressed in terms of the scattering data for $\widetilde{L}(\lambda)$:

$$\begin{aligned} \widetilde{\Omega}^{(k)}_{\rm MHF} &= \frac{c_k}{2\pi}\sum_{\nu=1}^{M}\int_{\lambda\in l_\nu\cup l_{M+\nu}} d\lambda\lambda^k\left(\widetilde{\Omega}^+_{0,\nu}(\lambda) - \widetilde{\Omega}^-_{0,\nu}(\lambda)\right), \\ \widetilde{\Omega}^{\pm}_{0,\nu}(\lambda) &= \left\langle \hat{\widetilde{D}}^{\pm}_{J,\nu}(\lambda)\hat{\widetilde{T}}^{\mp}_{J,\nu}(\lambda)\delta \widetilde{T}^{\mp}_{J,\nu}(\lambda) \widetilde{D}^{\pm}_{J,\nu}(\lambda) \wedge \hat{\widetilde{S}}^{\pm}_{J,\nu}(\lambda)\delta \widetilde{S}^{\pm}_{J,\nu}(\lambda)\right\rangle. \end{aligned} \tag{4.19}$$

The spectral theory of these two operators Λ and $\widetilde{\Lambda}$ underlie all the fundamental properties of these two classes of gauge equivalent NLEE, for details see [33]. Note that the gauge transformation relates, in a nontrivial manner, the symplectic structures, i.e. $\Omega^{(k)}_{\rm MNLS} \simeq \widetilde{\Omega}^{(k+2)}_{\rm MHF}$ [46, 33].

Figure 1: The continuous spectrum of the Lax operator $L(\lambda)$ related to (5.1).



5 Example $\mathfrak{g} \simeq so(5, \mathbb{C})$

Here, we consider a MNLS model with the Lax operators $L(\lambda)$ and $M(\lambda)$ belonging to $so(5, \mathbb{C})$ Lie algebra.

This algebra has 4 positive roots: $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = \alpha_1 + 2\alpha_2$. Let us choose also J to be a degenerate Cartan element ($\alpha_1(J) = 0$) so that the set of roots $\Delta_1^+ = \{\alpha_2, \alpha_3, \alpha_4\}$ of $so(5)$, for which $\alpha(J) \neq 0$ labels the coefficients of the potential $q(x,t)$:

$$q(x,t) \equiv \sum_{\alpha \in \Delta_1^+} (q_\alpha E_\alpha + p_\alpha E_{-\alpha}) = \begin{pmatrix} 0 & 0 & q_{11} & q_{12} & 0 \\ 0 & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & 0 \\ 0 & p_{12} & -p_{11} & 0 & 0 \end{pmatrix}; \qquad J = \operatorname{diag}(a, a, 0, -a, -a).$$

Here, q_1 and p_1 are related to the root α_2; the labels mn in $q_{mn}(x,t)$ and $p_{mn}(x,t)$ refer to the roots $(mn) \leftrightarrow m\alpha_1 + n\alpha_2$. The continuous spectrum of the Lax operator $L(\lambda)$ related to (5.1) is depicted on fig.1. Then, the corresponding MNLS type system is of the form:

$$\begin{aligned}
i\frac{\partial q_{12}}{\partial t} + \frac{1}{2a}\frac{\partial^2 q_{12}}{\partial x^2} + \frac{1}{a}q_{12}(q_1p_1 + q_{11}p_{11} + q_{12}p_{12}) + \frac{i}{a}q_1q_{11,x} - \frac{i}{a}q_{11}q_{1,x} &= 0, \\
i\frac{\partial q_{11}}{\partial t} + \frac{1}{a}\frac{\partial^2 q_{11}}{\partial x^2} + \frac{1}{a}q_{11}(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) + \frac{i}{a}q_{12}p_{1,x} + \frac{i}{2a}q_{12,x}p_1 &= 0, \\
i\frac{\partial q_1}{\partial t} + \frac{1}{a}\frac{\partial^2 q_1}{\partial x^2} + \frac{1}{a}q_1(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) - \frac{i}{a}q_{12}p_{11,x} - \frac{i}{2a}q_{12,x}p_{11} &= 0, \qquad (5.1) \\
i\frac{\partial p_1}{\partial t} - \frac{1}{a}\frac{\partial^2 p_1}{\partial x^2} - \frac{1}{a}p_1(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) - \frac{i}{a}p_{12}q_{11,x} - \frac{i}{2a}p_{12,x}q_{11} &= 0, \\
i\frac{\partial p_{11}}{\partial t} - \frac{1}{a}\frac{\partial^2 p_{11}}{\partial x^2} - \frac{1}{a}p_{11}(q_1p_1 + q_{11}p_{11} + \frac{1}{2}q_{12}p_{12}) + \frac{i}{a}p_{12}q_{1,x} + \frac{i}{2a}p_{12,x}q_1 &= 0,
\end{aligned}$$

$$i\frac{\partial p_{12}}{\partial t} - \frac{1}{2a}\frac{\partial^2 p_{12}}{\partial x^2} - \frac{1}{a}p_{12}(q_1p_1 + q_{11}p_{11} + q_{12}p_{12}) + \frac{i}{a}p_1p_{11,x} - \frac{i}{a}p_{11}p_{1,x} = 0.$$

In order to evaluate the gauge equivalent recursion operator, we use the gauge covariant approach. First, we need to express any function $f(K)$ of the the operator $K = \mathrm{ad}_J$ through the projectors onto its eigensubspaces. Our choice of J means that $K = \mathrm{ad}_J$ has five different eigenvalues: $-2a$, $-a$, 0, a and $2a$. Then, the minimal characteristic polynomial for K is $K(K^2 - a^2)(K^2 - 4a^2) = 0$. Let us also introduce the projectors onto the eigensubspaces of K as follows:

$$\pi_{\pm 2} = \frac{K(K^2 - a^2)(K \pm 2a)}{24a^4}, \quad \pi_{\pm 1} = -\frac{K(K^2 - 4a^2)(K \pm a)}{6a^4}, \quad \pi_0 = \frac{(K^2 - a^2)(K^2 - 4a^2)}{4a^4}.$$

Using the characteristic equation $K(K^2 - a^2)(K^2 - 4a^2) = 0$, it is easy to check that π_j are orthogonal projectors; i.e. they satisfy: $\pi_j\pi_k = \delta_{jk}\pi_j$ for all $j, k = \pm 2, \pm 1, 0$ and that $K\pi_{\pm 2} = \pm 2a\pi_{\pm 2}$, $K\pi_{\pm 1} = \pm a\pi_{\pm 1}$, $K\pi_0 = 0$. Thus any function $f(K)$ can be expressed in terms of these projectors: $f(K) = f(2a)\pi_2 + f(a)\pi_1 + f(0)\pi_0 + f(-a)\pi_{-1}f(-2a)\pi_{-2}$ provided $f(\lambda)$ is regular for $\lambda = \pm 2a$, $\pm a$ and 0. Note also that $\mathrm{ad}_J = K$ introduces a grading on $\mathfrak{g} = \overset{2}{\underset{j=-2}{\oplus}} \mathfrak{g}_j$ and the projectors π_j project precisely onto $\mathfrak{g}_j$. Obviously, $\mathfrak{g}_j = \pi_j\mathfrak{g}$, $\mathfrak{g}_0 \equiv \mathfrak{g}_J$ and $\mathcal{M}_J \simeq \mathfrak{g}\backslash\mathfrak{g}_J$.

Then, applying a gauge transformation, one can recalculate easily the projectors on the eigensubspaces of $\mathrm{ad}_{S(x)} \equiv \widetilde{K}(x) = g_0^{-1}Kg_0(x,t)$:

$$\widetilde{\pi}_{\pm 2} = \frac{\widetilde{K}(\widetilde{K}^2 - a^2)(\widetilde{K} \pm 2a)}{24a^4}, \quad \widetilde{\pi}_{\pm 1} = -\frac{\widetilde{K}(\widetilde{K}^2 - 4a^2)(\widetilde{K} \pm a)}{6a^4}, \quad \widetilde{\pi}_0 = \frac{(\widetilde{K}^2 - a^2)(\widetilde{K}^2 - 4a^2)}{4a^4}.$$

Using these formulae, one can cast also, the gauge-equivalent MHF-type system (4.3) in the form:

$$iS_t - \frac{5}{4a^2}[S, S_{xx}] + \frac{1}{4a^4}\left((\mathrm{ad}_S)^3 S_x\right)_x = 0, \tag{5.2}$$

where $\mathcal{S}$ is constrained by $\mathcal{S}(\mathcal{S}^2 - a^2)^2 = 0$. In addition, the operator $\widetilde{K}(x,t)$ satisfies the equation $\widetilde{K}(\widetilde{K}^2 - a^2)(\widetilde{K}^2 - 4a^2) = 0$.

Now, let us apply to the system (5.1) the following reduction: $L(\lambda) = -L(\lambda^*)^\dagger$. This implies, that the potential matrix $Q(x,t)$ is hermitian, i.e. that $p_\alpha = q_\alpha^*$, and, that the matrix elements of the Cartan element J are real. As a result, we get the following 3-component MNLS system for the complex-valued fields $q_1(x,t)$, $q_{11}(x,t)$ and $q_{12}(x,t)$:

$$\begin{aligned}
&i\frac{dq_{12}}{dt} + \frac{1}{2a}\frac{d^2q_{12}}{dx^2} - \frac{1}{a}q_{12}(|q_1|^2 + |q_{11}|^2 + |q_{12}|^2) + \frac{i}{a}q_1q_{11,x} - \frac{i}{a}q_{11}q_{1,x} = 0\\
&i\frac{dq_{11}}{dt} + \frac{1}{a}\frac{d^2q_{11}}{dx^2} - \frac{1}{a}q_{11}(|q_1|^2 + |q_{11}|^2 + \frac{1}{2}|q_{12}|^2) + \frac{i}{a}q_{12}q^*_{1,x} + \frac{i}{2a}q_{12,x}q^*_1 = 0\\
&i\frac{dq_1}{dt} + \frac{1}{a}\frac{d^2q_1}{dx^2} - \frac{1}{a}q_1(|q_1|^2 + |q_{11}|^2 + \frac{1}{2}|q_{12}|^2) - \frac{i}{a}q_{12}q^*_{11,x} - \frac{i}{2a}q_{12,x}q^*_{11} = 0,
\end{aligned} \tag{5.3}$$

Using the well-known isomorphism between the algebras $so(5,\mathbb{C})$ and $sp(4,\mathbb{C})$ [38], due to the purely Lie-algebraic nature of the Lax representation, one can convert this 3-component MNLS system in the typical representation of $sp(4,\mathbb{C})$[1]. This system is equivalent to the 3-component one, describing $F = 1$ spinor Bose-Einstein condensate [40, 42] in one dimensional approximation.

Finally, applying the reduction $\widetilde{L}(\lambda) = -\widetilde{L}(\lambda^*)^\dagger$ we obtain that the reduced model, that corresponds to (5.2) will be constrained by the condition, that the matrix $\mathcal{S}$ must be hermitian: $\mathcal{S} = \mathcal{S}^\dagger$.

[1]this representation is equivalent to the spinor representation of $so(5,\mathbb{C})$ [36, 38]

6 Conclusions

We will finish this article with several concluding remarks. In order to obtain the soliton solutions for the gauge equivalent MHF systems, one needs to apply the Zakharov–Shabat dressing method to a regular FAS $\widetilde{\chi}^{\pm}_{(0)}(x,\lambda)$ of $\widetilde{L}$ with potential $\mathcal{S}_{(0)}$. Thus, one gets a new singular solution $\widetilde{\chi}^{\pm}_{(1)}(x,\lambda)$ of the Riemann–Hilbert problem with singularities located at prescribed positions $\lambda_1^{\pm}$. It is related to the regular one by the dressing factors $\widetilde{u}(x,\lambda)$. The dressing method for the generalised Zakharov-Shabat systems (related to semi-simple Lie algebras) is developed in [53, 20] , [17], [37] and [41].

To MNLS systems and their gauge equivalent MHF ones, one can apply the analysis [33] and derive the completeness relations for the corresponding system of squared solutions. Such analysis will allow one to prove the pair-wise compatibility of the Hamiltonian structures and eventually, to derive their action-angle variables, see e.g. [51] and [7] for the $\mathbf{A}_r$-series.

The approach presented here allows one to consider CBC systems with more general λ-dependence, like the Principal Chiral field models and other relativistic invariant field theories [53].

Finally, some open problems are: 1) to study the internal structure of the soliton solutions and soliton interactions (for both types of systems); 2) to study reductions of the gauge equivalent MHF systems and the spectral decompositions for the relevant recursion operator.

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