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Completeness of δ-interacting particles

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A THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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Abstract

This thesis concerns the completeness of scattering states of n δ-interacting particles in one dimension. Only the repulsive case is treated, where there are no bound states and the spectrum is entirely absolutely continuous, so the scattering Hilbert space is the whole of $L^2(\mathbb{R}^n)$.

The thesis consists of 4 chapters: The first chapter describes the model, the scattering states as given by the Bethe Ansatz, and the main completeness problem. The second chapter contains the proof of the completeness relation in the case of two particles: $n = 2$. This case had in fact already been treated by B. Smit (1997), [17], but it is useful to include this case as it clarifies the more general case. In particular, the more algebraic approach used for the n-particle case is illustrated in this simple example. In Chapter 3 the case $n = 3$ is examined. This is useful for illustrative purposes as the scattering states can still be written explicitly term by term and it is not yet necessary to introduce the complicated notation used in the general case. On the other hand, this case shows up certain technical difficulties to do with the non-commutativity of the permutation group (S_3) which do not occur in the 2-particle case. Finally, Chapter 4 contains the proof of the completeness relation in the general n-particle case. The method used is the same as in the 3-particle case, but the algebra is much more complicated. In particular a number of interesting lemmas and one theorem is proved. The first lemma for 3-particle case and its generalisation - theorem for n-particle case essentially concerns the Yang-Baxter relation for this model, as first written by Yang. Indeed, Yang proposed his version of these relations as a consistency condition for the Bethe Ansatz solution of the model but never actually gave a complete proof of the consistency given these relations. Here a complete inductive proof is given. Some algebraic manipulation reduces the left-hand side of the completeness relation to a simpler form. Another lemma, which seems to be new, then shows that this expression does not contain divergent terms and consists of a sum of integrals similar to those encountered in the 2- and 3-particle cases. Evaluation of these integrals then leads to the required δ-relation.

I certify that this thesis which I now submit for examination for the award of Doctor of Philosophy Degree, is entirely my own work and has not been taken from the work of others, save and to the extend that such work has been cited and acknowledged within the text of my work.

This thesis was prepared according to the regulations for postgraduate study by research of Dublin Institute of Technology and has not been submitted in whole or in part for another award in any other third level institution.

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Candidate

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Chapter 1

EXPLICIT FORMULA FOR THE WAVE FUNCTION OF THE SYSTEM OF 3 AND n δ-INTERACTING PARTICLES IN ONE DIMENSION

1.1 Introduction

We describe the scattering of particles by solving the stationary Schrödinger equation for the wave function of the system:

$$\left[-\sum_{\mu=1}^{n}\frac{\partial^2}{\partial x_\mu^2}+2c\sum_{\mu<\nu}^{n}\delta(x_\mu-x_\nu)\right]\psi(x_1...x_n)=E\,\psi(x_1...x_n), \qquad (1.1)$$

where $x_1,...,x_n$ are the coordinates of the particles, $\delta(x)$ is the Dirac delta-function and $c>0$ is the coupling constant for the a repulsive interaction potential. We assume that:
1) All particles are numbered, and distinguishable as in [20].
2) Interaction potential is the same for all pairs.
3) Due to the local nature of interaction a probability of appearance of 3 or more particles in one point of x is negligible small (corresponding terms are $\delta(x_\mu-x_\nu)=\delta(x_\nu-x_\eta)=\delta(x_\mu-x_\eta)$, $\mu\neq\nu\neq\eta\neq\mu$) and this leads to absence of such terms in equation.
With the free Hamiltonian $H_0=-\sum_{\mu=1}^{n}\frac{\partial^2}{\partial x_\mu^2}$, the Hamiltonian

$$H=H_0+2c\sum_{\mu<\nu}^{n}\delta(x_\mu-x_\nu), \qquad (1.2)$$

is uniquely defined as a self-adjoint operator in the Hilbert space $L^2(\mathbb{R}^n)$ as was shown in [10] with the use of [14]. For that Hamiltonian, a set of eigenfunctions can be written out explicitly by guessing what it looks like, namely as a sum of waves with different phases obtained by $n!$ permutations of n momenta $k_1, ..., k_n$ with different amplitudes $F(\sigma)$:

$$\psi(x_1...x_n \mid k_1...k_n) = \sum_{\sigma \in S_n} F(\sigma) \exp\left(i \sum_{\rho=1}^{n} k_{\sigma_\rho} x_\rho\right).$$

This set of the waves is known as the Bethe Ansatz [8], but it was applied to the Hamiltonian (1.2) in [13]. There are many applications of this method and its generalisation for solving different problems, for example in [1, 12, 11, 21], also for the difference equations in the form of the quantum inverse problem method or matrix Bethe Ansatz [19, 22], etc. In particular, the infinite discrete spin chain in [1, 2, 3, 4, 5] provides a model for spin chain scattering, which has many features of the problem of n δ-interacting particles (due to the assumption that only neighbouring particles can interact in both models). In a number of cases, completeness was proved, notably scattering completeness for the Heisenberg chain in [2, 3], and with periodic boundary conditions in [12, 18], and for the δ-interacting Bose gas in [10] (see also [9] for an improved version).

The case of two particles is well known and can be easily simplified by changing coordinates of the particles into relative coordinates:

$$X = \frac{1}{2}(x_1 + x_2), \; x = (x_1 - x_2).$$

Hence the motion of two particles is equal to the motion of a combined particle interacting with a δ-potential placed at $x = 0$, and with the free Hamiltonian

$$-\frac{1}{2}\frac{\partial^2}{\partial X^2} - 2\frac{\partial^2}{\partial x^2}.$$

If we have an incoming wave in the form $\psi(x \mid k) = e^{ikx}$ from the left ($k > 0$), then the solution is a scattering wave-function

$$\psi_{in}(x \mid k) = \begin{cases} e^{ikx} + \frac{B_k}{A_k} e^{-ikx}, & x < 0 \\ \frac{1}{A_k} e^{ikx}, & x > 0 \end{cases}, \tag{1.3}$$

where

$$B_k = 1 - A_k = -\frac{ic}{k},$$

obtained from the boundary conditions at $x = 0$, by [13]:

$$\psi(x - 0 \mid k) = \psi(x + 0 \mid k),$$
$$\psi'(x - 0 \mid k) = \psi'(x + 0 \mid k) + c.$$

We can consider this solution as a combination of transmitted and reflected waves in each region defined by the relative position of particles ($x_1 < x_2$ and $x_1 > x_2$ for this case).

With the use of notation

$[12] := e^{i(k_1x_1+k_2x_2)}$, $[21] := e^{i(k_2x_1+k_1x_2)}$ for the wave phases,

$\chi(12) := \chi(x_1 > x_2)$, $\chi(21) := \chi(x_2 > x_1)$ for characteristic function of relative coordinate regions and

$$B_{12} = 1 - A_{12} = -\frac{ic}{k_1 - k_2},$$

the expression for the wave function is given by [7]:

$$\psi(x_1x_2 \mid k_1k_2) = \chi(12)[12] + \chi(21)\{A_{12}[12] + B_{12}[21]\},$$

and the scattering wave function is constructed as:

$$\psi_{in}(x_1x_2 \mid k_1k_2) = \frac{\psi(x_1x_2 \mid k_1k_2)}{A_{12}}\theta(12) + \frac{\psi(x_2x_1 \mid k_2k_1)}{A_{21}}\theta(21)$$

$$\begin{aligned} &= \left\{\frac{1}{A_{12}}[12]\,\chi(12) + \left([12] + \frac{B_{12}}{A_{12}}[21]\right)\chi(21)\right\}\theta(12) \\ &+ \left\{\frac{1}{A_{21}}[12]\,\chi(21) + \left([12] + \frac{B_{21}}{A_{21}}[21]\right)\chi(12)\right\}\theta(21), \end{aligned} \tag{1.4}$$

which is obviously non-symmetric with respect to interchange of coordinates x_1 and x_2, and this two-particle example illustrates the problem considered in this work.

For bosons the wave function needs to be symmetrized by adding the following expression with permuted coordinates x_1 and x_2 (relative to (1.4)):

$$\begin{aligned} \psi_{in}(x_2x_1 \mid k_1k_2) = &\left\{\frac{1}{A_{12}}[21]\,\chi(21) + \left([21] + \frac{B_{12}}{A_{12}}[12]\right)\chi(12)\right\}\theta(12) \\ &+ \left\{\frac{1}{A_{21}}[21]\,\chi(12) + \left([21] + \frac{B_{21}}{A_{21}}[12]\right)\chi(21)\right\}\theta(21), \end{aligned} \tag{1.5}$$

Adding (1.5) to (1.4) and using relations $1 + B_{12} = A_{21}$, $1 + B_{21} = A_{12}$, we obtain symmetric scattering wave function for bosonic particles:

$$\begin{aligned} \psi_{in\ \mathtt{sym}}(x_1x_2 \mid k_1k_2) = &\left\{\left(\frac{A_{21}}{A_{12}}[12] + [21]\right)\chi(12) + \left(\frac{A_{21}}{A_{12}}[21] + [12]\right)\chi(21)\right\}\theta(12) \\ &+ \left\{\left(\frac{A_{12}}{A_{21}}[21] + [12]\right)\chi(12) + \left(\frac{A_{12}}{A_{21}}[12] + [21]\right)\chi(21)\right\}\theta(21), \end{aligned} \tag{1.6}$$

which is not considered in this work.

Scattering completeness for the 2-particle case is proved in [17]. We briefly recall his approach, which is based on that for potential scattering developed by Povzner, Ikebe and Simon, see [15, 16]. Scattering completeness is defined as unitarity of the scattering operator. The scattering operator S is defined in terms of the wave operators $\Omega^\pm$ as

$$S = (\Omega^-)^* \Omega^+,$$

where the latter are defined by (see [4, 15])

$$\Omega^\pm(H, H_0) = \text{s-}\lim_{t \to \mp\infty} e^{itH} e^{-itH_0} P_{\rm ac}(H_0),$$

where $P_{\rm ac}(H_0)$ is the projection onto the absolutely continuous spectrum of H_0, which in our case is simply the identity. The wave operators, assuming they exist, are in general isometries onto their ranges. Scattering completeness is then defined as

$$\text{Ran}(\Omega^+) = \text{Ran}(\Omega^-) = \text{Ran}(P_{ac}(H)). \tag{1.7}$$

In the following we only consider the case of a repulsive interaction, i.e. $c > 0$. In that case, there are no bound states and the spectrum is entirely absolutely continuous, i.e. $P_{ac}(H) = L^2(\mathbb{R}^n)$. The scattering states ψ_{in} (and the associated ψ_{out}) are then given by

$$\psi_{in}(x \mid k) = \Omega^+ e^{i(k_1 x_1 + \ldots + k_n x_n)} \qquad \text{and} \qquad \psi_{out}(x \mid k) = \Omega^- e^{i(k_1 x_1 + \ldots + k_n x_n)}. \tag{1.8}$$

(Note that this requires justification as $\Omega^\pm f$ are originally only defined for $f \in L^2$.) One defines a scattering transform analogous to the Fourier transform, by

$$f^\#(k) = \frac{1}{(2\pi)^{n/2}} \int f(x)\, \overline{\psi_{in}(x \mid k)}\, d^n x. \tag{1.9}$$

(To be precise, the integral has to be interpreted as a limit 'in the mean' in general.) The main completeness identity can then be formulated as an orthonormality relation for the scattering waves:

$$\langle \psi_{in}(y \mid \cdot) \mid \psi_{in}(x \mid \cdot) \rangle = \delta^n(x - y) \tag{1.10}$$

(or in the form of (4.1)). From this relation (which has to be interpreted in distributional sense) it follows that the scattering transform can be inverted:

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int f^\#(k)\, \psi_{in}(x \mid k)\, d^n k. \tag{1.11}$$

Next, we prove that

$$(\Omega^+ f)^\# = \hat{f}, \tag{1.12}$$

where $\hat{f}$ is the Fourier transform of $f \in L^2$. To prove this it suffices to show that for all $f, g \in L^2$,

$$\langle f \,|\, \Omega^+ g\rangle = \langle f^\# \,|\, \hat{g}\rangle.$$

Indeed, (1.12) follows by replacing f with $\Omega^+ f$ and using the fact that Ω^+ is an isometry. Now, the latter identity follows roughly by inserting (1.8):

$$\begin{aligned} \langle f \,|\, \Omega^+ g\rangle &= \left\langle f \,\middle|\, \Omega^+ \int \hat{g}(k)\, e^{ikx} \frac{d^n k}{(2\pi)^{n/2}} \right\rangle \\ &= \left\langle f \,\middle|\, \int \hat{g}(k)\, \psi_{in}(x \,|\, k)\, \frac{d^n k}{(2\pi)^{n/2}} \right\rangle \\ &= \int d^n k\, \hat{g}(k) \int \overline{f(x)}\, \psi_{in}(x \,|\, k)\, \frac{d^n x}{(2\pi)^{n/2}} \\ &= \int \hat{g}(k)\, \overline{f^\#(k)}\, d^n k = \langle f^\# \,|\, \hat{g}\rangle. \end{aligned}$$

The identity (1.12) together with the invertibility of the scattering transform implies that $\mathrm{Ran}(\Omega^+) = L^2(\mathbb{R}^n)$. The analogous identity $\mathrm{Ran}(\Omega^-) = L^2(\mathbb{R}^n)$ follows from a similar argument using ψ_{out} instead of ψ_{in} together with the symmetry relation

$$\psi_{out}(x_1 ... x_n \mid k_1 ... k_n) = \psi_{in}(x_n ... x_1 \mid k_n ... k_1).$$

We elucidate this further in the case $n = 2$ by calculating the scattering matrix. We have (see [17]):

$$\psi_{in}(x \mid k < 0) = \psi_{out}(x \mid k > 0) = \begin{cases} \frac{1}{\overline{A_k}} e^{ikx}, & x < 0 \\ e^{ikx} + \frac{\overline{B_k}}{\overline{A_k}} e^{-ikx}, & x > 0 \end{cases},$$

$$\psi_{in}(x \mid k > 0) = \psi_{out}(x \mid k < 0) = \begin{cases} e^{ikx} + \frac{B_k}{A_k} e^{-ikx}, & x < 0 \\ \frac{1}{A_k} e^{ikx}, & x > 0 \end{cases}. \tag{1.13}$$

Noting that $\overline{B_k} = -B_k$, $\overline{B_{-k}} = B_k$ hence $\overline{A_{-k}} = A_k$, $1 + |B_k|^2 = |A_k|^2$, we express $\psi_{out}(x \mid k)$ in terms of combination of $\psi_{in}(x \mid k)$ and $\psi_{in}(x \mid -k)$ for all cases of $k > 0, k < 0$ and $x > 0, x < 0$.

1) For $k > 0$
a) $x > 0$:

$$\psi_{out}(x > 0 \mid k) = e^{ikx} + \frac{\overline{B_k}}{\overline{A_k}} e^{-ikx} = \frac{1}{\overline{A_k}} \frac{1}{A_k} e^{ikx} + \frac{\overline{B_k}}{\overline{A_k}} \left(e^{-ikx} + \frac{B_k}{A_k} e^{ikx} \right)$$

$$= \frac{1}{\overline{A_k}} \psi_{in}(x > 0 \mid k) + \frac{\overline{B_k}}{\overline{A_k}} \psi_{in}(x > 0 \mid -k),$$

b) $x < 0$:

$$\psi_{out}(x < 0 \mid k) = \frac{1}{\overline{A_k}} e^{ikx} = \frac{1}{\overline{A_k}} \left(e^{ikx} + \frac{B_k}{A_k} e^{-ikx} \right) + \frac{\overline{B_k}}{\overline{A_k}} \frac{1}{A_k} e^{-ikx}$$

$$= \frac{1}{\overline{A_k}} \psi_{in}(x < 0 \mid k) + \frac{\overline{B_k}}{\overline{A_k}} \psi_{in}(x < 0 \mid -k).$$

2) For $k < 0$
a) $x > 0$:

$$\psi_{out}(x > 0 \mid k) = \frac{1}{A_k} e^{ikx} = \frac{1}{A_k} \left(e^{ikx} + \frac{\overline{B_k}}{\overline{A_k}} e^{-ikx} \right) + \frac{B_k}{A_k} \frac{1}{\overline{A_k}} e^{-ikx}$$

$$= \frac{1}{A_k} \psi_{in}(x > 0 \mid k) + \frac{B_k}{A_k} \psi_{in}(x > 0 \mid -k),$$

b) $x < 0$:

$$\psi_{out}(x < 0 \mid k) = e^{ikx} + \frac{B_k}{A_k} e^{-ikx} = \frac{1}{A_k} \frac{1}{\overline{A_k}} e^{ikx} + \frac{B_k}{A_k} \left(e^{-ikx} + \frac{\overline{B_k}}{\overline{A_k}} e^{ikx} \right)$$

$$= \frac{1}{A_k} \psi_{in}(x < 0 \mid k) + \frac{B_k}{A_k} \psi_{in}(x < 0 \mid -k),$$

so that generally we have:

$$\psi_{out}(x \mid k) = \begin{cases} \frac{1}{\overline{A_k}} \psi_{in}(x \mid k) + \frac{\overline{B_k}}{\overline{A_k}} \psi_{in}(x \mid -k), & k > 0 \\ \frac{1}{A_k} \psi_{in}(x \mid k) + \frac{B_k}{A_k} \psi_{in}(x \mid -k), & k < 0 \end{cases}$$

$$= \frac{1}{\overline{A_{|k|}}} \psi_{in}(x \mid k) + \frac{\overline{B_{|k|}}}{\overline{A_{|k|}}} \psi_{in}(x \mid -k). \quad (1.14)$$

Now we can insert this into expression for scattering matrix

$$S(k, k') = \langle \psi_{out}(k') \mid \psi_{in}(k) \rangle$$

and, using the orthogonality relation

$$\langle \psi_{in}(k') \mid \psi_{in}(k) \rangle = \delta(k - k'),$$

obtain

$$S(k, k') = \langle \psi_{out}(k') \mid \psi_{in}(k) \rangle = \left\langle \left(\frac{1}{\overline{A_{|k'|}}} \psi_{in}(k') + \frac{\overline{B_{|k'|}}}{\overline{A_{|k'|}}} \psi_{in}(-k') \right) \middle| \psi_{in}(k) \right\rangle$$

$$= \frac{1}{A_{|k'|}} \langle \psi_{in}(k') \mid \psi_{in}(k) \rangle + \frac{B_{|k'|}}{A_{|k'|}} \langle \psi_{in}(-k') \mid \psi_{in}(k) \rangle$$

$$= \frac{1}{A_{|k'|}} \delta(k - k') + \frac{B_{|k'|}}{A_{|k'|}} \delta(k + k'). \quad (1.15)$$

Using the formula

$$\delta(k^2 - k'^2) = \frac{\delta(k - k')}{2\,|k'|} + \frac{\delta(k + k')}{2\,|k'|}$$

and substituting $B_{|k'|}/A_{|k'|} = (-ic/\,|k'|)/(1 - (-ic/\,|k'|)) = -ic/(|k'| + ic)$, $A_{|k'|} + B_{|k'|} = 1$, we rearrange (1.15) as follows:

$$\begin{aligned} S(k, k') &= \frac{1}{A_{|k|}}\,\delta(k - k') + \frac{B_{|k|}}{A_{|k|}}\,\delta(k + k') \\ &= \delta(k - k') + \frac{B_{|k'|}}{A_{|k'|}}\,[\delta(k - k') + \delta(k + k')] \\ &= \delta(k - k') - \frac{2ic\,|k'|}{|k'| + ic}\,\delta(k^2 - k'^2). \end{aligned}$$

In this work we prove the orthogonality relation explicitly, first for the 2-particle case, then for 3 particles, and finally for the n-particle case with arbitrary n. The motivation of this work is to consider the case with an arbitrary number of distinguishable particles for which completeness of explicit eigenstates has been proved in only very few models. The method of proof is essentially combinatorial and more direct than other approaches. Since the scattering wave function $\psi_{in}(x \mid k)$ is defined as a combination of permutations of the wave functions in all coordinate regions, in the rest of this chapter we evaluate the explicit form of the wave function for all regions of the relative positions of the particles, as a preliminary to the scattering wave-function expressions.

1.2 3-particle case with δ-shape potential of interaction

In the case of more than two particles the situation is much more complicated since we have a system of waves with several transmissions and reflections. We are looking for an explicit formula for the wave function of the n-particle system with δ-interaction, or equivalently: suppose we know an expression for the wave function in any region defined by the relative positions of the particles, then we wish to establish a general rule of calculation of that function in any arbitrary region.
Starting with the 3-particle case, we use the notation of [7]:

$$[\beta_1\beta_2\beta_3] = \exp\left(i\sum_{\rho=1}^{3} k_{\beta_\rho} x_\rho\right)$$

- the wave-functions which are eigenfunctions of the corresponding Hamiltonian,

and $\chi(x_{\sigma_1} > x_{\sigma_2} > x_{\sigma_3}) =: \chi(\sigma_1\sigma_2\sigma_3)$, for $\sigma \in S_3$ - a characteristic function of an arbitrary coordinate region, defined by the particular relative position of particles on the x-axis.
Then the following diagrams apply:

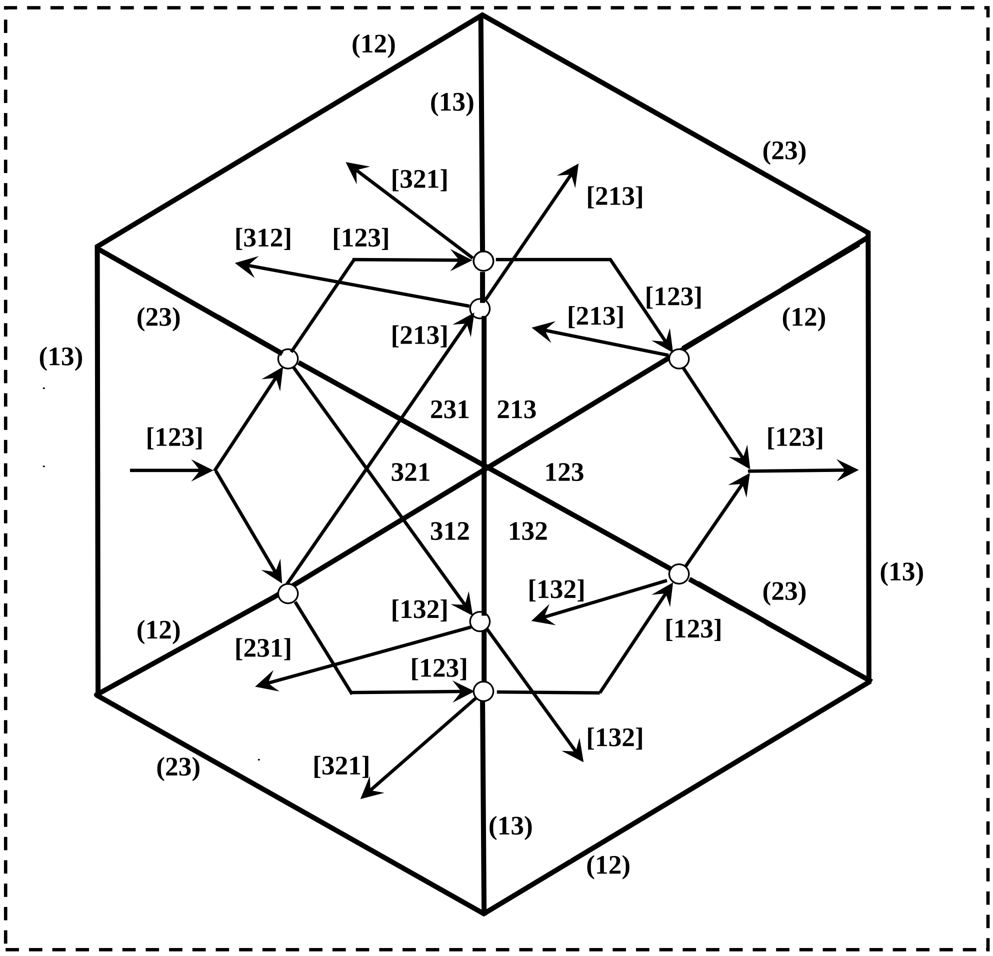

Figure 1.1: The phases of waves in coordinate regions.

The first diagram shows the phases of all waves in the regions (123), (213), (132), (231) and (312). Each arrow with the number in square brackets represents the wave of the particular phase in the particular region. However, some waves are shown twice, representing the different possible ways of realisation of the *same* wave. The mentioned regions contain combinations of 1, 2, 2, 4 and 4 *different* waves, respectively. (The round brackets for the numbers of regions are not shown for simplicity.)
Notation (12), (13), (23) used for the regions boundaries corresponding to transposition of the pair particles with respective numbers.
Note that reflections and transmissions are possible only for internal borders between regions of the diagram. It follows directly from the fact that in one dimension interaction between particles is restricted to only nearest neighbour particles. Each reflection is in correspondence with momentum

transpositions between two particles without coordinate transposition (with effective change of the wave phase), and each transmission through a border between regions is in correspondence with simultaneous coordinate and momentum transpositions for a particular pair of particles (without effective change of the wave phase).

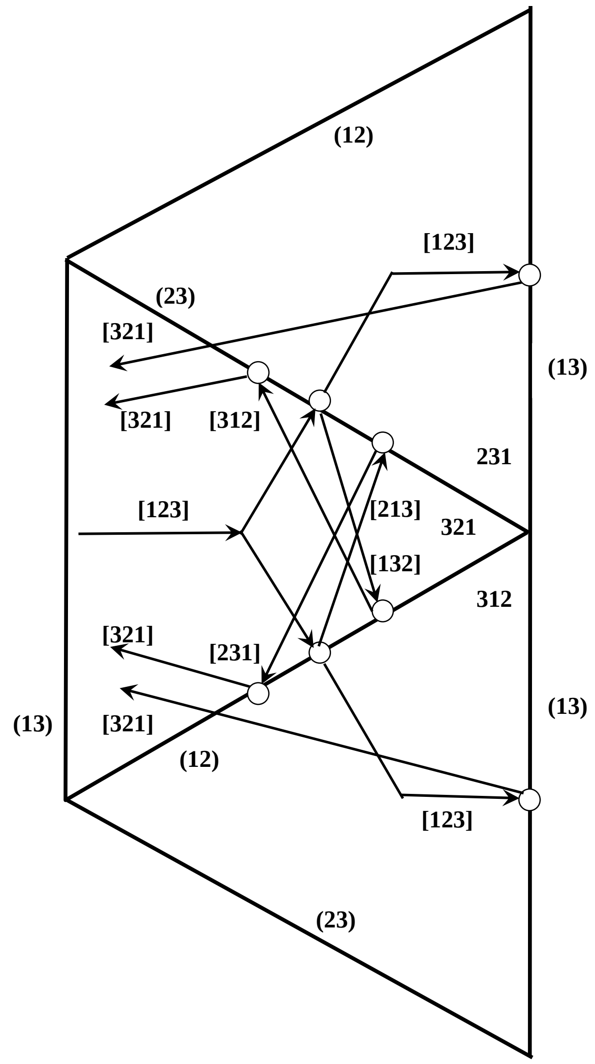

Figure 1.2: The phases of waves in coordinate region (321).

The second diagram shows the phases of waves for the region (321) separately. There are 9 waves but only 7 of them are different (two pairs of identical waves are shown for symmetry).

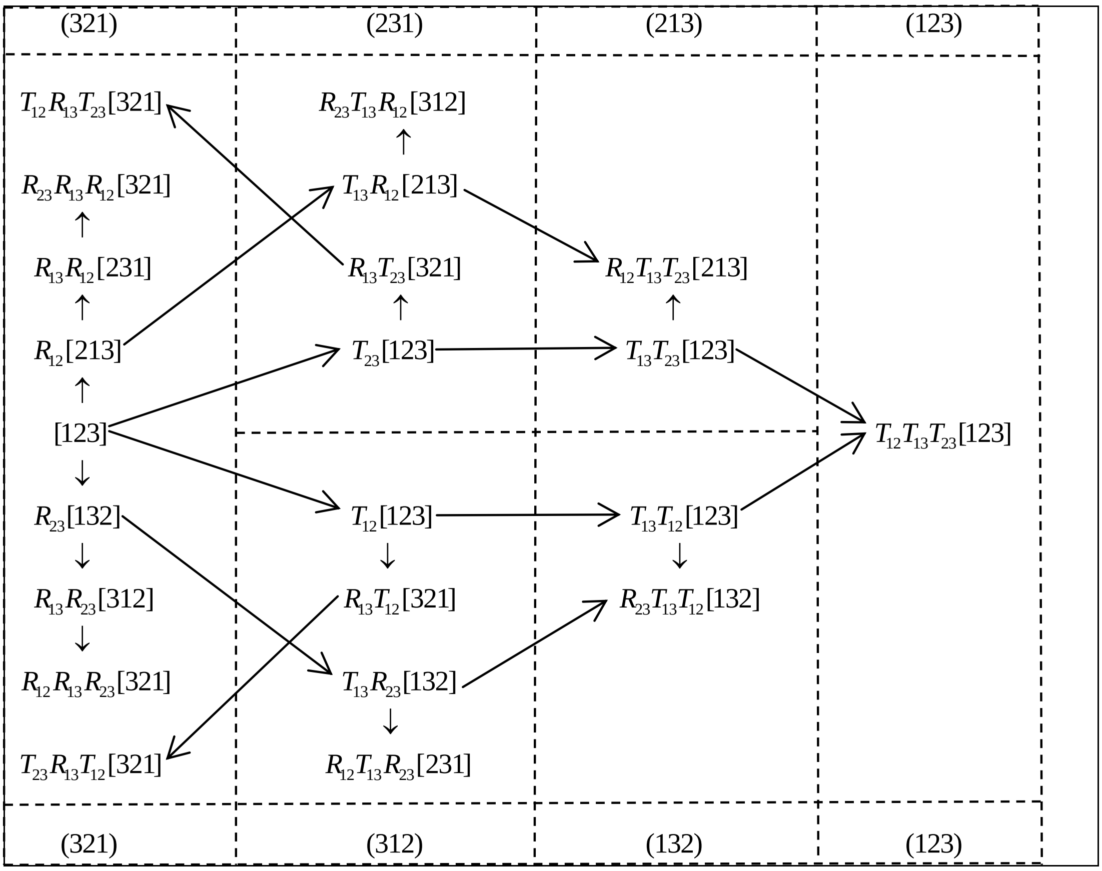

Figure 1.3: The system of waves in coordinate regions.

The diagram on Figure 1.3 shows the actual relation (including corresponding reflection and transmission coefficients) between the waves in all regions. The transmission $T_{\mu\nu}$ and reflection $R_{\mu\nu}$ coefficients are defined by:

$$T_{\mu\nu} = \frac{1}{A_{\mu\nu}}, \quad R_{\mu\nu} = \frac{B_{\mu\nu}}{A_{\mu\nu}}, \quad \text{where} \ \ \mu, \nu = 1, 2, 3, \ \ \mu < \nu. \tag{1.16}$$

Each coordinate region is in correspondence with a combination of elementary waves with some particular amplitudes. The way of obtaining such combinations is the following:

1) We assume that incoming wave in the region $(x_3 > x_2 > x_1) =: (321)$ has the structure $\exp[i(k_1x_1 + k_2x_2 + k_3x_3)] =: [123]$. When this wave is transmitted through the borders (23), (23)(13), (12), (12)(13), it does not change its phase but changes the amplitudes to respectively $T_{23}, T_{13}T_{23}, T_{12}, T_{13}T_{12}$ for regions (231), (213), (312) and (132), respectively.
The same wave can reach the last region (123) in two ways: either (23)(13)(12) or (12)(13)(23) with the same result $T_{12}T_{13}T_{23} = T_{23}T_{13}T_{12}$.

2) Consider the border (12) between regions (213) and (123). It follows from $A_{\mu\nu} + B_{\mu\nu} = 1$ that $T_{\mu\nu} = 1 + R_{\mu\nu}$ so that the wave reflected from (12) in the region (213) has amplitude $R_{12}T_{13}T_{23}$ and the phase [213] corresponding to interchanged momenta for incoming wave [123], so that the full expression for the wave in region (213) is $T_{13}T_{23}[123] + R_{12}T_{13}T_{23}[213]$.

3) Similarly for the region (132) we get $T_{13}T_{12}[123] + R_{23}T_{13}T_{12}[132]$.

4) For the region (231) there are four waves:
a) transmitted $T_{23}[123]$;
b) its reflection from the border (13), namely $R_{13}T_{23}[321]$;
c) reflected from (12) in the region (321) wave $[123] \rightarrow R_{12}[213]$, then transmitted through the border (23) between regions (321) and (231):
$R_{12}[213] \rightarrow T_{13}R_{12}[213]$, where indices 13 represent a pair of interchanging momenta between corresponding particles $\#2, 3$;
d) its reflection from the border (13) between regions (231) and (213):
$T_{13}R_{12}[213] \rightarrow R_{23}T_{13}R_{12}[312]$, where indices 23 is a pair of momenta for the pair of corresponding particles $\#1, 3$.
Finally, the full expression for the region (231) is:
$T_{23}[123] + R_{13}T_{23}[321] + T_{13}R_{12}[213] + R_{23}T_{13}R_{12}[312]$.

5) Similarly for the region (312) we get:
$T_{12}[123] + R_{13}T_{12}[321] + T_{13}R_{23}[132] + R_{12}T_{13}R_{23}[231]$.

6) For the region (123) there are 9, but only 7 different waves:
a) Incoming wave [123].
b) Reflection of [123] from (23) between (321) and (231) regions:
$[123] \rightarrow R_{23}[132]$.
c) Reflection of [123] from (12) between (321) and (312) regions:
$[123] \rightarrow R_{12}[213]$.
d) Further reflection of $R_{23}[132]$ from (12) between (321) and (312) regions: $R_{23}[132] \rightarrow R_{13}R_{23}[312]$, where indices 13 represent a pair of interchanging momenta between corresponding particles $\#1, 2$.
e) Further reflection of $R_{12}[213]$ from (23) between (321) and (231) regions: $R_{12}[213] \rightarrow R_{13}R_{12}[231]$, where indices 13 represent a pair of interchanging momenta between corresponding particles $\#2, 3$.
f) Repeated reflection of $R_{13}R_{23}[312]$ from (23) between (321) and (231) regions: $R_{13}R_{23}[312] \rightarrow R_{12}R_{13}R_{23}[321]$, where indices 12 represent a pair of interchanging momenta between corresponding particles $\#2, 3$.
g) Repeated reflection of $R_{13}R_{12}[231]$ from (12) between (321) and (312) regions: $R_{13}R_{12}[231] \rightarrow R_{23}R_{13}R_{12}[321]$, where indices 23 represent a pair of interchanging momenta between corresponding particles $\#1, 2$.
Actually, f) and g) are different interpretations of the same wave.

h) Transmitted through the border (23) between (231) and (321) regions, wave $R_{13}T_{23}[321] \to T_{12}R_{13}T_{23}[321]$, where indices 12 represent a pair of interchanging momenta between corresponding particles $\#2, 3$.
i) Transmitted through the border (12) between (312) and (321) regions, wave $R_{13}T_{12}[321] \to T_{23}R_{13}T_{12}[321]$, where indices 23 represent a pair of interchanging momenta between corresponding particles $\#1, 2$.
Actually, h) and i) are different interpretation of the same wave.
Finally, for the region (123) we get:
$[123]{+}R_{23}[132]{+}R_{12}[213]{+}R_{13}R_{23}[312]{+}R_{13}R_{12}[231]{+}(R_{12}R_{13}R_{23}{+}T_{12}R_{13}T_{23})[321]$.

Assume the amplitude normalisation such that outgoing wave for the region (123) has unit amplitude. Then multiplying all amplitudes in the previous table by $A_{12}A_{13}A_{23}$ and using (1.16) we obtain the similar diagram but in terms of coefficients $A_{\mu\nu}$,$B_{\mu\nu}$, where $\mu, \nu = 1, 2, 3, \;\; \mu < \nu$:

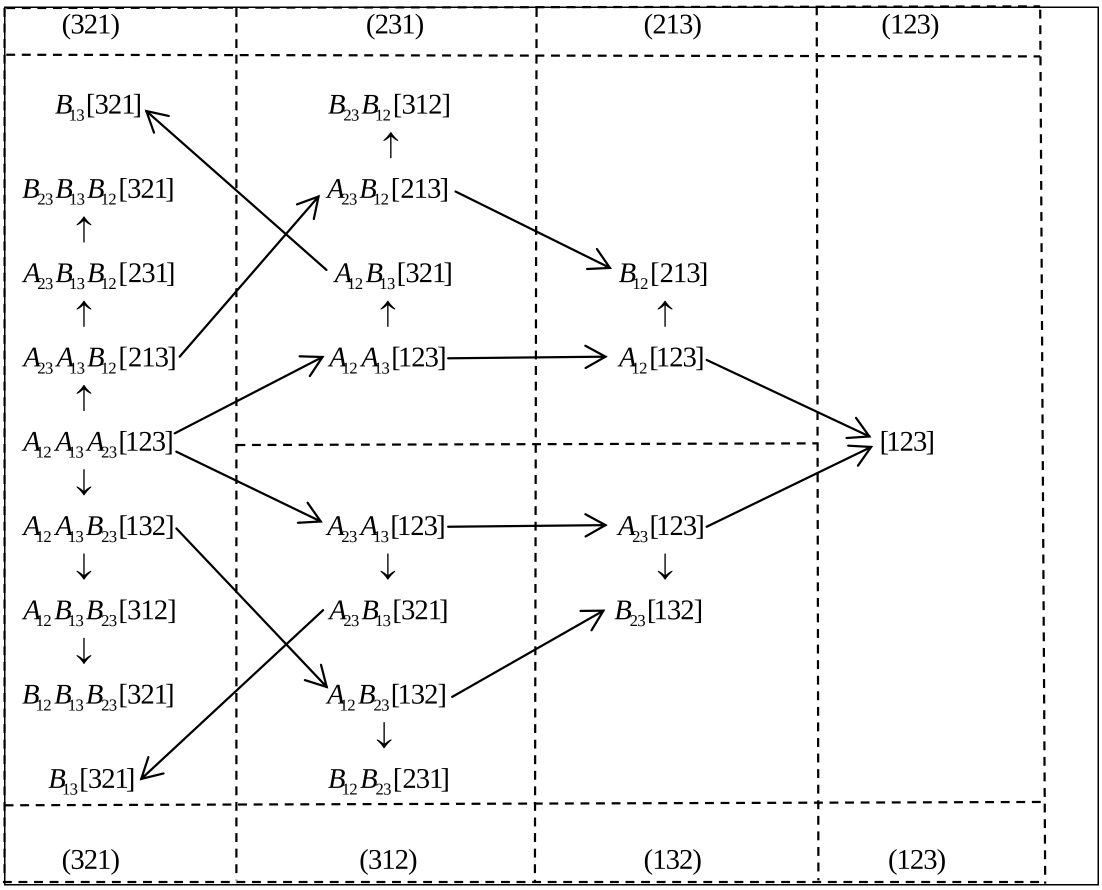

Figure 1.4: The system of waves in coordinate regions, with amplitudes in terms of $A_{\mu\nu}$,$B_{\mu\nu}$.

1.3 Transposition matrices for 3-particle case

The wave function for all regions is just the sum over all regions of all wave combinations in each region:

$$
\begin{aligned}
&\psi(x_1x_2x_3 \mid k_1k_2k_3)\\
&= \chi(123)[123] + \chi(213)\big(A_{12}[123] + B_{12}[213]\big) + \chi(132)\big(A_{23}[123] + B_{23}[132]\big)\\
&+ \chi(231)\big(A_{12}A_{13}[123] + A_{12}B_{13}[321] + A_{23}B_{12}[213] + B_{12}B_{23}[312]\big)\\
&+ \chi(312)\big(A_{23}A_{13}[123] + A_{23}B_{13}[321] + A_{12}B_{23}[132] + B_{12}B_{23}[231]\big)\\
&+ \chi(321)\big(A_{12}A_{13}A_{23}[123] + A_{23}A_{13}B_{12}[213] + A_{12}A_{13}B_{23}[132]\\
&+ A_{23}B_{13}B_{12}[231] + A_{12}B_{13}B_{23}[312] + B_{13}(1 + B_{12}B_{23})[321]\big). \qquad (1.17)
\end{aligned}
$$

For the case of more than 3 particles the diagrams similar to Figures 1.1-1.4 become much more complicated. However, it is possible to evaluate a certain algorithm for construction of the wave system in any region. We will start with the same three-particle case.

The wave function of arbitrary region $\alpha \in S_3$ can be expressed, in particular, as a product $I^0 \cdot G^\alpha$, where $I^0 = (1\,1\,1\,1\,1\,1)$,

G^α is a column of the waves with all possible phases:

$$
G^\alpha = \begin{pmatrix} A^\alpha_{[123]}[123] \\ A^\alpha_{[213]}[213] \\ A^\alpha_{[231]}[231] \\ A^\alpha_{[321]}[321] \\ A^\alpha_{[312]}[312] \\ A^\alpha_{[132]}[132] \end{pmatrix},
$$

and $A^\alpha_{[\beta_1\beta_2\beta_3]}$ is a corresponding amplitude for the wave with phase $[\beta_1\beta_2\beta_3]$ in coordinate region α. All possible regions of the relative positions for 3 particles are: (123), (213), (231), (321), (312), (132). The order of particles permutations is taken for convenience in such a way that any two nearest regions differ from each other by only a pair of nearest particles.

Each border between two coordinate regions represents interchange of positions of two neighbouring particles transforming the system of waves in one region to the system of waves in another region, by means of transmission and reflection of each elementary wave. In fact, we can reduce the number of different possible transpositions between particles (or, equivalently, transformations between neighbouring regions) by the following convention: instead of labeling each transposition by initial numbers of a pair of corresponding particles, we label them by the relative positions of these particles. Since for 3 particles there are always two pairs of nearest neighbour positions, there are two matrices each of which represents a transition of the system of waves through the boundary between classes of regions: for one matrix that is a transposition of two particles between the first and second current positions, and for the second matrix - between particles on the second and third current positions.

$$
\mathrm{Q}_{(\alpha_1\alpha_2)} = \begin{pmatrix} \begin{pmatrix} B_{12} & 1+B_{12} \\ 1-B_{12} & -B_{12} \end{pmatrix} & (0) & (0) \\ (0) & \begin{pmatrix} B_{23} & 1+B_{23} \\ 1-B_{23} & -B_{23} \end{pmatrix} & (0) \\ (0) & (0) & \begin{pmatrix} -B_{13} & 1-B_{13} \\ 1+B_{13} & B_{13} \end{pmatrix} \end{pmatrix},
$$

$$
Q_{(\alpha_2\alpha_3)} = \begin{pmatrix} B_{23} & 0 & 0 & 0 & 0 & 1+B_{23} \\ 0 & \begin{pmatrix} B_{13} & 1+B_{13} \\ 1-B_{13} & -B_{13} \end{pmatrix} & & (0) & & 0 \\ 0 & & & & & 0 \\ 0 & & (0) & \begin{pmatrix} -B_{12} & 1-B_{12} \\ 1+B_{12} & B_{12} \end{pmatrix} & & 0 \\ 0 & & & & & 0 \\ 1-B_{23} & 0 & 0 & 0 & 0 & -B_{23} \end{pmatrix},
$$

where $(0) := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

and, in particular, $B_{13} \equiv -B_{31}$, $B_{12} \equiv -B_{21}$.

The matrix $Q_{(\alpha_1\alpha_2)}$ represents transformation of the system of waves on the boundary between two regions with the first and second positions of particles transposed, namely $(123) \to (213), (231) \to (321), (312) \to (132)$.
Similarly, the matrix $Q_{(\alpha_2\alpha_3)}$ represents transformation of the system of waves on the boundary between two regions with the second and third positions of particles, namely $(123) \to (132), (213) \to (231), (321) \to (312)$.
With the change of phases corresponding to this transformation, the column of waves in any region can be expressed as a product of a matrix related to the boundary with previous region and the column of waves in previous region: $G^{\alpha_2\alpha_1\alpha_3} = Q_{(\alpha_1\alpha_2)}G^{\alpha_1\alpha_2\alpha_3}$ and $G^{\alpha_1\alpha_3\alpha_2} = Q_{(\alpha_2\alpha_3)}G^{\alpha_1\alpha_2\alpha_3}$. Since a particular order of permutations for regions was chosen, then a particular order of indices for coefficients $B_{\beta_1\beta_2}$, $\beta_1 < \beta_2$, $\beta_1, \beta_2 = 1, 2, 3$, occurs in

the corresponding entries of matrices: in the case $\beta_1\beta_2 \to \beta_2\beta_1$ in transition between two regions, the corresponding coefficient on the upper left position of smaller matrix 2×2 is $B_{\beta_1\beta_2}$. However, since inverse order of transposition $\beta_2\beta_1 \to \beta_1\beta_2$ corresponds to the inverse order of indices in $B_{\beta_2\beta_1}$ which is also pure imaginary, then $B_{\beta_2\beta_1} = \overline{B_{\beta_1\beta_2}} = -B_{\beta_1\beta_2}$, and $Q_{(\alpha_1\alpha_2)}, Q_{(\alpha_2\alpha_3)}$ can be adjusted by changing signs in corresponding entries in the case of inverse transpositions or, generally, in the case of any permutations of regions different from the initially chosen order. In fact, for the practical purpose of solving the problem of finding the system of waves for all coordinate regions such non-uniqueness of the order of transformations is not important, because it suffices to choose only one path between regions with some particular order of transpositions on the region boundaries, with the corresponding Q-matrices, and there is no need to find all other paths and adjust matrices for other possible orders of permutations (see *Theorem (Yang)*, p.71, Section 4.1). With this remark, which is also true for n-particle case, we can find a column of waves for all regions in 3-particle case.
Assume such normalisation of the wave in region (123), that

$$G^{123} = \begin{pmatrix} 1 \cdot [123] \\ 0 \cdot [213] \\ 0 \cdot [231] \\ 0 \cdot [321] \\ 0 \cdot [312] \\ 0 \cdot [132] \end{pmatrix}.$$

Then the expression for each of 6 coordinate regions can be obtained consequently from the previous one:

$$\begin{aligned}
G^{123} &= \mathbb{I}, \\
G^{213} &= Q_{(\alpha_1\alpha_2)}G^{123}, \\
G^{132} &= Q_{(\alpha_2\alpha_3)}G^{123}, \\
G^{231} &= Q_{(\alpha_2\alpha_3)}G^{213}, \\
G^{312} &= Q_{(\alpha_1\alpha_2)}G^{132}, \\
G^{321} &= Q_{(\alpha_1\alpha_2)}G^{231} = Q_{(\alpha_2\alpha_3)}G^{312}.
\end{aligned}$$

After substituting explicit forms of G^{123}, $Q_{(\alpha_1\alpha_2)}$, $Q_{(\alpha_2\alpha_3)}$ and applying phase changes according to transpositions in pairs of particles, we obtain:

$$G^{123} = \begin{pmatrix} 1\cdot[123] \\ 0\cdot[213] \\ 0\cdot[231] \\ 0\cdot[321] \\ 0\cdot[312] \\ 0\cdot[132] \end{pmatrix},$$

$$G^{213} = Q_{(\alpha_1\alpha_2)}G^{123} = \begin{pmatrix} B_{12}\cdot[213] \\ A_{12}\cdot[123] \\ 0\cdot[321] \\ 0\cdot[231] \\ 0\cdot[132] \\ 0\cdot[312] \end{pmatrix}, \qquad G^{132} = Q_{(\alpha_2\alpha_3)}G^{123} = \begin{pmatrix} B_{23}\cdot[132] \\ 0\cdot[231] \\ 0\cdot[213] \\ 0\cdot[312] \\ 0\cdot[321] \\ A_{23}\cdot[123] \end{pmatrix},$$

$$G^{231} = Q_{(\alpha_2\alpha_3)}G^{213} = \begin{pmatrix} B_{23}B_{12}\cdot[312] \\ B_{13}A_{12}\cdot[321] \\ A_{13}A_{12}\cdot[123] \\ 0\cdot[132] \\ 0\cdot[231] \\ A_{23}B_{12}\cdot[213] \end{pmatrix}, \quad G^{312} = Q_{(\alpha_1\alpha_2)}G^{132} = \begin{pmatrix} B_{12}B_{23}\cdot[231] \\ A_{12}B_{23}\cdot[132] \\ 0\cdot[312] \\ 0\cdot[213] \\ A_{13}A_{23}\cdot[123] \\ B_{13}A_{23}\cdot[321] \end{pmatrix},$$

$$G^{321} = Q_{(\alpha_1\alpha_2)}G^{231} = \begin{pmatrix} \left(B_{12}B_{23}B_{12} + \overline{A_{12}}B_{13}A_{12}\right)\cdot[321] \\ \left(A_{12}B_{23}B_{12} + \overline{B_{12}}B_{13}A_{12}\right)\cdot[312] \\ B_{23}A_{13}A_{12}\cdot[132] \\ A_{23}A_{13}A_{12}\cdot[123] \\ A_{13}A_{23}B_{12}\cdot[213] \\ B_{13}A_{23}B_{12}\cdot[231] \end{pmatrix}. \qquad (1.18)$$

Using the easily verified equality $B_{12}B_{23} = B_{12}B_{13} + B_{13}B_{23}$, the expression for G^{321} can be simplified:

$$B_{12}B_{23}B_{12} + \overline{A_{12}}B_{13}A_{12} = B_{12}B_{23}B_{12} + (1 - B_{12}^2)B_{13}$$
$$= B_{13} + B_{12}(B_{12}B_{23} - B_{12}B_{13}) = B_{13} + B_{12}B_{13}B_{23} = B_{13}(1 + B_{12}B_{23}),$$

and $A_{12}B_{23}B_{12} + \overline{B_{12}}B_{13}A_{12} = A_{12}(B_{23}B_{12} - B_{12}B_{13}) = A_{12}B_{13}B_{23}$.

Substituting this in (1.18) we obtain

$$G^{321} = Q_{(\alpha_1\alpha_2)}G^{231} = \begin{pmatrix} B_{13}(1+B_{12}B_{23})[321] \\ A_{12}B_{13}B_{23}[312] \\ A_{12}A_{13}B_{23}[132] \\ A_{12}A_{13}A_{23}[123] \\ A_{23}A_{13}B_{12}[213] \\ A_{23}B_{13}B_{12}[231] \end{pmatrix}. \tag{1.19}$$

Now all G^α in (1.18) and (1.19) are in correspondence with the diagram on Figure 1.4. However, the order of wave phases are different in all columns G^α due to permutations of pairs of particles at each boundary between coordinate regions. To preserve the same order of phases as in G^{123} in all coordinate regions G^α we need, simultaneously with the transpositions of particles $Q_{(\alpha_j\alpha_{j+1})}$, $j = 1, ..., n-1$, to perform a transposition of corresponding momenta applying momentum transposition matrices $I_{(\beta_{\alpha_j}\beta_{\alpha_{j+1}})}$, where, for the 3-particle case there are only two such matrices $I_{(\beta_{\alpha_1}\beta_{\alpha_2})}$ and $I_{(\beta_{\alpha_2}\beta_{\alpha_3})}$:

$$I_{(\beta_{\alpha_1}\beta_{\alpha_2})} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & (0) & (0) \\ (0) & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & (0) \\ (0) & (0) & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}, \quad I_{(\beta_{\alpha_2}\beta_{\alpha_3})} = \begin{pmatrix} 0 & 0 \quad 0 & 0 \quad 0 & 1 \\ 0 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & (0) & 0 \\ 0 & & & 0 \\ 0 & (0) & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & & & 0 \\ 1 & 0 \quad 0 & 0 \quad 0 & 0 \end{pmatrix}.$$

Then, for example, a transposition between regions (123) and (213) with the boundary (12) achieved by applying the matrix $Y_{(12)} =: I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}$ etc. To construct composition Y-matrices corresponding to double and triple transpositions of relative coordinate positions with the use of available pairs of Q- and I-matrices it is necessary to satisfy the following conditions:

a) Any composition of transpositions in pairs of coordinates has to be accompanied by antisymmetric (inverse) composition of transpositions for corresponding pairs of momenta, to insure retaining the same phases.

b) Constructed with two pairs $Q_{(\alpha_1\alpha_2)}$, $Q_{(\alpha_2\alpha_3)}$ and $I_{(\beta_{\alpha_1}\beta_{\alpha_2})}$, $I_{(\beta_{\alpha_2}\beta_{\alpha_3})}$ the matrices $Y_{(12)}$, $Y_{(13)}$, $Y_{(23)}$ are different from each other:
$Y_{(12)} \neq Y_{(13)} \neq Y_{(23)} \neq Y_{(12)}$.

Then, according to these conditions and definition of $Y_{(12)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}$, we obtain:

$$Y_{(13)}Y_{(12)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}Q_{(\alpha_1\alpha_2)},$$

$$Y_{(13)}Y_{(23)} = I_{(\beta_{\alpha_2}\beta_{\alpha_3})}I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}Q_{(\alpha_2\alpha_3)},$$

$$Y_{(23)}Y_{(13)}Y_{(12)} = I_{(\beta_{\alpha_2}\beta_{\alpha_3})}I_{(\beta_{\alpha_1}\beta_{\alpha_2})}I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}Q_{(\alpha_1\alpha_2)}Q_{(\alpha_2\alpha_3)}$$
$$= Y_{(12)}Y_{(13)}Y_{(23)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}I_{(\beta_{\alpha_2}\beta_{\alpha_3})}I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}Q_{(\alpha_2\alpha_3)}Q_{(\alpha_1\alpha_2)}.$$

Using $Y_{(12)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}$ and $I^2_{(\beta_{\alpha_1}\beta_{\alpha_2})} = I^2_{(\beta_{\alpha_2}\beta_{\alpha_3})} = \mathbb{1}$, this can be written as:

$$Y_{(13)}Y_{(12)} = \big[I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big)I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big]\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big),$$

$$Y_{(13)}Y_{(23)} = \big[I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big)I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big]\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big),$$

$$Y_{(23)}Y_{(13)}Y_{(12)} = \big\{I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big[I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big)I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big]I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big\}$$
$$\times\big[I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big)I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big]\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big)$$

$$= Y_{(12)}Y_{(13)}Y_{(23)} = \big\{I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big[I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big)I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big]I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big\}$$
$$\times\big[I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big)I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big]\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big).$$

Comparing these formulas with the definition of $Y_{(12)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}$ we conclude that:

$$Y_{(12)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)} = I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big[I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big)I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big]I_{(\beta_{\alpha_2}\beta_{\alpha_3})},$$

$$Y_{(23)} = I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big[I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big)I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big]I_{(\beta_{\alpha_1}\beta_{\alpha_2})},$$

$$Y_{(13)} = I_{(\beta_{\alpha_1}\beta_{\alpha_2})}\big(I_{(\beta_{\alpha_2}\beta_{\alpha_3})}Q_{(\alpha_2\alpha_3)}\big)I_{(\beta_{\alpha_1}\beta_{\alpha_2})} = I_{(\beta_{\alpha_2}\beta_{\alpha_3})}\big(I_{(\beta_{\alpha_1}\beta_{\alpha_2})}Q_{(\alpha_1\alpha_2)}\big)I_{(\beta_{\alpha_2}\beta_{\alpha_3})}, \tag{1.20}$$

where the explicit form of $Y_{(12)}$, $Y_{(23)}$, $Y_{(13)}$ is given by:

$$
\begin{array}{c}
(\sigma_1\sigma_2\sigma_3)\rightarrow \quad (123) \qquad (213) \qquad (231) \qquad (321) \qquad (312) \qquad (132) \qquad (\beta_1\beta_2\beta_3)\downarrow \\
Y_{(12)} = \begin{pmatrix}
\begin{pmatrix} 1-B_{12} & -B_{12} \\ B_{12} & 1+B_{12} \end{pmatrix} & (0) & (0) \\
(0) & \begin{pmatrix} 1-B_{23} & -B_{23} \\ B_{23} & 1+B_{23} \end{pmatrix} & (0) \\
(0) & (0) & \begin{pmatrix} 1+B_{13} & B_{13} \\ -B_{13} & 1-B_{13} \end{pmatrix}
\end{pmatrix}
\begin{array}{c} (123) \\ (213) \\ (231) \\ (321) \\ (312) \\ (132) \end{array},
\end{array}
$$

$$
\begin{array}{c}
(\sigma_1\sigma_2\sigma_3)\rightarrow \quad (123) \qquad (213) \qquad (231) \qquad (321) \qquad (312) \qquad (132) \qquad (\beta_1\beta_2\beta_3)\downarrow \\
Y_{(23)} = \begin{pmatrix}
1-B_{23} & 0 & 0 & 0 & 0 & -B_{23} \\
0 & 1-B_{13} & -B_{13} & \multicolumn{2}{c}{\multirow{2}{*}{(0)}} & 0 \\
0 & B_{13} & 1+B_{13} & & & 0 \\
0 & \multicolumn{2}{c}{\multirow{2}{*}{(0)}} & 1+B_{12} & B_{12} & 0 \\
0 & & & -B_{12} & 1-B_{12} & 0 \\
B_{23} & 0 & 0 & 0 & 0 & 1+B_{23}
\end{pmatrix}
\begin{array}{c} (123) \\ (213) \\ (231) \\ (321) \\ (312) \\ (132) \end{array},
\end{array}
$$

$$
\begin{array}{c}
(\sigma_1\sigma_2\sigma_3)\rightarrow \quad (123) \qquad (213) \qquad (231) \qquad (321) \qquad (312) \qquad (132) \qquad (\beta_1\beta_2\beta_3)\downarrow \\
Y_{(13)} = \begin{pmatrix}
1-B_{13} & 0 & 0 & -B_{13} & 0 & 0 \\
0 & 1-B_{23} & 0 & 0 & -B_{23} & 0 \\
0 & 0 & 1+B_{12} & 0 & 0 & B_{12} \\
B_{13} & 0 & 0 & 1+B_{13} & 0 & 0 \\
0 & B_{23} & 0 & 0 & 1+B_{23} & 0 \\
0 & 0 & -B_{12} & 0 & 0 & 1-B_{12}
\end{pmatrix}
\begin{array}{c} (123) \\ (213) \\ (231) \\ (321) \\ (312) \\ (132) \end{array},
\end{array}
$$

where $(0) := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then applying a particular composition of $Y_{(12)}$, $Y_{(23)}$, $Y_{(13)}$,
G^{α} can be obtained from G^{123} as follows:

$$
\begin{aligned}
G^{123} &= \mathbb{1}\, G^{123}, \\
G^{213} &= Y_{(12)} G^{123}, \\
G^{132} &= Y_{(23)} G^{123}, \\
G^{231} &= Y_{(13)} Y_{(12)} G^{123}, \\
G^{312} &= Y_{(13)} Y_{(23)} G^{123}, \\
G^{321} &= Y_{(23)} Y_{(13)} Y_{(12)} G^{123} = Y_{(12)} Y_{(13)} Y_{(23)} G^{123}.
\end{aligned}
$$

1.4 Construction of n-particle wave function in general form

As it is shown in (1.20), each of $Y_{(12)}$, $Y_{(23)}$, $Y_{(13)}$ can be expressed in different ways: namely, with the use of $Q_{(\alpha_1\alpha_2)}$ or $Q_{(\alpha_2\alpha_3)}$. Each way of representation corresponds to a particular order and combination of particles transpositions and their momenta transpositions but the result for $Y_{(12)}$, $Y_{(23)}$, $Y_{(13)}$ is independent of the particular way of representation. Also, the matrix form for $Y_{(12)}$, $Y_{(23)}$, $Y_{(13)}$ is independent of the chosen order of phases in G^{α} and each of that matrices can be transformed to another one by simple permutation of one or two indices in index pairs: $(12) \leftrightarrow (13) \leftrightarrow (23)$. Therefore, there exists a general form for all Y-matrices independent of any particular order of phases in G^{α}. For n-particle case (including $n = 3$) this form is constructed as follows. Let $Y_{(lm)}$, where $l, m \in \{1, ..., n\}$, be any arbitrary Y-matrix. Let $\chi_{\beta\sigma}$ be a characteristic function of an arbitrary entry, where the first index β is in correspondence with the row number, the second one σ - with the column number. The positions of the diagonal elements $1 - B_{\beta_l\beta_m}$ is defined by $\delta_{\beta\sigma}$. By convention, we define $B_{\beta_l\beta_m}$ in a such way that $l < m$, however since $B_{\beta_l\beta_m}$ is pure imaginary, then for $l > m$ we replace it by $-B_{\beta_m\beta_l}$. The position of any off-diagonal non-zero element $B_{\beta_l\beta_m}$ is defined by such elementary transposition either of β or σ that the l-th and m-th components (in β or σ, respectively) are interchanged so that generally the non-zero off-diagonal entry can be expressed as $\delta_{\beta'(lm)\sigma}$, where all the components of $\beta'(lm)$ are the same as in β except the l-th and m-th which are interchanged relatively their position in β. It follows that the expression for any $Y_{(lm)}$-matrix and arbitrary $l, m = 1, ..., n$ is given by

$$
Y_{(lm)} = \sum_{\beta\in S_n} \sum_{\sigma\in S_n} \chi_{\beta\sigma} \left[\delta_{\beta\sigma}(1 - B_{\beta_l\beta_m}) + \delta_{\beta'(lm)\sigma} B_{\beta_l\beta_m} \right]. \tag{1.21}
$$

In particular, for $l = m$, if we define $A_{lm} = 1$, then $Y_{(lm)}$ is equal to identity matrix.
According to (1.18), the region $(1...n)$ contains only one wave $[1...n]$, for

$\beta = (1...n)$. It follows that the column of waves for that region can be expressed as $\sum\limits_{\gamma\in S_n} \chi_{\gamma\sigma_0}\delta_{\gamma\sigma_0}[1...n] \equiv \sum\limits_{\gamma\in S_n} \chi_{\gamma\sigma_0}\delta_{\gamma\sigma_0} \exp\left(i\sum_{\rho=1}^{n} k_{\beta_\rho}x_\rho\right)$,

because the first row of this matrix corresponds to $\beta = (1...n)$, here $\sigma_0 := (1...n)$ is corresponding to only the first one column of this matrix. Applying (1.21) to this column of waves for the region $(1...lm...n)$, we obtain the column of waves for the region $(1...ml...n)$, $m = l+1$, $l = 1, ..., n-1$:

$$\psi^{(1...ml...n)}(x_1...x_n \mid k_1...k_n) = Y_{(lm)}\psi^{(1...lm...n)}(x_1...x_n \mid k_1...k_n)$$

$$= \sum_{\beta\in S_n}\sum_{\sigma\in S_n} \chi_{\beta\sigma}\left[\delta_{\beta\sigma}(1 - B_{\beta_l\beta_m}) + \delta_{\beta'(lm)\sigma}B_{\beta_l\beta_m}\right]\sum_{\gamma\in S_n}\chi_{\gamma\sigma_0}\delta_{\gamma\sigma_0}\exp\left(i\sum_{\rho=1}^{n} k_{\beta_\rho}x_\rho\right)$$

$$= \sum_{\beta\in S_n}\sum_{\sigma\in S_n} \chi_{\beta\sigma}\left[\delta_{\beta\sigma}(1 - B_{\beta_l\beta_m}) + \delta_{\beta'(lm)\sigma}B_{\beta_l\beta_m}\right]\chi_{\sigma\sigma_0}\delta_{\sigma\sigma_0}\exp\left(i\sum_{\rho=1}^{n} k_{\beta_\rho}x_\rho\right)$$

$$= \sum_{\beta\in S_n} \chi_{\beta\sigma_0}\left[\delta_{\beta\sigma_0}(1 - B_{\beta_l\beta_m}) + \delta_{\beta'(lm)\sigma_0}B_{\beta_l\beta_m}\right]\exp\left(i\sum_{\rho=1}^{n} k_{\beta_\rho}x_\rho\right). \qquad (1.22)$$

To obtain a general form for the wave function in any given coordinate region, we have to multiply the matrices $Y_{(l_jm_j)}$ over some path between the region $(1...n)$ and this given region $\alpha = (\alpha_1...\alpha_n)$. Let $j = 1, ..., j(\alpha)$ be the sequence of elementary transpositions such that

$$\psi^{\alpha}(x_1...x_n \mid k_1...k_n) = \overleftarrow{\prod_{j=1,...,j(\alpha)}} Y_{(l_jm_j)}\psi^{(1...n)}(x_1...x_n \mid k_1...k_n). \qquad (1.23)$$

Then substituting (1.22) in (1.23) we obtain:

$$\psi^{\alpha}(x_1...x_n \mid k_1...k_n)$$
$$= \sum_{\beta\in S_n}\chi_{\beta\sigma_0}\overleftarrow{\prod_{j=1,...,j(\alpha)}}\left[\delta_{\beta\sigma_0}(1 - B_{\beta_{l_j}\beta_{m_j}}) + \delta_{\beta'(l_jm_j)\sigma_0}B_{\beta_{l_j}\beta_{m_j}}\right]\exp\left(i\sum_{\rho=1}^{n} k_{\beta_\rho}x_\rho\right), \qquad (1.24)$$

and finally, summing over all coordinate regions α with the corresponding characteristic functions $\chi(\alpha)$,

$$\psi(x_1...x_n \mid k_1...k_n) = \sum_{\alpha\in S_n}\chi(\alpha)\,\psi^{\alpha}(x_1...x_n \mid k_1...k_n)$$
$$= \sum_{\alpha\in S_n}\chi(\alpha)\sum_{\beta\in S_n}\chi_{\beta\sigma_0}\overleftarrow{\prod_{j=1,...,j(\alpha)}}\left[\delta_{\beta\sigma_0}(1 - B_{\beta_{l_j}\beta_{m_j}}) + \delta_{\beta'(l_jm_j)\sigma_0}B_{\beta_{l_j}\beta_{m_j}}\right]\exp\left(i\sum_{\rho=1}^{n} k_{\beta_\rho}x_\rho\right). \qquad (1.25)$$

Chapter 2

COMPLETENESS OF δ-INTERACTING PARTICLES IN THE 2-PARTICLE CASE

2.1 Transpositions operators, the wave function in terms of transpositions operators

The fact that the Bethe Ansatz is valid for δ-interacting particles in one dimension means essentially that a system of n interacting particles can be considered as consisting of independent 2-particle systems with δ-interaction. In order to understand the n-particle case, therefore, it is crucial first to understand the 2-particle case. This 2-particle case is well known by [17] but we wish to consider it from a new point of view, which is useful for the generalisation to the cases of more than two particles. We want to prove the following completeness relation:

$$\frac{1}{(2\pi)^2}\int \psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\, d^2k = \delta^2(x-y). \tag{2.1}$$

There are only two possible relative positions of two particles in one dimension, namely $x_1 > x_2$ and $x_2 > x_1$. We associate these positions with corresponding regions $(x_1 > x_2) =: (12)$ and $(x_2 > x_1) =: (21)$ and their characteristic functions $\chi(x_1 > x_2) =: \chi(12)$ and $\chi(x_2 > x_1) =: \chi(21)$ respectively. Then, as was shown in the previous chapter, the wave function of the system of two δ-interacting particles has the form

$$\psi(x_1x_2 \mid k_1k_2) = \chi(12)[12] + \chi(21)\{A_{12}[12] + B_{12}[21]\}, \tag{2.2}$$

where $[12] := e^{i(k_1x_1+k_2x_2)}$, $[21] := e^{i(k_2x_1+k_1x_2)}$, and

$$B_{12} = 1 - A_{12} = -\frac{ic}{k_1 - k_2}.$$

The terms of (2.2) can be arranged in matrix form such that each row corresponds to a particular coordinate region $\chi(\sigma_1\sigma_2)$, where $\sigma \in S_2$, and each of its columns corresponds to an ordered pair of momenta $(k_{\sigma'_1}k_{\sigma'_2})$, where $\sigma' \in S_2$. Then a wave corresponding to any entry of the matrix has a phase $i(k_{\sigma'_1}x_{\sigma_1} + k_{\sigma'_2}x_{\sigma_2})$ and the wave function $\psi(x_1x_2 \mid k_1k_2)$ is equal to the sum of *all entries* of this matrix:

$$\begin{array}{cc} (k_1k_2) & (k_2k_1) \end{array}$$
$$\begin{pmatrix} [12] & 0 \\ B_{12}[21] & A_{12}[12] \end{pmatrix} \begin{array}{c} \chi(12) \\ \chi(21) \end{array} .$$

According to the definition of transmission and reflection coefficients, given by (1.16), assuming normalisation of the amplitude for the wave function without interaction, this matrix may be rewritten as:

$$\begin{array}{cc} k_1>k_2 & k_2>k_1 \end{array}$$
$$\begin{pmatrix} \frac{1}{A_{12}}[12] & 0 \\ \frac{B_{12}}{A_{12}}[21] & [12] \end{pmatrix} \begin{array}{c} x_1 > x_2 \\ x_2 > x_1 \end{array} .$$

This has a clear physical interpretation. As was assumed in the previous chapter, the initial position (before interaction) of particle #1 is to the left of the position of particle #2. Positive values of momenta correspond to a wave moving to the right on the x-axis. Each of four entries corresponds to a particular combination of relative coordinate sign and relative momentum sign.

For $k_2 > k_1$ there is no interaction possible between the particles because particle #1 moves slower than particle #2. It follows that there exists only a wave with unit amplitude in the region $x_2 > x_1$ and no wave in the region $x_1 > x_2$, i.e. the coefficient of the latter equals 0. If $k_1 > k_2$, then, as a result of the interaction of the faster particle #1 and the slower particle #2 there are a transmitted wave $\frac{1}{A_{12}}[12]$ in region $x_1 > x_2$, with unchanged phase due to simultaneous transposition of coordinates and momenta, and a reflected wave $\frac{B_{12}}{A_{12}}[21]$ in region $x_2 > x_1$, with transposed phase due to transposition of momenta without transposition of coordinates.

It is natural and convenient to introduce the following three operators:

1) Operator of momentum transposition P_{12} defined as

$$P_{12}f(x_1x_2; k_1k_2) = f(x_1x_2; k_2k_1).$$

2) Operator of coordinate transposition R_{12} defined as

$R_{12}f(x_1x_2;k_1k_2) = f(x_2x_1;k_1k_2)$.

3) Operator of index transposition τ_{12} defined as

$\tau_{12}f(x_1x_2;k_1k_2) = f(x_2x_1;k_2k_1)$.

Obviously, these operators relate as:

$P_{12}R_{12} = R_{12}P_{12} = \tau_{12},\ \ \tau_{12}P_{12} = P_{12}\tau_{12} = R_{12},\ \ \tau_{12}R_{12} = R_{12}\tau_{12} = P_{12}$,

and also equal to their inverses:

$P_{21} = P_{12}^{-1} = P_{12},\ R_{21} = R_{12}^{-1} = R_{12},\ \tau_{21} = \tau_{12}^{-1} = \tau_{12}$.

Then (2.2) can be rewritten in the form:

$$\psi(x_1x_2 \mid k_1k_2) = \{\mathbb{1} + (A_{12} + B_{12}P_{12})\,\tau_{12}\}[12]\,\chi(12). \tag{2.3}$$

2.2 Construction $\psi_{in}(x_1x_2 \mid k_1k_2)$ in terms of transposition operators

First and until further notice we consider $\psi_{in}(x \mid k)$ in the coordinate region $x_1 > x_2$, with characteristic function $\chi(12)$. Now, we are using the definition of $\psi_{in}(x \mid k)$ for the 2-particle case,

$$\psi_{in}(x_1x_2 \mid k_1k_2) = \frac{\psi(x_1x_2 \mid k_1k_2)}{A_{12}}\,\theta(k_1 > k_2) + \frac{\psi(x_2x_1 \mid k_2k_1)}{A_{21}}\,\theta(k_2 > k_1), \tag{2.4}$$

with the notation $\theta(k_1 > k_2) =: \theta(12)$, $\theta(k_2 > k_1) =: \theta(21)$. Hence we obtain, for region $\chi(12)$,

$$\psi_{in}(x_1x_2 \mid k_1k_2)\,\chi(12) = \chi(12)\left\{\frac{1}{A_{12}}\,[12]\,\theta(12) + \frac{1}{A_{21}}(A_{21}[12] + B_{21}[21])\,\theta(21)\right\}. \tag{2.5}$$

In operator form, using the relation $P_{12}, R_{12}, \tau_{12}$, and also the δ-relation for characteristic functions of coordinate regions, i.e. $\chi(12)\chi(21) = 0$, this can be rewritten as:

$$\psi_{in}(x_1x_2 \mid k_1k_2)\,\chi(12)$$

$$= \chi(12)\left\{\frac{\psi(x_1x_2 \mid k_1k_2)}{A_{12}}\,\theta(12) + \tau_{12}\,\frac{\psi(x_1x_2 \mid k_1k_2)}{A_{12}}\,\theta(12)\right\}(\chi(12)+\chi(21))$$

$$= \chi(12)\left\{\theta(12)\,\frac{1}{A_{12}}\,\chi(12) + \tau_{12}\,\theta(12)\,\frac{1}{A_{12}}(A_{12}+B_{12}P_{12})\,\tau_{12}\,\chi(12)\right\}[12]$$

$$= \chi(12)\left\{\theta(12)\,\frac{1}{A_{12}} + \tau_{12}\,\theta(12)\,\frac{1}{A_{12}}(A_{12}+B_{12}P_{12})\,\tau_{12}\right\}[12], \tag{2.6}$$

with the corresponding matrix

$$\begin{pmatrix} \frac{1}{A_{12}}[12] & 0 \\ \frac{B_{21}}{A_{21}}[21] & [12] \end{pmatrix} \begin{matrix} \theta(12) \\ \theta(21) \end{matrix}.$$

Similar to (2.6), we construct an expression for $\psi_{in}(y_1y_2 \mid k_1k_2)\,\chi(12)$. Then the combined function in the common region $\chi^x(12)\chi^y(12) = \chi(12)$ has the form:

$$\begin{aligned} &\psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\,\chi(12) \\ &= \chi(12)\left\{\theta(12)\,\frac{1}{A_{12}} + \tau_{12}\,\theta(12)\,\frac{1}{A_{12}}(A_{12}+B_{12}P_{12})\,\tau_{12}\right\}[12](x) \\ &\times \left\{\theta(12)\,\frac{1}{\overline{A_{12}}} + \tau_{12}\,\theta(12)\,\frac{1}{\overline{A_{12}}}(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}\right\}[12](-y), \end{aligned} \tag{2.7}$$

where all operators act only on corresponding x- or y-components of the formula. Then using the δ-relation for characteristic functions of momentum regions, in particular $\theta(12)\theta(21) = 0$, and relations between $P_{12}, R_{12}, \tau_{12}$, this can be rearranged as:

$$\begin{aligned} &\psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\,\chi(12) \\ &= \chi(12)\Big\{\theta(12)\,\frac{1}{A_{12}}\,[12](x)\,\frac{1}{\overline{A_{12}}}\,[12](-y) \\ &+ \tau_{12}\,\theta(12)\,\frac{1}{A_{12}}\,(A_{12}+B_{12}P_{12})\,\tau_{12}\,[12](x)\,\tau_{12}\,\frac{1}{\overline{A_{12}}}(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}[12](-y)\Big\}, \end{aligned} \tag{2.8}$$

where all operators still act only on corresponding x- or y-components of the formula. Further rearrangement brings it into a form where all the operators act from the left on the complete expression to the right of the operator. To achieve this we replace $P_{12}\tau_{12}$ in x-component of the second term of (2.8) by

R^x_{12}, and similarly $P_{12}\tau_{12}$ in the y-component by R^y_{12} acting only on x- and y-component respectively. Since the operator τ_{12} in front of each component in (2.8) acts separately to these components, we leave only one operator τ_{12} in front of the whole second term, now acting to the right on both x- and y-components, and obtain:

$$\begin{aligned}
&\psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\,\chi(12) \qquad (2.9)\\
&= \chi(12)\bigg\{\frac{1}{A_{12}}\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}\,[12](-y)\\
&+ \tau_{12}\,\frac{1}{A_{12}}(A_{12}+B_{12}R^x_{12})\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}(\overline{A_{12}}+\overline{B_{12}}R^y_{12})\,[12](-y)\bigg\}.
\end{aligned}$$

The corresponding matrices are:

$$\begin{pmatrix} \frac{1}{A_{12}}[12](x) & 0 \\ \frac{B_{21}}{A_{21}}[21](x) & [12](x) \end{pmatrix} \quad \begin{matrix} \theta(12) \\ \theta(21) \end{matrix} \quad \begin{pmatrix} \frac{1}{\overline{A_{12}}}[12](-y) & 0 \\ \frac{\overline{B_{21}}}{\overline{A_{21}}}[21](-y) & [12](-y) \end{pmatrix},$$

and that can be verified directly with use of (2.9):

$$\begin{aligned}
&\chi(12)\bigg\{\frac{1}{A_{12}}\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}\,[12](-y)\\
&+ \tau_{12}\,\frac{1}{A_{12}}(A_{12}+B_{12}R^x_{12})\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}(\overline{A_{12}}+\overline{B_{12}}R^y_{12})\,[12](-y)\bigg\}\\
&= \frac{[12](x-y)}{|A_{12}|^2}\,\theta(12) + \left([12](x)+\frac{B_{21}}{A_{21}}[21](x)\right)\left([12](-y)+\frac{\overline{B_{21}}}{\overline{A_{21}}}[21](-y)\right)\theta(21).
\end{aligned}$$

2.3 Modification of integrand: $\psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)} \to \widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)$

We now wish to evaluate the integral of (2.8) over whole momentum space, i.e. over the two momentum regions $\theta(12)$ and $\theta(21)$. The standard way to do this is to insert (2.5) for the x- or y-components in (2.1):

$$\frac{1}{(2\pi)^2}\int d^2k\,\psi_{in}(x_1x_2\mid k_1k_2)\overline{\psi_{in}(y_1y_2\mid k_1k_2)} \tag{2.10}$$

$$=\frac{1}{(2\pi)^2}\Bigg\{\int_{k_1>k_2} d^2k\,\frac{1}{A_{12}}\,[12](x)\,\frac{1}{\overline{A_{12}}}\,[12](-y)$$

$$+\int_{k_2>k_1} d^2k\Bigg([12](x)+\frac{B_{21}}{A_{21}}\,[21](x)\Bigg)\Bigg([12](-y)+\frac{\overline{B_{21}}}{\overline{A_{21}}}\,[21](-y)\Bigg)\Bigg\}$$

$$=\frac{1}{(2\pi)^2}\Bigg\{\int_{k_1>k_2} d^2k\,\frac{1}{|A_{12}|^2}\,[12](x-y)+\int_{k_2>k_1} d^2k\Bigg([12](x-y)$$

$$+\frac{\overline{B_{21}}}{\overline{A_{21}}}\,[12](x)[21](-y)+\frac{B_{21}}{A_{21}}\,[21](x)[12](-y)+\frac{|B_{12}|^2}{|A_{12}|^2}\,[21](x-y)\Bigg)\Bigg\}$$

$$=\frac{1}{(2\pi)^2}\Bigg\{\int_{k_1>k_2} d^2k\Bigg(\Bigg(\frac{1}{|A_{12}|^2}+\frac{|B_{12}|^2}{|A_{12}|^2}\Bigg)[12](x-y)+\frac{B_{12}}{A_{12}}\,[12](x)[21](-y)\Bigg)$$

$$+\int_{k_2>k_1} d^2k\Bigg([12](x-y)+\frac{B_{12}}{A_{12}}\,[12](x)[21](-y)\Bigg)$$

$$=\frac{1}{(2\pi)^2}\int d^2k\,\Bigg([12](x-y)+\frac{B_{12}}{A_{12}}\,[12](x)[21](-y)\Bigg),$$

with the use of $\overline{A_{21}}\equiv A_{12}$, $\overline{B_{21}}\equiv B_{12}$.
It is clear that we need to apply a change of variables in the last two of four integrals over the region $k_2>k_1$ to obtain equal integrands for both regions $k_1>k_2$ and $k_2>k_1$. This method of solving requires special attention to each term: whether it already has the desired limit of integration or not, and so whether or not to apply a change of variables. To simplify the procedure we shall instead aim to express the sum of all integrals in an operator form that already includes these changes of variable. The following diagram illustrates schematically the necessary change in operator formula (2.8):

$$\begin{pmatrix} \bullet\bullet & \\ \bullet(\bullet+\bullet) & \bullet(\bullet+\bullet)\end{pmatrix}\begin{matrix}1\\4\end{matrix}\;\longrightarrow\;\begin{pmatrix}\bullet\bullet & \bullet(\bullet+\bullet)\\ & \bullet(\bullet+\bullet)\end{pmatrix}\begin{matrix}3\\2\end{matrix}$$

Figure 2.1: Modification of combined scattering wave function for 2-particle case.

Here the numbers to the right of each matrix are the numbers of integrals for each particular row. In fact, we wish to redistribute the integrals between momentum regions (according to (2.10)) by particular transformation. The combined x, y-matrix after such transformation we call *modified ψ_{in}-matrix*, the transformation we call **modification** *of* $\psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}$ *matrix* (and modification of corresponding *function*) and we use the notation $\widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)$ for it. In fact, the diagram above does not yet correspond to the modified matrix, because each entry of the second column still contains a sum of y-waves with different phases. The change of variables also reduces the number of different phases from 2×2 (two phases for each x- and y-wave: ($[12](x)$ and $[21](x)$) $\times$ ($[12](-y)$ and $[21](-y)$)) to only 1×2 by changing $[21](x) \to [12](x)$. Therefore the resulting combined matrix contains only 2 different phases: $[12](x-y)$ and $[12](x)[21](-y)$ in each row (region of momentum). Adjusting each particular element in such a way that any entry of combined matrix contains only one phase of combined x, y-wave, and this phase is associated with the position of that entry, gives the modified matrix as shown in the diagram below:

$$\begin{pmatrix} \frac{1}{A_{12}}[12](x)\frac{1}{\overline{A_{12}}}[12](-y) & 0 \\ \frac{B_{21}}{A_{21}}[21](x)\left(\frac{\overline{B_{21}}}{\overline{A_{21}}}[21](-y)+[12](-y)\right) & [12](x)\left(\frac{\overline{B_{21}}}{\overline{A_{21}}}[21](-y)+[12](-y)\right) \end{pmatrix} \begin{matrix} \theta(12) \\ \theta(21) \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} \frac{1}{|A_{12}|^2}[12](x-y) & \frac{B_{12}}{A_{12}}[12](x)\left(\frac{\overline{B_{12}}}{\overline{A_{12}}}[12](-y)+[21](-y)\right) \\ 0 & [12](x)\left(\frac{\overline{B_{21}}}{\overline{A_{21}}}[21](-y)+[12](-y)\right) \end{pmatrix} \begin{matrix} \theta(12) \\ \theta(21) \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} \left(\frac{1}{|A_{12}|^2}+\frac{|B_{12}|^2}{|A_{12}|^2}\right)[12](x-y) & \frac{B_{12}}{A_{12}}[12](x)[21](-y) \\ \frac{\overline{B_{21}}}{\overline{A_{21}}}[12](x)[21](-y) & [12](x-y) \end{pmatrix} \begin{matrix} \theta(12) \\ \theta(21) \end{matrix}$$

$$= \begin{pmatrix} [12](x-y) & \frac{B_{12}}{A_{12}}[12](x)[21](-y) \\ \frac{B_{12}}{A_{12}}[12](x)[21](-y) & [12](x-y) \end{pmatrix} \begin{matrix} \theta(12) \\ \theta(21) \end{matrix} .$$

The transformation performed on the diagram is precisely the action of the operator P_{12} on off-diagonal element(s) of the initial matrix which clearly leaves the integral invariant. To show that, it suffices to consider the integral

$$\int d^2k\,\theta(21)\,\frac{B_{21}}{A_{21}}[21](x)\left(\frac{\overline{B_{21}}}{\overline{A_{21}}}\,[21](-y)+[12](-y)\right),$$

where the integrand is the only non-zero off-diagonal element in the upper matrix above. Then

$$\int d^2k\,P_{12}\,\theta(21)\,\frac{B_{21}}{A_{21}}\,[21](x)\left(\frac{\overline{B_{21}}}{\overline{A_{21}}}\,[21](-y)+[12](-y)\right)$$
$$=\int d^2k\,\theta(12)\,\frac{B_{12}}{A_{12}}\,[12](x)\left(\frac{\overline{B_{12}}}{\overline{A_{12}}}\,[12](-y)+[21](-y)\right),$$

and this is equal to the change of variables $(k_1,k_2)\mapsto(k_2,k_1)$ in the integral. Applying the operator P_{12} to the off-diagonal element we rearrange the x-component for the off-diagonal element as

$$P_{12}\left(\tau_{12}\,\frac{B_{12}}{A_{12}}R^x_{12}[12](x)\right)=\left(\frac{B_{12}}{A_{12}}R_{12}R^x_{12}\tau_{12}\right)\tau_{12}[12](x)=\left(\frac{B_{12}}{A_{12}}R^y_{12}\tau_{12}\right)\tau_{12}[12](x).$$

Also, in order to write the diagonal element of the x-component in a similar form, we move the operator τ_{12} from the left through the brackets to the middle. The operator $R^y_{12}\tau_{12}\tau_{12}$ in front of the y-component acts as $P_{12}\tau_{12}$ on the whole y-component and this describes exactly a change of variables in the y-component. However, here we replace $\frac{\overline{B_{12}}}{\overline{A_{12}}}\,R^y_{12}$ by $\frac{\overline{B_{12}}}{\overline{A_{12}}}\,P_{12}\tau_{12}$, and then use the identity:

$$\frac{1}{\overline{A_{12}}}(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}=\tau_{12}\,(\overline{A_{12}}+P_{12}\overline{B_{12}})\frac{1}{\overline{A_{12}}}.$$

The full transformation is expressed as:

$$\begin{aligned}
&\psi_{in}(x_1x_2\mid k_1k_2)\overline{\psi_{in}(y_1y_2\mid k_1k_2)}\,\chi(12)\\
&=\chi(12)\Bigg\{\frac{1}{A_{12}}\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}\,[12](-y)\\
&+\tau_{12}\,\frac{1}{A_{12}}(A_{12}+B_{12}R^x_{12})\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}(\overline{A_{12}}+\overline{B_{12}}R^y_{12})\,[12](-y)\Bigg\}\to
\end{aligned}$$

$$\begin{aligned}
&\to\widetilde{\psi_{in}}(x_1x_2,y_1y_2\mid k_1k_2)\,\chi(12)\\
&=\chi(12)\Bigg\{\frac{1}{A_{12}}\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}\,[12](-y)\\
&+\frac{1}{A_{12}}(A_{12}+B_{12}R^y_{12}\tau_{12})\,\tau_{12}\,[12](x)\,\theta(12)\,\tau_{12}(\overline{A_{12}}+P_{12}\overline{B_{12}})\frac{1}{\overline{A_{12}}}\,[12](-y)\Bigg\}.
\end{aligned}\tag{2.11}$$

After transformation all the elements have the same phase $[12](x)$ of x-component, according to (2.11), which can therefore be moved to the front of the formula. After that, the operator $R^y_{12}\tau_{12}$ acts only on the y-component of the formula and can be replaced by $R_{12}\tau_{12} = P_{12}$. It follows that

$$\widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)\,\chi(12) = \chi(12)\bigg\{\frac{[12](x)}{A_{12}}\,\theta(12)\,\frac{[12](-y)}{\overline{A_{12}}} \tag{2.12}$$
$$+\,\frac{[12](x)}{A_{12}}(A_{12}+B_{12}P_{12})\,\tau_{12}\,\theta(12)\,\tau_{12}\,(\overline{A_{12}}+P_{12}\overline{B_{12}})\,\frac{[12](-y)}{\overline{A_{12}}}\bigg\},$$

and using $\overline{A_{21}} = A_{12}$, $\overline{B_{21}} = B_{12}$, its evaluation gives:

$$\widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)\,\chi(12) = \chi(12)\bigg\{\bigg[\frac{1}{|A_{12}|^2}\,\theta(12) + \theta(21) + \frac{|B_{12}|^2}{|A_{12}|^2}\,\theta(12)\bigg][12](x-y)$$
$$+\bigg[\frac{B_{12}}{A_{12}}\,\theta(12) + \frac{\overline{B_{21}}}{\overline{A_{21}}}\,\theta(21)\bigg][12](x)[21](-y)\bigg\}$$
$$= \chi(12)\bigg\{[12](x-y) + \frac{B_{12}}{A_{12}}\,[12](x)[21](-y)\bigg\}\big(\theta(12)+\theta(21)\big)$$
$$= \chi(12)\bigg([12](x-y) + \frac{B_{12}}{A_{12}}\,[12](x)[21](-y)\bigg), \tag{2.13}$$

which is in correspondence with the result of (2.10) for the integrand.

2.4 Construction of $\widetilde{\psi_{in}}(x, y \mid k)$ for different regions of coordinate space

We now consider the coordinate region $\chi^x(21)\chi^y(21) = \chi(21)$. For this region it suffices to apply the operator τ_{12} to all components of (2.8) so that (2.13) transforms into:

$$\widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)\,\chi(21)$$
$$= \tau_{12}\,\chi(12)\bigg([12](x-y) + \frac{B_{12}}{A_{12}}\,[12](x)[21](-y)\bigg)$$
$$= \chi(21)\bigg([12](x-y) + \frac{B_{21}}{A_{21}}\,[12](x)[21](-y)\bigg). \tag{2.14}$$

Next we consider the cases when x- and y-components of $\widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)$ are in different coordinate regions.
In the case $\chi^x(12)\chi^y(21)$, using the definition (2.4) but for the y-component in the coordinate region $\chi^y(21)$ we obtain, similar to (2.5),

$$\psi_{in}(y_1y_2 \mid k_1k_2)\,\chi^y(21) \tag{2.15}$$
$$= \chi^y(21)\bigg\{\frac{1}{A_{12}}\big(A_{12}\,[12](y) + B_{12}\,[21](y)\big)\,\theta(12) + \frac{1}{A_{21}}\,[12](y)\,\theta(21)\bigg\},$$

and its complex conjugate in operator form is equal to:

$$\begin{aligned}
&\overline{\psi_{in}(y_1 y_2 \mid k_1 k_2)}\, \chi^y(21) \\
&= \tau_{12}\, \chi^y(12) \Big\{ \theta(12)\, \frac{1}{\overline{A_{12}}} + \tau_{12}\, \theta(12)\, \frac{1}{\overline{A_{12}}} \big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big)\, \tau_{12} \Big\} [12](-y) \\
&= \chi^y(21) \Big\{ \theta(12)\, \frac{1}{\overline{A_{12}}} \big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big)\, \tau_{12} + \tau_{12}\, \theta(12)\, \frac{1}{\overline{A_{12}}} \Big\} [12](-y). \qquad (2.16)
\end{aligned}$$

Now we wish to express (2.16) in the form of composition of certain operators with the y-component of formula (2.8) (for the case $\chi^y(12)$). To get this form we use the properties $\Re B_{12} = 0$, $P_{12} B_{12} = \overline{B_{12}} P_{12}$, $P_{12} A_{12} = \overline{A_{12}} P_{12}$, $\tau_{12}^2 = \mathbb{I}$, relation $A_{12} = 1 - B_{12}$ and also the identity:

$$\begin{aligned}
&\big(A_{12} + B_{12} P_{12}\big)\big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big) \\
&= A_{12} \overline{A_{12}} + B_{12} P_{12} \overline{A_{12}} + A_{12} \overline{B_{12}} P_{12} + B_{12} P_{12} \overline{B_{12}} P_{12} \\
&= (1 - B_{12})(1 + B_{12}) + A_{12} B_{12} P_{12} + A_{12} \overline{B_{12}} P_{12} + B_{12}^2 P_{12}^2 \\
&= 1 - B_{12}^2 + A_{12}(B_{12} + \overline{B_{12}}) P_{12} + B_{12}^2 = \mathbb{I}. \qquad (2.17)
\end{aligned}$$

It follows that the first term of (2.16) in the brackets is

$$\theta(12)\, \frac{1}{\overline{A_{12}}} \big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big)\, \tau_{12} = \theta(12)\, \frac{1}{\overline{A_{12}}} \tau_{12} \big(A_{12} + B_{12} P_{12}\big), \qquad (2.18)$$

and the second term is

$$\tau_{12}\, \theta(12)\, \frac{1}{\overline{A_{12}}} = \tau_{12}\, \theta(12)\, \frac{1}{\overline{A_{12}}}\, \tau_{12} \big(A_{12} + B_{12} P_{12}\big)\big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big)\, \tau_{12}\,. \qquad (2.19)$$

Inserting (2.18) and (2.19) in (2.16) we obtain the terms of each momentum region transformed by the operator $\big[\tau_{12} \big(A_{12} + B_{12} P_{12}\big)\big]$,

$$\begin{aligned}
&\overline{\psi_{in}(y_1 y_2 \mid k_1 k_2)}\, \chi^y(21) = \chi^y(21) \Big\{ \theta(12)\, \frac{1}{\overline{A_{12}}} \big[\tau_{12} \big(A_{12} + B_{12} P_{12}\big)\big] \qquad (2.20) \\
&+ \tau_{12}\, \theta(12)\, \frac{1}{\overline{A_{12}}} \big[\tau_{12} \big(A_{12} + B_{12} P_{12}\big)\big] \big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big)\, \tau_{12} \Big\} [12](-y),
\end{aligned}$$

and after use of δ-relations for characteristic functions of momentum regions,

$$\begin{aligned}
&\psi_{in}(x_1 x_2 \mid k_1 k_2) \overline{\psi_{in}(y_1 y_2 \mid k_1 k_2)}\, \chi^x(12)\, \chi^y(21) \qquad (2.21) \\
&= \chi^x(12)\, \chi^y(21) \Big\{ \frac{1}{A_{12}}\, \theta(12)\, [12](x)\, \frac{1}{\overline{A_{12}}} \big[\tau_{12} \big(A_{12} + B_{12} P_{12}\big)\big] [12](-y) \\
&+ \Big(\tau_{12}\, \theta(12)\, \frac{1}{A_{12}} \big(A_{12} + B_{12} P_{12}\big)\, \tau_{12}\, [12](x) \Big) \\
&\times \Big(\tau_{12}\, \frac{1}{\overline{A_{12}}} \big[\tau_{12} \big(A_{12} + B_{12} P_{12}\big)\big] \big(\overline{A_{12}} + \overline{B_{12}} P_{12}\big)\, \tau_{12}\, [12](-y) \Big) \Big\},
\end{aligned}$$

where all operators act only on corresponding x- or y-components of formula. In fact, $\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right]\left(\overline{A_{12}}+\overline{B_{12}}P_{12}\right)\tau_{12}=\mathbb{1}$ in (2.21) because of (2.17) but we keep it in expanded form to show that expressions for the y-component in both momentum regions $\theta(12)$ and $\tau_{12}\,\theta(12)$ are transformed by the same operator $\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right]$ due to the transformation of coordinate region for y-component $\chi^y(12)\rightarrow\chi^y(21)$. The significance of this approach will be more apparent for more than two particles where there are more than two possible regions of relative coordinate positions. Applying the same algorithm as for (2.8), to bring (2.21) into a form where all the operators act from the left on the complete expression to the right of the operator, we obtain similar to (2.9) but for the case $\chi^x(12)\,\chi^y(21)$:

$$\begin{aligned}&\psi_{in}(x_1x_2\mid k_1k_2)\overline{\psi_{in}(y_1y_2\mid k_1k_2)}\,\chi^x(12)\,\chi^y(21)\\&=\chi^x(12)\,\chi^y(21)\Bigg\{\frac{1}{A_{12}}\,[12](x)\,\theta(12)\,\frac{1}{\overline{A_{12}}}\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right][12](-y)\\&+\tau_{12}\,\frac{1}{A_{12}}\left(A_{12}+B_{12}R^x_{12}\right)[12](x)\,\theta(12)\\&\times\frac{1}{\overline{A_{12}}}\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right]\left(\overline{A_{12}}+\overline{B_{12}}P_{12}\right)\tau_{12}\,[12](-y)\Bigg\}.\end{aligned}\tag{2.22}$$

Under transformation which was defined as a modification of ψ_{in}-function:

$$\psi_{in}(x_1x_2\mid k_1k_2)\overline{\psi_{in}(y_1y_2\mid k_1k_2)}\,\chi^x(12)\,\chi^y(21)\rightarrow\widetilde{\psi_{in}}(x_1x_2,y_1y_2\mid k_1k_2)\,\chi^x(12)\,\chi^y(21),$$

similar to (2.11) and (2.12), we apply the operator P_{12} to the elements with off-diagonal x-component. Also, for convenience, we rewrite the operators for the y-component in a form that enables us to move the operator $\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right]$, common to both terms, outside the brackets, contrary to the expression in (2.21)-(2.22), where its position is between other operators. To perform this transformation we need to transpose the y-component of (2.22) but keep equivalence for each term and with use of inverted identity (2.17): $\left(\overline{A_{12}}+P_{12}\overline{B_{12}}\right)\left(A_{12}+P_{12}B_{12}\right)=\mathbb{1}$, so that (2.22) transforms into:

$$\begin{aligned}&\widetilde{\psi_{in}}(x_1x_2,y_1y_2\mid k_1k_2)\,\chi^x(12)\,\chi^y(21)=\chi^x(12)\,\chi^y(21)\,\frac{[12](x)}{A_{12}}\Bigg\{\theta(12)\\&+\left(A_{12}+B_{12}P_{12}\right)\tau_{12}\,\theta(12)\,\tau_{12}\left(\overline{A_{12}}+P_{12}\overline{B_{12}}\right)\Bigg\}\left[\left(A_{12}+P_{12}B_{12}\right)\tau_{12}\right]\frac{[12](-y)}{\overline{A_{12}}}\\&=\chi^x(12)\,\chi^y(21)\Bigg\{\theta(12)\Bigg(\frac{1}{A_{12}}\,[12](x-y)+\frac{1}{A_{12}}\frac{B_{21}}{A_{21}}\,[12](x)[21](-y)\\&+\frac{B_{12}}{A_{12}}\frac{1}{A_{21}}\,[12](x)[21](-y)\Bigg)+\theta(21)\,\frac{1}{A_{12}}\,[12](x-y)\Bigg\}\\&=\chi^x(12)\,\chi^y(21)\,\frac{1}{A_{12}}\,[12](x-y).\end{aligned}\tag{2.23}$$

2.5 Evaluation of integrals

2.5.1 Integration for the case $\chi^x(12)\chi^y(12) = \chi(12)$

The first integral in (2.10) (for coordinate region $\chi(12)$) is equal to:

$$\chi(12)\,\frac{1}{(2\pi)^2}\int d^2k\,[12](x-y) \tag{2.24}$$

$$= \chi(12)\,\frac{1}{(2\pi)^2}\int d^2k\,e^{i(k_1(x_1-y_1)+k_2(x_2-y_2))} = \chi(12)\,\delta^2(x-y).$$

After substituting the value of

$$\frac{B_{12}}{A_{12}} = \frac{-ic}{k_1-k_2}\,\frac{1}{1-(-ic)/(k_1-k_2)} = \frac{-ic}{k_1-k_2+ic} = \frac{-ic}{2q_1+ic},$$

where $k_1 - k_2 =: 2q_1$, $k_1 + k_2 =: 2q_2$, and using
$k_1(x_1-y_2)+k_2(x_2-y_1) = \big\{q_1[(x_1-x_2)+(y_1-y_2)]+q_2[(x_1+x_2)-(y_1+y_2)]\big\}$,
the second integral in (2.10) becomes (up to a constant coefficient)

$$\int dq_2\ \exp\big[i\big(q_2((x_1+x_2)-(y_1+y_2))\big)\big]\int \frac{dq_1 e^{iqz}}{q_1+ic/2},$$

where $z = (x_1 - x_2) + (y_1 - y_2) > 0$ in region $\chi(12)$. Set $q_1 = q$, $c/2 = b$.
To evaluate the integral

$$\int \frac{dq\,e^{iqz}}{q+ib} \tag{2.25}$$

we first note that its real part

$$\Re\int \frac{dq\,e^{iqz}}{q+ib} = \int_{-\infty}^{+\infty}\frac{q\cos(qz)+b\sin(qz)}{q^2+b^2}\,dq = 0, \tag{2.26}$$

because the integrand is an odd function. Next, we find a bound for its imaginary part:

$$\Im\int \frac{dq\,e^{iqz}}{q+ib} = \int_{-\infty}^{+\infty}\frac{q\sin(qz)-b\cos(qz)}{q^2+b^2}\,dq. \tag{2.27}$$

Note that $z > 0$ and integrating by parts, we set $u = q/(q^2 + b^2)$,
$dv = dq\,\sin(qz)$, so that $du = dq\,(b^2-q^2)/(q^2+c^2)^2$, $v = -(1/z)\cos(qz)$, and

$$\int_{-\infty}^{+\infty}\frac{q\sin(qz)}{q^2+b^2}\,dq = -\frac{1}{z}\,\frac{q\cos(qz)}{q^2+b^2}\bigg|_{-\infty}^{+\infty} + \frac{1}{z}\int_{-\infty}^{+\infty}\frac{(b^2-q^2)\cos(qz)}{(q^2+b^2)^2}\,dq$$

$$= \frac{1}{z}\int_{-\infty}^{+\infty}\frac{(b^2-q^2)\cos(qz)}{(q^2+b^2)^2}\,dq \quad \text{is bounded,}$$

$$\int_{-\infty}^{+\infty} \frac{b\cos(qz)}{q^2+b^2}dq$$ is bounded too, therefore the integral (2.25) is bounded.

Next, we use analytical continuation $q \mapsto q + i\kappa$, where $\kappa > 0$. Since there is only one pole which is negative imaginary: $q = -ib$ and $\kappa > 0$, a shift of integration contour from the axis $\kappa = 0$ to the upper half-plane does not change the result of integration and

$$\int \frac{dq\, e^{iqz}}{q+ib} = \int \frac{dq\, e^{i(q+i\kappa)z}}{q+ib} = e^{-\kappa z} \int \frac{dq\, e^{iqz}}{q+ib}, \qquad \text{where } 0 < e^{-\kappa z} < 1.$$

It follows that
$$\int \frac{dq\, e^{iqz}}{q+ib} = 0 \tag{2.28}$$

and
$$\frac{1}{(2\pi)^2} \int d^2k\, \widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)\, \chi(12) = \chi(12)\, \delta^2(x-y), \tag{2.29}$$

i.e. (2.1) holds for the coordinate region $\chi(12)$.

2.5.2 Integration for the cases of different coordinate regions

In the case of $\chi^x(21)\, \chi^y(21) = \chi(21)$ due to (2.14) and with use of (2.29) the integral can be expressed as

$$\tau_{12} \frac{1}{(2\pi)^2} \int d^2k\, \widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)\, \chi(12) = \tau_{12}\, \chi(12)\, \delta^2(x-y) = \chi(21)\, \delta^2(x-y). \tag{2.30}$$

In the case of $\chi^x(12)\, \chi^y(21)$ we use the relation $A_{12} = 1 - B_{12}$, so that the integral of (2.23) becomes

$$\begin{aligned} &\frac{\chi^x(12)\, \chi^y(21)}{(2\pi)^2} \int \psi_{in}(x_1x_2 \mid k_1k_2) \overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\, d^2k \\ &= \frac{\chi^x(12)\, \chi^y(21)}{(2\pi)^2} \int \left(1 + \frac{B_{12}}{A_{12}}\right) [12](x-y)\, d^2k. \end{aligned} \tag{2.31}$$

The first integral in (2.31) is

$$\frac{1}{(2\pi)^2} \int [12](x-y)\, d^2k = \delta^2(x-y).$$

In particular, for the region $\chi^x(12)\, \chi^y(21)$,

$$\frac{\chi^x(12)\, \chi^y(21)}{(2\pi)^2} \int [12](x-y)\, d^2k = 0,$$

because for that coordinate region if $x_1 = y_1$, then $x_2 < x_1 = y_1 < y_2$ and hence $x_2 \neq y_2$, or, similarly if $x_2 = y_2$, then $x_1 > x_2 = y_2 > y_1$ and hence $x_1 \neq y_1$. However, since $\chi^x(12)\,\chi^y(21)\,\delta^2(x-y) = 0$, we still can formally write

$$\frac{\chi^x(12)\,\chi^y(21)}{(2\pi)^2}\int [12](x-y)\,d^2k = \chi^x(12)\,\chi^y(21)\,\delta^2(x-y).$$

After substituting the value of: $B_{12}/A_{12} = -ic/(2q_1 + ic)$,
where $k_1 - k_2 =: 2q_1$, $k_1 + k_2 =: 2q_2$, and using
$k_1(x_1-y_1)+k_2(x_2-y_2) = \big\{q_1[(x_1-x_2)+(y_2-y_1)]+q_2[(x_1+x_2)-(y_1+y_2)]\big\}$,
the second integral in (2.12) becomes (up to a constant coefficient)

$$\int dq_2\;\exp\big[i\big(q_2((x_1+x_2)-(y_1+y_2))\big)\big]\int \frac{dq_1 e^{iqz}}{q_1 + ic/2},$$

where $z = (x_1 - x_2) + (y_2 - y_1) > 0$ for the region $\chi^x(12)\,\chi^y(21)$. Then the internal integral reduces to (2.25) which is equal 0 as it was proved in the previous subsection. It follows that

$$\widetilde{\psi_{in}}(x_1x_2, y_1y_2 \mid k_1k_2)\,\chi^x(12)\,\chi^y(21) = \chi^x(12)\,\chi^y(21)\,\delta^2(x-y),$$

i.e. (2.1) holds for coordinate region $\chi^x(12)\,\chi^y(21)$.

In the case $\chi^x(21)\,\chi^y(12)$, since $\chi^x(21)\,\chi^y(12) = \tau_{12}\,\chi^x(12)\,\chi^y(21)$ it follows that

$$\frac{\chi^x(21)\,\chi^y(12)}{(2\pi)^2}\int \psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\,d^2k$$

$$= \tau_{12}\,\frac{\chi^x(12)\,\chi^y(21)}{(2\pi)^2}\int \psi_{in}(x_1x_2 \mid k_1k_2)\overline{\psi_{in}(y_1y_2 \mid k_1k_2)}\,d^2k$$

$$= \tau_{12}\,\chi^x(12)\,\chi^y(21)\,\delta^2(x-y) = \chi^x(21)\,\chi^y(12)\,\tau_{12}\,\delta(x_1-y_1)\,\delta(x_2-y_2)$$

$$= \chi^x(21)\,\chi^y(12)\,\delta(x_2-y_2)\,\delta(x_1-y_1) = \chi^x(21)\,\chi^y(12)\,\delta^2(x-y).$$

Summing over all four coordinate regions we obtain (2.1).

Chapter 3

COMPLETENESS OF δ-INTERACTING PARTICLES IN THE 3-PARTICLE CASE

3.1 The wave function in terms of transposition operators

We have an explicit formula for the wave function in the 3-particle case:

$$\begin{aligned}
\psi(x_1x_2x_3 \mid k_1k_2k_3) &= \chi(x_1 > x_2 > x_3)[123] \\
&+ \chi(x_2 > x_1 > x_3)\{A_{12}[123] + B_{12}[213]\} \\
&+ \chi(x_1 > x_3 > x_2)\{A_{23}[123] + B_{23}[132]\} \\
&+ \chi(x_2 > x_3 > x_1)\{A_{12}A_{13}[123] + A_{12}B_{13}[321] + A_{23}B_{12}[213] + B_{12}B_{23}[312]\} \\
&+ \chi(x_3 > x_1 > x_2)\{A_{23}A_{13}[123] + A_{23}B_{13}[321] + A_{12}B_{23}[132] + B_{12}B_{23}[231]\} \\
&+ \chi(x_3 > x_2 > x_1)\{A_{12}A_{13}A_{23}[123] + A_{23}A_{13}B_{12}[213] + A_{12}A_{13}B_{23}[132] \\
&\qquad + A_{23}B_{13}B_{12}[231] + A_{12}B_{13}B_{23}[312] + B_{13}(1 + B_{12}B_{23})[321]\},
\end{aligned} \tag{3.1}$$

where

$$[\beta_1\beta_2\beta_3] = \exp\left(i\sum_{\rho=1}^{3} k_{\beta_\rho}x_\rho\right),$$

$\chi(x_{\sigma_1} > x_{\sigma_2} > x_{\sigma_3})$ is the characteristic function of a region in coordinate space and

$$B_{\mu\nu} = 1 - A_{\mu\nu} = -\frac{ic}{k_\mu - k_\nu}, \quad \mu, \nu = 1, 2, 3, \ \ \mu < \nu.$$

The terms of $\psi(x_1x_2x_3 \mid k_1k_2k_3)$ can be arranged in the form of a matrix such that each row corresponds to a particular coordinate region

$\chi(x_{\sigma_1} > x_{\sigma_2} > x_{\sigma_3}) =: \chi(\sigma_1\sigma_2\sigma_3) =: \chi(\sigma)$, where $\sigma \in S_3$; and each column corresponds to a particular transposition of momenta $(\sigma_1'\sigma_2'\sigma_3') =: \sigma'$, with $\sigma' \in S_3$. The wave corresponding to any entry of the matrix has a phase $\exp\left[i(k_{\sigma_1'}x_{\sigma_1} + k_{\sigma_2'}x_{\sigma_2} + k_{\sigma_3'}x_{\sigma_3})\right]$ and the wave function $\psi(x_1x_2x_3 \mid k_1k_2k_3)$ is equal to the sum of all entries of the matrix:

$$
\begin{array}{cccccccc}
 & & & \leftarrow (\sigma_1'\sigma_2'\sigma_3') \rightarrow & & & & \\
(123) & (213) & (132) & (231) & (312) & (321) & \chi(\sigma_1\sigma_2\sigma_3) \\
 & & & & & & \downarrow
\end{array}
$$

$$
\left(\begin{array}{cccccc}
[123] & 0 & 0 & 0 & 0 & 0 \\
B_{12}[213] & A_{12}[123] & 0 & 0 & 0 & 0 \\
B_{23}[132] & 0 & A_{23}[123] & 0 & 0 & 0 \\
B_{12}B_{23}[312] & A_{12}B_{13}[321] & A_{23}B_{12}[213] & A_{12}A_{13}[123] & 0 & 0 \\
B_{12}B_{23}[231] & A_{12}B_{23}[132] & A_{23}B_{13}[321] & 0 & A_{23}A_{13}[123] & 0 \\
\begin{array}{c} B_{13}(1+B_{12} \\ \times B_{23})[321] \end{array} & \begin{array}{c} A_{12}B_{13}B_{23} \\ \times[312] \end{array} & \begin{array}{c} A_{23}B_{13}B_{12} \\ \times[231] \end{array} & \begin{array}{c} A_{12}A_{13}B_{23} \\ \times[132] \end{array} & \begin{array}{c} A_{23}A_{13}B_{12} \\ \times[213] \end{array} & \begin{array}{c} A_{12}A_{13}A_{23} \\ \times[123] \end{array}
\end{array}\right)
\begin{array}{c} (123) \\ (213) \\ (132) \\ (231) \\ (312) \\ (321) \end{array}
$$

In this chapter we prove the following completeness relation for the 3-particle case:

$$\frac{1}{(2\pi)^3}\int \psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}\, d^3k = \delta^3(x-y). \qquad (3.2)$$

Similar to the 2-particle case, but for all three pairs of particles, we define transposition operators for each pair of momenta and corresponding pair of particles. Since
1) every arbitrary permutation of 3 elements can be expressed as a composition of elementary transpositions of pairs from all 3 elements in such way that only neighbouring elements are allowed to interchange their positions and/or momenta in an elementary transposition;
2) each entry of the above matrix corresponds to a particular combination of momentum and coordinate permutation;
we define the following elementary transposition operators for each pair of momenta $k_ik_j,\ i<j,\ \ i,j = 1,2,3$.

a) Operators of momentum transposition P_{ij} - numbered by the indices of transposed momenta:

$$
\begin{aligned}
P_{12}\, f(x_1x_2x_3; k_1k_2k_3) &= f(x_1x_2x_3; k_2k_1k_3), \\
P_{23}\, f(x_1x_2x_3; k_1k_2k_3) &= f(x_1x_2x_3; k_1k_3k_2), \\
P_{13}\, f(x_1x_2x_3; k_1k_2k_3) &= f(x_1x_2x_3; k_3k_2k_1).
\end{aligned}
$$

(Since elementary transpositions do not depend on the order of the pair of indices, i.e. $P_{ij} = P_{ji}$, there are only three different operators: P_{12}, P_{13}, P_{23}.)

b) Operators of coordinate transposition R_{ij}:

$$
\begin{aligned}
R_{12}\, f(x_1x_2x_3;k_1k_2k_3) &= f(x_2x_1x_3;k_1k_2k_3),\\
R_{23}\, f(x_1x_2x_3;k_1k_2k_3) &= f(x_1x_3x_2;k_1k_2k_3),\\
R_{13}\, f(x_1x_2x_3;k_1k_2k_3) &= f(x_3x_2x_1;k_1k_2k_3).
\end{aligned}
$$

c) Operators of index transposition τ_{ij}:

$$
\begin{aligned}
\tau_{12}\, f(x_1x_2x_3;k_1k_2k_3) &= f(x_2x_1x_3;k_2k_1k_3),\\
\tau_{23}\, f(x_1x_2x_3;k_1k_2k_3) &= f(x_1x_3x_2;k_1k_3k_2),\\
\tau_{13}\, f(x_1x_2x_3;k_1k_2k_3) &= f(x_3x_2x_1;k_3k_2k_1).
\end{aligned}
$$

Obviously, these operators are related as follows:

$$P_{ij}R_{ij} = R_{ij}P_{ij} = \tau_{ij};\;\; \tau_{ij}P_{ij} = P_{ij}\tau_{ij} = R_{ij};\;\; \tau_{ij}R_{ij} = R_{ij}\tau_{ij} = P_{ij};$$

$$P_{ij}P_{jl} = P_{jl}P_{il} = P_{il}P_{ij};\;\; R_{ij}R_{jl} = R_{jl}R_{il} = R_{il}R_{ij};\;\; \tau_{ij}\tau_{jl} = \tau_{jl}\tau_{il} = \tau_{il}\tau_{ij}.$$

With the help of these operators, the wave function can be expressed in the following form:

$$
\begin{aligned}
&\psi(x_1x_2x_3 \mid k_1k_2k_3) = \{\mathbb{1}\\
&+ (A_{12} + B_{12}P_{12})\,\tau_{12}\\
&+ (A_{23} + B_{23}P_{23})\,\tau_{23}\\
&+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{12}\\
&+ (A_{23} + B_{23}P_{23})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{23}\\
&+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23})\,\tau_{23}\tau_{13}\tau_{12}\}[123]\,\chi(123).
\end{aligned}
\tag{3.3}
$$

3.2 Construction of $\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)$ in terms of transposition operators

The scattering wave $\psi_{in}(x \mid k)$ for 3-particle case is defined as follows:

$$\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3) = \sum_{\sigma' \in S_3} \frac{\psi(x_{\sigma'_1}x_{\sigma'_2}x_{\sigma'_3} \mid k_{\sigma'_1}k_{\sigma'_2}k_{\sigma'_3})}{A_{\sigma'_1\sigma'_2}A_{\sigma'_1\sigma'_3}A_{\sigma'_2\sigma'_3}}\,\theta(k_{\sigma'_1} > k_{\sigma'_2} > k_{\sigma'_3}). \tag{3.4}$$

This is a linear combination of the wave functions (3.1) which transforms the system of wave functions for all regions of coordinate space into a system of waves in momentum space for all regions of momentum space in a given region of coordinate space. Since there are at most $3! = 6$ different waves in each of the 6 regions of coordinate space, we construct at most 6 different waves for each of the 6 regions of momentum space and this is for each of 6 regions of coordinate space.

From now on and until further notice, we will consider $\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)$ only in the region $\chi(123)$ of coordinate space. Arbitrary regions $\chi(\sigma^x)$, $\sigma^x \in S_3$ will be considered later.

We make use of the shortened notations

$\chi(x_{\sigma_1} > x_{\sigma_2} > x_{\sigma_3}) =: \chi(\sigma_1\sigma_2\sigma_3) =: \chi(\sigma)$ - characteristic function of corresponding region in coordinate space,

$\theta(k_{\sigma'_1} > k_{\sigma'_2} > k_{\sigma'_3}) =: \theta(\sigma'_1\sigma'_2\sigma'_3) =: \theta(\sigma')$ - characteristic function of corresponding region in momentum space.

In (3.4), ψ_{in} is defined in terms of wave functions for the different momentum regions and this can equivalently be expressed in the form of momentum transpositions, since every momentum region is a momentum transposition of another region, and eventually of $\theta(123)$. However, we only have an expression (3.1) for the wave function $\psi(x_1x_2x_3 \mid k_1k_2k_3)$ in different coordinate regions $\chi(\sigma_1\sigma_2\sigma_3)$, i.e. in terms of coordinate transpositions of coordinate regions. Therefore we wish to express ψ_{in} in the form of known transpositions of coordinate regions instead of transpositions of momentum regions. This can be done with the use of antisymmetric property of transpositions in momentum and coordinate space. According to (3.4), ψ_{in} for coordinate region $\chi(123)$ can be expressed as:

$$\begin{aligned}\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)\chi(123) &= \chi(123) \sum_{\sigma' \in S_3} \frac{\psi(x_{\sigma'_1}x_{\sigma'_2}x_{\sigma'_3} \mid k_{\sigma'_1}k_{\sigma'_2}k_{\sigma'_3})}{A_{\sigma'_1\sigma'_2}A_{\sigma'_1\sigma'_3}A_{\sigma'_2\sigma'_3}}\,\theta(\sigma') \\ &= \chi(123) \sum_{\sigma' \in S_3} \tau_{\sigma'}\,\theta(123)\,\frac{\psi(x_1x_2x_3 \mid k_1k_2k_3)}{A_{12}A_{13}A_{23}},\end{aligned}$$

where $\tau_{\sigma'}$ is a permutation of indices, defined by
$\tau_{\sigma'}\, f(x_1x_2x_3; k_1k_2k_3) = f(x_{\sigma'_1}x_{\sigma'_2}x_{\sigma'_3}; k_{\sigma'_1}k_{\sigma'_2}k_{\sigma'_3})$ and equal to a particular composition of elementary transpositions τ_{ij}, $i < j$, $i, j = 1, 2, 3$.
Since the sum over σ' does not depend on the order of summation we can replace the permutations $\tau_{\sigma'}$ by their inverse $\tau_{\sigma'}^{-1}$. Also, we multiply both sides by $\mathbb{I} = \sum_{\sigma \in S_3} \chi(\sigma)$ and use the identity $\chi(123) = \tau_{\sigma'}^{-1}\,\chi(\sigma')$. It follows that

$$\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)\chi(123)$$

$$= \chi(123) \sum_{\sigma' \in S_3} \tau_{\sigma'}^{-1}\, \theta(123)\, \frac{\psi(x_1x_2x_3 \mid k_1k_2k_3)}{A_{12}A_{13}A_{23}} \sum_{\sigma \in S_3} \chi(\sigma)$$

$$= \sum_{\sigma' \in S_3} \tau_{\sigma'}^{-1}\, \theta(123)\, \frac{\psi(x_1x_2x_3 \mid k_1k_2k_3)}{A_{12}A_{13}A_{23}}\, \chi(\sigma') \sum_{\sigma \in S_3} \chi(\sigma)$$

$$= \sum_{\sigma \in S_3} \tau_{\sigma}^{-1}\, \theta(123) \frac{\psi(x_1x_2x_3 \mid k_1k_2k_3)}{A_{12}A_{13}A_{23}}\, \chi(\sigma), \tag{3.5}$$

where we use the δ-relation for characteristic functions $\chi(\sigma')\,\chi(\sigma) = \chi(\sigma)\,\delta(\sigma', \sigma) = \chi(\sigma)\,\delta(\tau_{\sigma'}, \tau_\sigma)$, where τ_σ is defined similar to $\tau_{\sigma'}$. Using the identity $\tau_{12}\tau_{13}\tau_{23} = \tau_{23}\tau_{13}\tau_{12}$, which is easily verified directly, and inserting (3.3) into (3.5) we obtain:

$$\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)\,\chi(123) = \chi(123) \Bigg\{ \frac{\theta(123)}{A_{12}A_{13}A_{23}} \tag{3.6}$$

$$+ \tau_{12}\, \frac{\theta(123)}{A_{12}A_{13}A_{23}} (A_{12} + B_{12}P_{12})\, \tau_{12}$$

$$+ \tau_{23}\, \frac{\theta(123)}{A_{12}A_{13}A_{23}} (A_{23} + B_{23}P_{23})\, \tau_{23}$$

$$+ \tau_{12}\tau_{13}\, \frac{\theta(123)}{A_{12}A_{13}A_{23}} (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})\, \tau_{13}\tau_{12}$$

$$+ \tau_{23}\tau_{13}\, \frac{\theta(123)}{A_{12}A_{13}A_{23}} (A_{23} + B_{23}P_{23})(A_{13} + B_{13}P_{13})\, \tau_{13}\tau_{23}$$

$$+ \tau_{12}\tau_{13}\tau_{23}\, \frac{\theta(123)}{A_{12}A_{13}A_{23}} (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23})\, \tau_{23}\tau_{13}\tau_{12} \Bigg\} [123].$$

By moving τ_σ^{-1} operators on the left through each term to the right we evaluate an explicit form of each term and obtain the corresponding matrix, where each entry is defined by the combination of momentum region and phase of wave:

$$
\begin{array}{ccccccc}
\multicolumn{7}{c}{\leftarrow (\sigma_1\sigma_2\sigma_3) \rightarrow} \\
(123) & (213) & (132) & (231) & (312) & (321) & \theta(\sigma_1'\sigma_2'\sigma_3') \\
 & & & & & & \downarrow
\end{array}
$$

$$
\begin{pmatrix}
\frac{1}{A_{12}A_{13}A_{23}}[123] & 0 & 0 & 0 & 0 & 0 \\
\frac{B_{21}}{A_{21}A_{13}A_{23}}[213] & \frac{1}{A_{13}A_{23}}[123] & 0 & 0 & 0 & 0 \\
\frac{B_{32}}{A_{32}A_{13}A_{12}}[132] & 0 & \frac{1}{A_{13}A_{12}}[123] & 0 & 0 & 0 \\
\frac{B_{31}B_{23}}{A_{31}A_{23}A_{21}}[231] & \frac{B_{31}}{A_{31}A_{21}}[321] & \frac{B_{21}}{A_{21}A_{23}}[213] & \frac{1}{A_{23}}[123] & 0 & 0 \\
\frac{B_{31}B_{12}}{A_{31}A_{12}A_{32}}[312] & \frac{B_{32}}{A_{32}A_{12}}[132] & \frac{B_{31}}{A_{31}A_{32}}[321] & 0 & \frac{1}{A_{12}}[123] & 0 \\
\frac{B_{31}(1+B_{21}B_{32})}{A_{21}A_{31}A_{32}}[321] & \frac{B_{21}B_{31}}{A_{21}A_{31}}[231] & \frac{B_{32}B_{31}}{A_{32}A_{31}}[312] & \frac{B_{32}}{A_{32}}[132] & \frac{B_{21}}{A_{21}}[213] & [123]
\end{pmatrix}
\begin{array}{c}
(123) \\ (213) \\ (132) \\ (231) \\ (312) \\ (321)
\end{array}
$$

3.3 Modification of the integrand: $\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3))\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}$ $\rightarrow \widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)$

Similar to (3.6) we construct an expression for $\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}$, and then, with the use of a δ-relation for characteristic functions of momentum regions $\theta(\sigma')$, a combined function $\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}$ in the common coordinate region $\chi^y(123)\,\chi^x(123) = \chi(123)$. After rearrangement, this combined function has a form similar to (2.8) for the 2-particle case:

$$\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}\,\chi(123) \tag{3.7}$$

$$= \chi(123)\Bigg\{ \frac{\theta(123)}{A_{12}A_{13}A_{23}}\,[123](x)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}$$

$$+ \left(\tau_{12}\,\frac{\theta(123)}{A_{12}A_{13}A_{23}}\,(A_{12}+B_{12}P_{12})\,\tau_{12}\,[123](x)\right)\left(\tau_{12}\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}\right)$$

$$+ \left(\tau_{23}\,\frac{\theta(123)}{A_{12}A_{13}A_{23}}(A_{23}+B_{23}P_{23})\,\tau_{23}\,[123](x)\right)\left(\tau_{23}\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}\right)$$

$$+ \left(\tau_{12}\tau_{13}\,\frac{\theta(123)}{A_{12}A_{13}A_{23}}\,(A_{12}+B_{12}P_{12})(A_{13}+B_{13}P_{13})\tau_{13}\tau_{12}\,[123](x)\right)$$
$$\times \left(\tau_{12}\tau_{13}\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{12}\right)$$

$$+ \left(\tau_{23}\tau_{13}\,\frac{\theta(123)}{A_{12}A_{13}A_{23}}(A_{23}+B_{23}P_{23})\,(A_{13}+B_{13}P_{13})\tau_{13}\tau_{23}[123](x)\right)$$
$$\times \left(\tau_{23}\tau_{13}\,\frac{\theta(123)}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\right)$$

$$+ \left(\tau_{12}\tau_{13}\tau_{23}\,\frac{\theta(123)}{A_{12}A_{13}A_{23}}\,(A_{12}+B_{12}P_{12})(A_{13}+B_{13}P_{13})(A_{23}+B_{23}P_{23})\,\tau_{23}\tau_{13}\tau_{12}[123](x)\right)$$
$$\times \left(\tau_{12}\tau_{13}\tau_{23}\;\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}\tau_{13}\tau_{12}\right)\Bigg\}$$
$$\times\,[123](-y),$$

where all operators act only on corresponding x- or y-components of the formula. Since the x-component of (3.7) for each region contains terms with different combinations of P_{ij}, $i<j,\ i,j=1,2,3$, rearrangement of (3.7) to a form where all the operators act from the left on the complete expression to the right of the operator, needs additional individual adjustment after each x-term, in front of the y-terms, by the inverse of the combination of P_{ij} in that particular term, because we cannot separate the action of a momentum transposition operator P_{ij} on the x- and y-component. It would be possible to adjust the part of (3.7), in particular the terms for regions $\theta(123)$, $\tau_{12}\,\theta(123)$, $\tau_{23}\,\theta(123)$, where there is no more than one elementary transposition P_{ij} in the same way as it was done in the 2-particle case. However, for the other regions with compositions of more than one transposition it is impossible to separate the action of different combinations of P_{ij} on the

terms of the y-component. Nevertheless, it is still possible to transform (3.7) to a form where all the operators act from the left on the complete expression to the right of the operator, but only *after* a transformation which we call **modification** of ψ_{in}.

Similar to the 2-particle case, we need to apply a change of variables to the off-diagonal elements of (3.7) in such a way that the phases of the x-components are transformed into $[123](x)$. Since off-diagonal elements have generally different phases of x-components, depending on the position of the entry in the matrix, or according to the formula (3.7), where the phase $[123](x)$ is transformed by the particular combination of operators on the left of $[123](x)$, each off-diagonal element has to be transformed by the inverse of that combination of operators. Applying the inverse combination of momentum transposition operators and rearranging the full combination of operators for each element for all lines in (3.7), we obtain an expression for all off-diagonal elements. Action of this inverse combination on arbitrary off-diagonal elements is equal to a corresponding change of variables, i.e. we have to apply the system of different changes of variables to (3.7). Since each change of variables also changes the limits of integration, the corresponding element changes its position (momentum region $\theta(\sigma')$) in the above matrix. The following diagram illustrates the resulting change in the form of the matrix due to the required transformation of formula (3.7):

$$\begin{pmatrix}
\bullet(\bullet\circ\circ\circ\circ\circ) & & & & & \\
\bullet(\bullet\bullet\circ\circ\circ\circ) & \bullet(\bullet\bullet\circ\circ\circ\circ) & & & & \\
\bullet(\bullet\circ\bullet\circ\circ\circ) & & \bullet(\bullet\circ\bullet\circ\circ\circ) & & & \\
\bullet(\bullet\bullet\bullet\bullet\circ\circ) & \bullet(\bullet\bullet\bullet\bullet\circ\circ) & \bullet(\bullet\bullet\bullet\bullet\circ\circ) & \bullet(\bullet\bullet\bullet\bullet\circ\circ) & & \\
\bullet(\bullet\bullet\bullet\circ\bullet\circ) & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & \\
\bullet(\bullet\bullet\bullet\bullet\bullet\bullet) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet)
\end{pmatrix}
\begin{matrix} 1 \\ 4 \\ 4 \\ 16 \\ 16 \\ 36 \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix}
\bullet(\bullet\circ\circ\circ\circ\circ) & \bullet(\bullet\bullet\circ\circ\circ\circ) & \bullet(\bullet\circ\bullet\circ\circ\circ) & \bullet(\bullet\bullet\bullet\bullet\circ\circ) & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) \\
 & \bullet(\bullet\bullet\circ\circ\circ\circ) & \bullet(\bullet\circ\bullet\circ\circ\circ) & & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) \\
 & & \bullet(\bullet\circ\bullet\circ\circ\circ) & \bullet(\bullet\bullet\bullet\bullet\circ\circ) & & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) \\
 & & & \bullet(\bullet\bullet\bullet\bullet\circ\circ) & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) \\
 & & & & \bullet(\bullet\bullet\bullet\circ\bullet\circ) & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet) \\
 & & & & & \bullet(\bullet\bullet\bullet\bullet\bullet\bullet)
\end{pmatrix}
\begin{matrix} 19 \\ 16 \\ 16 \\ 10 \\ 10 \\ 6 \end{matrix}$$

Figure 3.1: Modification of combined scattering wave function for 3-particle case.

Black circles outside brackets correspond to the x-component of the entries in the combined matrix and the modified combined matrix respectively. Black circles inside brackets correspond to the y-components with non-zero amplitude. White circles correspond to y-components with zero amplitude. The number to the right of each row is the number of terms in the corresponding momentum region and equal to the number of integrals to evaluate. The total number of integrals is 77 and this is the same for the original matrix and its modification. As can be seen from (3.7), the number of different terms of each x- and y-component in the combined matrix is equal to $2^3 = 8$ which brings the total number of combined terms to 105, but since there are at most only $3! = 6$ different phases, some of the 8 terms have the same phase and can be reduced in correspondence with the diagram. Transformation of the diagram is equivalent to the action of such a combination of operators P_{ij} on each off-diagonal element of the combined matrix, that brings the x-phase of that element back to $[123](x)$.

First consider the x-terms of (3.7) for the region $\tau_{12}\,\theta(123) = \theta(213)$:

$$\tau_{12}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,(A_{12} + B_{12}P_{12})\,\tau_{12}\,[123](x).$$

Applying the operator P_{12} to the off-diagonal element only, we obtain the same phase $[123](x)$ for both elements and can therefore move it to the front so that, after rearrangement, the modified x-term becomes

$$[123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,(A_{12} + B_{12}P_{12})\,\tau_{12}\,\theta(123).$$

This can be verified directly separately for the diagonal and off-diagonal term:

$$\tau_{12}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,A_{12}\,\tau_{12}\,[123](x) = [123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,A_{12}\,\tau_{12}\,\theta(123)$$

and

$$P_{12}\tau_{12}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,B_{12}P_{12}\,\tau_{12}\,[123](x) = [123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,B_{12}P_{12}\,\tau_{12}\,\theta(123).$$

Since we wish to extend the action of all the operators to the whole expression to the right of the operator, we have to compensate the action of the operator in the x-term by applying the inverse operator $\tau_{12}^{-1} = \tau_{12}$ in the front of y-term. Note that the transposition operator is equal to $\mathbb{I}$ for the diagonal term whereas the transposition P_{12} in the off-diagonal term should not be compensated because, according to the diagram above, the change of variables applies simultaneously to the x- and y-components. It follows that the expression for the y-component is

$$\tau_{12}\tau_{12}\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}\,[123](-y).$$

Then, after moving one of the operators τ_{12} to the right it cancels out with the τ_{12} on the right and we get

$$\tau_{12}\,\frac{1}{\overline{A_{21}A_{23}A_{13}}}\,(\overline{A_{21}}+\overline{B_{21}}P_{12})[123](-y) = \tau_{12}\,(\overline{A_{12}}+P_{12}\overline{B_{12}})\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,[123](-y).$$

It follows that whole modified expression corresponding to the region $\tau_{12}\,\theta(123) = \theta(213)$ is

$$[123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,(A_{12}{+}B_{12}P_{12})\,\tau_{12}\,\theta(123)\,\tau_{12}\,(\overline{A_{12}}{+}P_{12}\overline{B_{12}})\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,[123](-y).$$

Next consider the x-component of (3.7) for the region $\tau_{12}\tau_{13}\,\theta(123) = \theta(312)$:

$$\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,(A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{12}[123](x).$$

The diagonal element does not change after modification but can be rearranged into the form

$$\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,A_{12}A_{13}\,\tau_{13}\tau_{12}[123](x)$$

$$= [123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,A_{23}A_{13}\,\tau_{13}\tau_{23}\,\theta(123).$$

To achieve the phase $[123](x)$ in the main off-diagonal element it has to be transformed by the application of the combination $P_{13}P_{12}$:

$$P_{13}P_{12}\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,B_{12}P_{12}B_{13}P_{13}\tau_{13}\tau_{12}[123](x).$$

With the use of the identities $P_{13}P_{12} = P_{23}P_{13}$ and $\tau_{12}\tau_{13} = \tau_{13}\tau_{23}$ we rearrange it into the form:

$$[123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,B_{23}P_{23}B_{13}P_{13}\,\tau_{13}\tau_{23}\,\theta(123).$$

Since $\tau_{12}\tau_{13}P_{13} = P_{23}\tau_{12}\tau_{13}$ and $\tau_{12}\tau_{13}P_{12} = P_{13}\tau_{12}\tau_{13}$, to achieve the phase $[123](x)$ in the other two off-diagonal elements they have to be transformed respectively as:

$$P_{23}\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,A_{12}B_{13}P_{13}\,\tau_{13}\tau_{12}[123](x)$$

$$= [123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,B_{23}P_{23}A_{13}\,\tau_{13}\tau_{23}\,\theta(123),$$

$$P_{13}\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{A_{12}A_{13}A_{23}}\,B_{12}P_{12}A_{13}\,\tau_{13}\tau_{12}[123](x)$$

$$= [123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,A_{23}B_{13}P_{13}\,\tau_{13}\tau_{23}\,\theta(123).$$

Summing all four elements we obtain the modified expression for the x-component in the region $\tau_{12}\tau_{13}\,\theta(123)$:

$$[123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,(A_{23}+B_{23}P_{23})(A_{13}+B_{13}P_{13})\,\tau_{13}\tau_{23}\,\theta(123).$$

Similar to the case of the region $\tau_{12}\,\theta(123)$, we compensate the action of the operator $\tau_{12}\tau_{13}$ on the y-component by applying its inverse $(\tau_{12}\tau_{13})^{-1} = \tau_{13}\tau_{12} = \tau_{23}\tau_{13}$ to the whole y-component of (3.7). Then, for each element of the x-component, the corresponding y-component is transformed by the same combination of operators, $\mathbb{I}$ or $P_{23}P_{13}$ or P_{23} or P_{13} respectively, and the y-component becomes equal to

$$(\tau_{23}\tau_{13})(\tau_{12}\tau_{13})\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{12}\,[123](-y).$$

Moving the operator $\tau_{12}\tau_{13}$ to the right it cancels out with $\tau_{13}\tau_{12}$ and we obtain

$$\tau_{23}\tau_{13}\,\frac{1}{\overline{A_{31}A_{32}A_{12}}}\,(\overline{A_{31}}+\overline{B_{31}}P_{13})(\overline{A_{32}}+\overline{B_{32}}P_{23})[123](-y)$$

$$=\tau_{23}\tau_{13}\,(\overline{A_{13}}+P_{13}\overline{B_{13}})(\overline{A_{23}}+P_{23}\overline{B_{23}})\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,[123](-y).$$

It follows that whole modified expression corresponding to the region $\tau_{12}\tau_{13}\,\theta(123) = (\tau_{23}\tau_{13})^{-1}\,\theta(123) = \theta(312)$ is

$$[123](x)\,\frac{1}{A_{12}A_{13}A_{23}}\,(A_{23}+B_{23}P_{23})(A_{13}+B_{13}P_{13})\,\tau_{13}\tau_{23}\,\theta(123)$$

$$\times\tau_{23}\tau_{13}\,(\overline{A_{13}}+P_{13}\overline{B_{13}})(\overline{A_{23}}+P_{23}\overline{B_{23}})\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,[123](-y).$$

Applying the same algorithm for all other elements in the expressions for all regions in (3.7) with the simultaneous transformation of x-wave to the phase $[123](x)$ and extending of action of the operators to the whole expression on the right to the operator and then adjusting and rearranging the corresponding y-components, we finally obtain:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi(123) = \chi(123)\,\frac{[123](x)}{A_{12}A_{13}A_{23}}\Big\{\theta(123) \tag{3.8}$$

$$+ (A_{12} + B_{12}P_{12})\,\tau_{12}\,\theta(123)\,\tau_{12}\,(\overline{A_{12}} + P_{12}\overline{B_{12}})$$

$$+ (A_{23} + B_{23}P_{23})\,\tau_{23}\,\theta(123)\,\tau_{23}\,(\overline{A_{23}} + P_{23}\overline{B_{23}})$$

$$+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{12}\,\theta(123)\,\tau_{12}\tau_{13}\,(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})$$

$$+ (A_{23} + B_{23}P_{23})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{23}\,\theta(123)\,\tau_{23}\tau_{13}\,(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{23}} + P_{23}\overline{B_{23}})$$

$$+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23})\,\tau_{23}\tau_{13}\tau_{12}\,\theta(123)$$

$$\times\;\tau_{12}\tau_{13}\tau_{23}\,(\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})\Big\}\,\frac{[123](-y)}{A_{12}A_{13}A_{23}}.$$

3.4 Construction of $\widetilde{\psi_{in}}(x, y \mid k)$ for different regions of coordinate space

Now consider another coordinate region such that $\chi^x(\sigma^x)\,\chi^y(\sigma^y) = \chi(\sigma)$, where $\sigma^x = \sigma^y = \sigma \in S_3$. It suffices to apply the operator τ_σ to all components of (3.8) so that:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi(\sigma) \tag{3.9}$$
$$= \widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\tau_\sigma\chi(123),$$

and this is equal to the same expression as right hand side of (3.8) but with all indices $1, 2, 3$ replaced by $\sigma_1, \sigma_2, \sigma_3$, respectively.

Next consider the cases where the x- and y-components of $\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)$ are in different coordinate regions but assume that the x-component is in the region (123): $\chi^x(123)\,\chi^y(\sigma^y)$.
For example, in the case $\chi^y(213)$ we can use the equations (2.15)-(2.20) for the 2-particle case. According to (2.20), the elementary transposition of the coordinate region $\chi^y(12) \to \chi^y(21)$ is associated with inserting an additional factor $\left[\tau_{12}\,(A_{12} + B_{12}P_{12})\right]$ in $\psi_{in}(y \mid k)$ for each momentum region. Since the similar transformation $\chi^y(123) \to \chi^y(213)$ does not affect particle #3 and this particle does not interact with the other two, we can assume the same transformation of $\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}$ and this is directly verified by

the definition (3.4). So, to justify this insertion and obtain an expression similar to (2.20) but for the 3-particle case, we insert the identity operator expanded as $1\!\!1 = (\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}\big[\tau_{12}\,(A_{12} + B_{12}P_{12})\big]$ in each of six terms of the expression similar to (3.6) but for y-component and then apply the index transposition operator τ_{12} to the whole formula. We obtain

$$\overline{\psi_{in}(y_1y_2y_3 \mid k_1k_2k_3)}\,\chi^y(213) \tag{3.10}$$

$$= \chi^y(213)\,\tau_{12}\Bigg\{\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}$$

$$+\,\tau_{12}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}$$

$$+\,\tau_{23}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}} + \overline{B_{23}}P_{23})\,\tau_{23}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}$$

$$+\,\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})(\overline{A_{13}} + \overline{B_{13}}P_{13})\,\tau_{13}\tau_{12}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}$$

$$+\,\tau_{23}\tau_{13}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}} + \overline{B_{23}}P_{23})(\overline{A_{13}} + \overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}$$

$$+\,\tau_{12}\tau_{13}\tau_{23}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})(\overline{A_{13}} + \overline{B_{13}}P_{13})(\overline{A_{23}} + \overline{B_{23}}P_{23})$$

$$\times\,\tau_{23}\tau_{13}\tau_{12}\,(\overline{A_{12}} + \overline{B_{12}}P_{12})\,\tau_{12}\Bigg\}\big[\tau_{12}\,(A_{12} + B_{12}P_{12})\big][123](-y).$$

Since the operator τ_{12} interchanges indices 1 and 2 in the whole formula (3.10), it also interchanges the order of momentum regions
$(123),(213),(132),(312),(231),(321) \longmapsto (213),(123),(231),(321),(132),(312)$.
Evaluating the result of application of τ_{12} in (3.10) using the identities $\tau_{12}\tau_{23}\tau_{13} = \tau_{23}$, $\tau_{12}\tau_{23} = \tau_{23}\tau_{13}$, $\tau_{13}\tau_{23} = \tau_{12}\tau_{13}$ and then interchanging the terms to restore the initial order of regions we obtain the new form for each momentum region. For example, the expression for the momentum region $\tau_{12}\tau_{23}\tau_{13}\,\theta(123)$ transforms as:

$$\tau_{12}\left\{\tau_{23}\tau_{13}\,\frac{\theta(123)}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}\right\}$$

$$=\tau_{23}\tau_{13}\tau_{13}\,\frac{\theta(123)}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}$$

$$=\tau_{23}\,\frac{\theta(123)}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}.$$

Rearranging similarly the terms for all regions we obtain:

$$\overline{\psi_{in}(y_1y_2y_3\mid k_1k_2k_3)}\,\chi^y(213) \tag{3.11}$$

$$=\chi^y(213)\left\{\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\right.$$

$$+\,\tau_{12}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}$$

$$+\,\tau_{23}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}$$

$$+\,\tau_{12}\tau_{13}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{12}$$

$$+\,\tau_{23}\tau_{13}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}$$

$$\left.+\,\tau_{12}\tau_{13}\tau_{23}\,\theta(123)\,\frac{1}{\overline{A_{12}A_{13}A_{23}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}\tau_{13}\tau_{12}\right\}$$

$$\times\left[\tau_{12}\,(A_{12}+B_{12}P_{12})\right][123](-y).$$

Thus we see, as claimed, that (3.11) is in correspondence with the expression for the y-component in (3.7) for coordinate region $\chi^y(123)$, but with an additional factor $\left[\tau_{12}\,(A_{12}+B_{12}P_{12})\right]$ representing the transformation of coordinate regions $\chi^y(123)\mapsto\chi^y(213)$. We can now apply the same modification to (3.11) as to (3.7) to obtain the analogue of expression (3.8) with the factor $\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right]$ at the end. Then using the identity:

$$\frac{1}{\overline{\overline{A_{12}A_{13}A_{23}}}}\left[\tau_{12}\left(A_{12}+B_{12}P_{12}\right)\right]=\left[\left(A_{12}+P_{12}B_{12}\right)\tau_{12}\right]\frac{1}{\overline{\overline{A_{12}A_{13}A_{23}}}}$$

we finally obtain for the region $\chi^x(123)\,\chi^y(213)$:

$$\widetilde{\psi_{in}}(x_1x_2x_3,y_1y_2y_3\mid k_1k_2k_3)\,\chi^x(123)\,\chi^y(213) \tag{3.12}$$

$$=\chi^x(123)\,\chi^y(213)\frac{[123](x)}{A_{12}A_{13}A_{23}}\Bigg\{\theta(123)$$

$$+\left(A_{12}+B_{12}P_{12}\right)\tau_{12}\,\theta(123)\,\tau_{12}\,(\overline{A_{12}}+P_{12}\overline{B_{12}})$$

$$+\left(A_{23}+B_{23}P_{23}\right)\tau_{23}\,\theta(123)\,\tau_{23}\,(\overline{A_{23}}+P_{23}\overline{B_{23}})$$

$$+\left(A_{12}+B_{12}P_{12}\right)(A_{13}+B_{13}P_{13})\,\tau_{13}\tau_{12}\,\theta(123)\,\tau_{12}\tau_{13}\,(\overline{A_{13}}+P_{13}\overline{B_{13}})(\overline{A_{12}}+P_{12}\overline{B_{12}})$$

$$+\left(A_{23}+B_{23}P_{23}\right)(A_{13}+B_{13}P_{13})\,\tau_{13}\tau_{23}\,\theta(123)\,\tau_{23}\tau_{13}\,(\overline{A_{13}}+P_{13}\overline{B_{13}})(\overline{A_{23}}+P_{23}\overline{B_{23}})$$

$$+\left(A_{12}+B_{12}P_{12}\right)(A_{13}+B_{13}P_{13})(A_{23}+B_{23}P_{23})\,\tau_{23}\tau_{13}\tau_{12}\,\theta(123)\,\tau_{12}\tau_{13}\tau_{23}$$

$$\times\,(\overline{A_{23}}+P_{23}\overline{B_{23}})(\overline{A_{13}}+P_{13}\overline{B_{13}})(\overline{A_{12}}+P_{12}\overline{B_{12}})\Bigg\}\left[\left(A_{12}+P_{12}B_{12}\right)\tau_{12}\right]\frac{[123](-y)}{\overline{\overline{A_{12}A_{13}A_{23}}}}.$$

For coordinate region $\chi^y(213)$ we define the operator

$$Z(213):=\left(A_{12}+P_{12}B_{12}\right)\tau_{12}.$$

To derive a formula similar to (3.12) for the coordinate region $\chi^y(231)$ we first apply the operator $\tau_{13}\tau_{12}$ to the expression for the y-component for $\chi^y(123)$ and insert

$$\mathbb{1}=(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\left[\tau_{23}\tau_{13}\left(A_{13}+B_{13}P_{13}\right)(A_{23}+B_{23}P_{23})\right]$$

at the end of the formula. Then making rearrangements similar to (3.10)-(3.12) we insert the expression for each momentum region of the y-component into the corresponding region of the x-component for $\chi^x(123)$. For example, the expression for momentum region $\tau_{12}\,\theta(123)$ transforms as:

$$\tau_{13}\tau_{12}\left\{\tau_{12}\frac{\theta(123)}{\overline{\overline{A_{12}A_{13}A_{23}}}}(\overline{A_{12}}+\overline{B_{12}}P_{12})\,\tau_{12}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\right\}$$

$$=\tau_{12}\tau_{13}\tau_{23}\frac{\theta(123)}{\overline{\overline{A_{12}A_{13}A_{23}}}}(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{23}}+\overline{B_{23}}P_{23})\,\tau_{23}\tau_{13}\tau_{12}.$$

However, for some of the momentum regions we need to use the identity

$(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{23}}+\overline{B_{23}}P_{23}) = (\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{12}}+\overline{B_{12}}P_{12})$.

This identity (in conjugated form) is proved in the following

Lemma 1
$z_{12}z_{13}z_{23} = z_{23}z_{13}z_{12}$, *where* $z_{ij} := A_{ij} + B_{ij}P_{ij}$, $i < j$, $i, j = 1, 2, 3$.

Proof

With the use of the easily verified properties
$P_{12}P_{13} = P_{13}P_{23} = P_{23}P_{12}$, $P_{23}P_{13} = P_{13}P_{12} = P_{12}P_{23}$, $P_{12}P_{13}P_{23} = P_{13}$, $B_{12}B_{23} = B_{12}B_{13} + B_{13}B_{23}$, we rearrange the expression

$$\begin{aligned}
z_{12}z_{13}z_{23} &:= (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23}) \qquad (3.13)\\
&= A_{12}A_{13}A_{23} + A_{12}A_{13}B_{23}P_{23} + A_{12}B_{13}P_{13}A_{23} + B_{12}P_{12}A_{13}A_{23}\\
&+ A_{12}B_{13}P_{13}B_{23}P_{23} + B_{12}P_{12}A_{13}B_{23}P_{23} + B_{12}P_{12}B_{13}P_{13}A_{23} + B_{12}P_{12}B_{13}P_{13}B_{23}P_{23}\\
&= A_{23}A_{13}A_{12} + B_{23}P_{23}A_{13}A_{12} + A_{23}A_{13}B_{12}P_{12} + B_{12}P_{12}A_{13}B_{23}P_{23}\\
&+ (A_{12}B_{13}P_{13}A_{23} + B_{12}P_{12}B_{13}P_{13}B_{23}P_{23}) + (A_{12}B_{13}P_{13}B_{23}P_{23} + B_{12}P_{12}B_{13}P_{13}A_{23}),
\end{aligned}$$

and then rewrite the term

$$\begin{aligned}
&B_{12}P_{12}A_{13}B_{23}P_{23} = B_{12}P_{12}B_{23}P_{23}A_{12} = B_{12}B_{13}P_{12}P_{23}A_{12}\\
&= (B_{12}B_{23} - B_{13}B_{23})P_{23}P_{13}A_{12} = B_{23}P_{23}B_{13}P_{13}A_{12} - B_{13}B_{23}P_{13}P_{12}A_{12}\\
&= B_{23}P_{23}B_{13}P_{13}A_{12} - A_{23}B_{13}P_{13}B_{21}P_{12} = B_{23}P_{23}B_{13}P_{13}A_{12} + A_{23}B_{13}P_{13}B_{12}P_{12}, \qquad (3.14)
\end{aligned}$$

the expression in the first brackets,

$$\begin{aligned}
&A_{12}B_{13}P_{13}A_{23} + B_{12}P_{12}B_{13}P_{13}B_{23}P_{23} = A_{12}A_{21}B_{13}P_{13} + B_{12}B_{23}B_{12}P_{12}P_{13}P_{23}\\
&= (1 - B_{12}^2)B_{13}P_{13} + B_{12}(B_{12}B_{13} + B_{13}B_{23})P_{13} = B_{13}P_{13} + B_{12}B_{13}B_{23}P_{13}\\
&= (1 - B_{23}^2)B_{13}P_{13} + B_{23}(B_{23}B_{13} + B_{13}B_{12})P_{13}\\
&= A_{23}A_{32}B_{13}P_{13} + B_{23}B_{12}B_{23}P_{23}P_{13}P_{12} = A_{23}B_{13}P_{13}A_{12} + B_{23}P_{23}B_{13}P_{13}B_{12}P_{12}, \qquad (3.15)
\end{aligned}$$

and that in the second brackets,

$$\begin{aligned}
&A_{12}B_{13}P_{13}B_{23}P_{23} + B_{12}P_{12}B_{13}P_{13}A_{23} = A_{12}(B_{13}B_{21}P_{13}P_{23} + B_{12}B_{23}P_{12}P_{13})\\
&= A_{12}(-B_{12}B_{13} + B_{12}B_{23})P_{23}P_{12} = A_{12}B_{23}B_{13}P_{23}P_{12} = B_{23}P_{23}A_{13}B_{12}P_{12}. \qquad (3.16)
\end{aligned}$$

Substituting (3.14)-(3.16) in (3.13) we obtain:

$$\begin{aligned}
z_{12}z_{13}z_{23} &= A_{23}A_{13}A_{12} + B_{23}P_{23}A_{13}A_{12} + A_{23}A_{13}B_{12}P_{12} + B_{23}P_{23}B_{13}P_{13}A_{12}\\
&+ A_{23}B_{13}P_{13}B_{12}P_{12} + A_{23}B_{13}P_{13}A_{12} + B_{23}P_{23}B_{13}P_{13}B_{12}P_{12} + B_{23}P_{23}A_{13}B_{12}P_{12}\\
&= (A_{23} + B_{23}P_{23})(A_{13} + B_{13}P_{13})(A_{12} + B_{12}P_{12}) = z_{23}z_{13}z_{12}. \qquad \square
\end{aligned}$$

With the use of *Lemma 1*, for example, the expression for the momentum region $\tau_{23}\tau_{13}\,\theta(123)$ transforms as:

$$\tau_{13}\tau_{12}\left\{\tau_{23}\tau_{13}\,\frac{\theta(123)}{\overline{\overline{A_{12}A_{13}A_{23}}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\right.$$

$$\left.\times(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{23}\right\}$$

$$=\tau_{12}\tau_{23}\tau_{23}\tau_{13}\,\frac{\theta(123)}{\overline{\overline{A_{12}A_{13}A_{23}}}}\,(\overline{A_{23}}+\overline{B_{23}}P_{23})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{12}}+\overline{B_{12}}P_{12})$$
$$\times(\overline{A_{32}}+\overline{B_{32}}P_{23})\,\tau_{13}\tau_{23}\tau_{13}\tau_{23}$$

$$=\tau_{12}\tau_{13}\,\frac{\theta(123)}{\overline{\overline{A_{12}A_{13}A_{23}}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})(\overline{A_{23}}+\overline{B_{23}}P_{23})$$
$$\times(\overline{A_{32}}+\overline{B_{32}}P_{23})\,\tau_{12}\tau_{13}\tau_{13}\tau_{23}$$

$$=\tau_{12}\tau_{13}\,\frac{\theta(123)}{\overline{\overline{A_{12}A_{13}A_{23}}}}\,(\overline{A_{12}}+\overline{B_{12}}P_{12})(\overline{A_{13}}+\overline{B_{13}}P_{13})\,\tau_{13}\tau_{12}.$$

Then after applying appropriate change of variables to each off-diagonal element we use the directly verified identity:

$$\frac{1}{\overline{\overline{A_{12}A_{13}A_{23}}}}\left[\tau_{23}\tau_{13}\,(A_{13}+B_{13}P_{13})(A_{23}+B_{23}P_{23})\right]$$

$$=\left[(A_{12}+P_{12}B_{12})(A_{13}+B_{13}P_{13})\,\tau_{13}\tau_{12}\right]\frac{1}{\overline{\overline{A_{12}A_{13}A_{23}}}},$$

and obtain for the region $\chi^x(123)\,\chi^y(231)$:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\, \chi^x(123)\, \chi^y(231) \tag{3.17}$$

$$= \chi^x(123)\, \chi^y(231) \frac{[123](x)}{A_{12}A_{13}A_{23}} \Big\{ \theta(123)$$

$$+ (A_{12} + B_{12}P_{12})\, \tau_{12}\, \theta(123)\, \tau_{12}\, (\overline{A_{12}} + P_{12}\overline{B_{12}})$$

$$+ (A_{23} + B_{23}P_{23})\, \tau_{23}\, \theta(123)\, \tau_{23}\, (\overline{A_{23}} + P_{23}\overline{B_{23}})$$

$$+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})\tau_{13}\tau_{12}\, \theta(123)\, \tau_{12}\tau_{13}\, (\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})$$

$$+ (A_{23} + B_{23}P_{23})(A_{13} + B_{13}P_{13})\tau_{13}\tau_{23}\, \theta(123)\, \tau_{23}\tau_{13}\, (\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{23}} + P_{23}\overline{B_{23}})$$

$$+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23})\, \tau_{23}\tau_{13}\tau_{12}\, \theta(123)\, \tau_{12}\tau_{13}\tau_{23}$$
$$\times (\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}}) \Big\}$$

$$\times \big[(A_{12} + P_{12}B_{12})(A_{13} + B_{13}P_{13})\, \tau_{13}\tau_{12}\big] \frac{[123](-y)}{A_{12}A_{13}A_{23}}.$$

In general, we extend the definition of $Z(213)$ to an arbitrary coordinate region and define $Z(\sigma^y)$ for all $\sigma^y \in S_3$, so that

$$\begin{aligned}
Z(123) &= \mathbb{1}, \\
Z(213) &= (A_{12} + P_{12}B_{12})\, \tau_{12}, \\
Z(132) &= (A_{23} + P_{23}B_{23})\, \tau_{23}, \\
Z(231) &= (A_{12} + P_{12}B_{12})(A_{13} + P_{13}B_{13})\, \tau_{13}\tau_{12}, \\
Z(312) &= (A_{23} + P_{23}B_{23})(A_{13} + P_{13}B_{13})\, \tau_{13}\tau_{23}, \\
Z(321) &= (A_{12} + P_{12}B_{12})(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23})\, \tau_{23}\tau_{13}\tau_{12}.
\end{aligned} \tag{3.18}$$

Similarly, for all momentum regions $\theta(\sigma')$, $\sigma' \in S_3$, we define the operators:

$$\begin{aligned}
W(123) &= \mathbb{1}, \\
W(213) &= \tau_{12}\, (\overline{A_{12}} + P_{12}\overline{B_{12}}), \\
W(132) &= \tau_{23}\, (\overline{A_{23}} + P_{23}\overline{B_{23}}), \\
W(231) &= \tau_{12}\tau_{13}\, (\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}}), \\
W(312) &= \tau_{23}\tau_{13}\, (\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{23}} + P_{23}\overline{B_{23}}), \\
W(321) &= \tau_{12}\tau_{13}\tau_{23}\, (\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}}).
\end{aligned} \tag{3.19}$$

With this notation we can describe the above transformation of the y-component under $\tau_{13}\tau_{12}$ in shortened form as:

$$\tau_{13}\tau_{12} \sum_{\sigma' \in S_3} \tau_{\sigma'} \frac{\theta(123)}{A_{12}A_{13}A_{23}} W(\sigma') Z^{-1}(231) Z(231)\, [123](-y) \longmapsto$$

$$\longmapsto \sum_{\sigma' \in S_3} \tau_{\sigma'}\, \theta(123)\, W(\sigma') Z(231)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}},$$

where the order of summation is modified by applying the operator $\tau_{13}\tau_{12}$, which is allowed because the sum is invariant under this reordering.

For an arbitrary permutation of the y-coordinate region, applying the general form of such transformation,

$$\tau_{\sigma^y} \sum_{\sigma' \in S_3} \tau_{\sigma'} \frac{\theta(\sigma^y)}{A_{12}A_{13}A_{23}} W(\sigma') Z^{-1}(\sigma^y) Z(\sigma^y)\, [123](-y) \longmapsto$$

$$\longmapsto \sum_{\sigma' \in S_3} \tau_{\sigma'}\, \theta(123)\, W(\sigma') Z(\sigma^y)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}},$$

we obtain the expression for the modified ψ_{in}-function in case of $\chi^x(123)\,\chi^y(\sigma^y)$:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\, \chi^x(123)\, \chi^y(\sigma^y) \tag{3.20}$$

$$= \chi^x(123)\, \chi^y(\sigma^y) \frac{[123](x)}{A_{12}A_{13}A_{23}} \sum_{\sigma' \in S_3} W^+(\sigma')\, \theta(123)\, W(\sigma') Z(\sigma^y)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}},$$

where $W^+(\sigma')$ is Hermitian conjugate of $W(\sigma')$.
In fact, $\widetilde{\psi_{in}}(x, y \mid k)$ for all other regions with an arbitrary $\chi^x(\sigma^x)$ can be obtained from (3.20) by acting of τ_{σ^x} from the left. Then due to the δ-relation for momentum regions the corresponding function $\widetilde{\psi_{in}}$ can be expressed again as a product of its x- and y-components:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\, \chi^x(\sigma^x)\, \chi^y(\sigma^y) \tag{3.21}$$

$$= \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, \tau_{\sigma^x} \frac{[123](x)}{A_{12}A_{13}A_{23}} \sum_{\sigma' \in S_3} W^+(\sigma')\, \theta(123) \sum_{\sigma' \in S_3} W(\sigma')\, Z(\sigma^y)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}}.$$

Inserting the identity operator $\mathbb{I} = Z^+(\sigma^x)[Z^+(\sigma^x)]^{-1}$, where $Z^+(\sigma^x)$ is Hermitian conjugate of $Z(\sigma^x)$, in the x-component of (3.21) we apply a similar reordering in summation over the terms of x-component as was performed for the y-component:

$$\chi^x(\sigma^x)\,\tau_{\sigma^x}\frac{[123](x)}{A_{12}A_{13}A_{23}}\,Z^+(\sigma^x)\sum_{\sigma'\in S_3}[Z^+(\sigma^x)]^{-1}W^+(\sigma')\,\theta(123) \tag{3.22}$$

$$= \chi^x(\sigma^x)\,\tau_{\sigma^x}\frac{[123](x)}{A_{12}A_{13}A_{23}}\,Z^+(\sigma^x)\sum_{\sigma'\in S_3}W^+(\sigma')\,\theta(123).$$

Now the original order of summation over σ' is restored, i.e. for both x- and y-components. Substituting the results for both components in (3.21) and using the δ-relation for momentum regions we obtain:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi^x(\sigma^x)\,\chi^y(\sigma^y) = \chi^x(\sigma^x)\,\chi^y(\sigma^y)\,\tau_{\sigma^x} \tag{3.23}$$

$$\times\,\frac{[123](x)}{A_{12}A_{13}A_{23}}\,Z^+(\sigma^x)\left\{\sum_{\sigma'\in S_3}W^+(\sigma')\,\theta(123)\,W(\sigma')\right\}Z(\sigma^y)\,\frac{[123](-y)}{A_{12}A_{13}A_{23}},$$

and in particular, for the case of $\sigma^x = \sigma^y = \sigma$ applying the similar transformations but in inverse order, (3.23) can be reduced to:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi(\sigma) \tag{3.24}$$

$$= \chi(\sigma)\,\tau_\sigma\,\frac{[123](x)}{A_{12}A_{13}A_{23}}\left\{\sum_{\sigma'\in S_3}W^+(\sigma')\,\theta(123)\,W(\sigma')\right\}\frac{[123](-y)}{A_{12}A_{13}A_{23}}.$$

3.5 Rearrangement of terms in $\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)$ according to the regions of momentum space

From the comparison of (3.8), (3.23), (3.24) we can conclude that the central part of these formulas, $\sum_{\sigma'\in S_3}W^+(\sigma')\,\theta(123)\,W(\sigma')$, depends only on transpositions of momentum regions in any particular combination of coordinate regions $\chi^x(\sigma^x)\,\chi^y(\sigma^y)$. However, each of the six terms in this sum (except the very first one) contains elements with different combinations of momentum transposition operators so that each of the operators $W, W^+(\sigma')$ (for $(\sigma') \neq (123)$) splits the wave into waves of a different momentum regions. This can be clearly seen from the explicit form of the sum:

$$\theta(123) + (A_{12} + B_{12}P_{12})\,\tau_{12}\,\theta(123)\,\tau_{12}\,(\overline{A_{12}} + P_{12}\overline{B_{12}}) \tag{3.25}$$

$$+ (A_{23} + B_{23}P_{23})\,\tau_{23}\,\theta(123)\,\tau_{23}\,(\overline{A_{23}} + P_{23}\overline{B_{23}})$$

$$+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{12}\,\theta(123)\,\tau_{12}\tau_{13}\,(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})$$

$$+ (A_{23} + B_{23}P_{23})(A_{13} + B_{13}P_{13})\,\tau_{13}\tau_{23}\,\theta(123)\,\tau_{23}\tau_{13}\,(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{23}} + P_{23}\overline{B_{23}})$$

$$+ (A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23})\,\tau_{23}\tau_{13}\tau_{12}\,\theta(123)$$

$$\times\ \tau_{12}\tau_{13}\tau_{23}\,(\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}}).$$

For further integration, we need to express this in a form of the sum over all momentum regions where each term contains only elements of one particular momentum region. In order to achieve this we use the identities

$$B_{ij}P_{ij} = -P_{ij}B_{ij}, \quad \big(A_{ij} + P_{ij}B_{ij}\big)\big(\overline{A_{ij}} + P_{ij}\overline{B_{ij}}\big) = \mathbb{1} \tag{3.26}$$

to get

$$\begin{aligned}
&\big\{\theta(123)(A_{12} + P_{12}B_{12})(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23}) \\
&+ (A_{12} - P_{12}B_{12})\theta(213)(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23}) \\
&+ (A_{23} - P_{23}B_{23})\theta(132)(A_{13} + P_{13}B_{13})(A_{12} + P_{12}B_{12}) \\
&+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})\theta(231)(A_{23} + P_{23}B_{23}) \\
&+ (A_{23} - P_{23}B_{23})(A_{13} - P_{13}B_{13})\theta(312)(A_{12} + P_{12}B_{12}) \\
&+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})(A_{23} - P_{23}B_{23})\theta(321)\big\} \\
&\times (\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}}).
\end{aligned} \tag{3.27}$$

Here the last common factor $(\overline{A_{23}}+P_{23}\overline{B_{23}})(\overline{A_{13}}+P_{13}\overline{B_{13}})(\overline{A_{12}}+P_{12}\overline{B_{12}})$ does not change the momentum regions because all the characteristic functions $\theta(\sigma')$ of momentum regions in (3.27) are to the left to this factor. Therefore, we consider and rearrange (3.27) without that factor. First, with the use of the identity $P_{\sigma'_1\sigma'_2}B_{\sigma'_1\sigma'_2}\theta(\sigma'_2\sigma'_1\sigma'_3) = \theta(\sigma'_1\sigma'_2\sigma'_3)P_{\sigma'_1\sigma'_2}B_{\sigma'_1\sigma'_2}$ we are grouping all the terms pairwise:

$$\begin{aligned}
&\theta(123)(A_{12} + P_{12}B_{12})(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23}) \\
&+ (A_{12} - P_{12}B_{12})\theta(213)(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23}) \\
&+ (A_{23} - P_{23}B_{23})\theta(132)(A_{13} + P_{13}B_{13})(A_{12} + P_{12}B_{12}) \\
&+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})\theta(231)(A_{23} + P_{23}B_{23}) \\
&+ (A_{23} - P_{23}B_{23})(A_{13} - P_{13}B_{13})\theta(312)(A_{12} + P_{12}B_{12}) \\
&+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})(A_{23} - P_{23}B_{23})\theta(321)
\end{aligned} \tag{3.28}$$

$$= [\theta(123) + \theta(213)]A_{12}(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23})$$
$$+ (A_{23} - P_{23}B_{23})[\theta(132) + \theta(312)]A_{13}(A_{12} + P_{12}B_{12})$$
$$+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})[\theta(231) + \theta(321)]A_{23}$$

$$= A_{12}\big\{[\theta(123) + \theta(213)]\big\}(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23})$$
$$+ (A_{23} - P_{23}B_{23})A_{13}\big\{P_{23}[\theta(123) + \theta(213)]P_{23}\big\}(A_{12} + P_{12}B_{12})$$
$$+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})A_{23}\big\{P_{13}P_{12}[\theta(123) + \theta(213)]P_{12}P_{13}\big\}.$$

It follows that there is the same set of terms for $\theta(123)$ as for $\theta(213)$ in (3.28). If we choose another pairing $\theta(\sigma_1'\sigma_2'\sigma_3'), \theta(\sigma_2'\sigma_1'\sigma_3')$, such that in each pair two first indices are interchanged, namely $(\theta(132), \theta(312))$ and $(\theta(231), \theta(321))$, then since we can express $\theta(132) + \theta(312) = P_{23}[\theta(123) + \theta(213)]P_{23}$, $\theta(231) + \theta(321) = P_{13}P_{12}[\theta(123) + \theta(213)]P_{12}P_{13}$, it follows that the same statement is also true for such pairs.
Next, with use of identity $P_{\sigma_2'\sigma_3'}B_{\sigma_2'\sigma_3'}\theta(\sigma_1'\sigma_3'\sigma_2') = \theta(\sigma_1'\sigma_2'\sigma_3')P_{\sigma_2'\sigma_3'}B_{\sigma_2'\sigma_3'}$ and also

$$(A_{12} \pm P_{12}B_{12})(A_{13} \pm P_{13}B_{13})(A_{23} \pm P_{23}B_{23}) \quad (3.29)$$
$$= (A_{23} \pm P_{23}B_{23})(A_{13} \pm P_{13}B_{13})(A_{12} \pm P_{12}B_{12}),$$

verified directly in the proof of *Lemma 1*, we group all the terms of the first expression of (3.28) pairwise in a different way:

$$\theta(123)(A_{23} + P_{23}B_{23})(A_{13} + P_{13}B_{13})(A_{12} + P_{12}B_{12}) \quad (3.30)$$
$$+ (A_{12} - P_{12}B_{12})\theta(213)(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23})$$
$$+ (A_{23} - P_{23}B_{23})\theta(132)(A_{13} + P_{13}B_{13})(A_{12} + P_{12}B_{12})$$
$$+ (A_{12} - P_{12}B_{12})(A_{13} - P_{13}B_{13})\theta(231)(A_{23} + P_{23}B_{23})$$
$$+ (A_{23} - P_{23}B_{23})(A_{13} - P_{13}B_{13})\theta(312)(A_{12} + P_{12}B_{12})$$
$$+ (A_{23} - P_{23}B_{23})(A_{13} - P_{13}B_{13})(A_{12} - P_{12}B_{12})\theta(321)$$

$$= [\theta(123) + \theta(132)]A_{23}(A_{13} + P_{13}B_{13})(A_{12} + P_{12}B_{12})$$
$$+ (A_{12} - P_{12}B_{12})[\theta(213) + \theta(231)]A_{13}(A_{23} + P_{23}B_{23})$$
$$+ (A_{23} - P_{23}B_{23})(A_{13} - P_{13}B_{13})[\theta(312) + \theta(321)]A_{23}$$

$$= A_{23}\big\{[\theta(123) + \theta(132)]\big\}(A_{13} + P_{13}B_{13})(A_{12} + P_{12}B_{12})$$
$$+ (A_{12} - P_{12}B_{12})A_{13}\big\{P_{12}[\theta(123) + \theta(132)]P_{12}\big\}(A_{23} + P_{23}B_{23})$$
$$+ (A_{23} - P_{23}B_{23})(A_{13} - P_{13}B_{13})A_{12}\big\{P_{13}P_{23}[\theta(123) + \theta(132)]P_{23}P_{13}\big\}.$$

Now, since we can express $\theta(213) + \theta(231) = P_{12}[\theta(123) + \theta(132)]P_{12}$, $\theta(312) + \theta(321) = P_{13}P_{23}[\theta(123) + \theta(132)]P_{23}P_{13}$, the set of terms in (3.30) is the same for the pairs $(\theta(123), \theta(132))$, $(\theta(213), \theta(231))$, $(\theta(312), \theta(321))$.

Comparing with the similar result for the pairs $(\theta(123), \theta(213))$, $(\theta(132), \theta(312))$, $(\theta(231), \theta(321))$ and using the fact that expression in (3.28) is equal to the expression in (3.30), we conclude that the set of terms for all six regions $\theta(\sigma')$, $\sigma' \in S_3$, is the same. It suffices therefore to evaluate these elements for just one of them, and we choose $\theta(321)$. According to the first expression of (3.28) or (3.30) and due to δ-relations for characteristic functions, there is only one term for $\theta(321)$, namely $A_{12}A_{13}A_{23}$ and the corresponding expression in (3.27) inside the brackets (i.e. for x-component) is: $A_{12}A_{13}A_{23}\,\theta(321)$.

In fact, we can verify explicitly that any region of momentum space has the same set of waves as region $\theta(321)$. For example, to show that this is true for the regions $\theta(123)$ and $\theta(213)$, we extract all the terms belonging to these regions, under the combination of transpositions from the left, from the last expression of (3.28), then using the directly verified equalities

$$P_{12}P_{13} = P_{13}P_{23} = P_{23}P_{12} \quad \text{and} \quad B_{12}B_{23} = B_{12}B_{13} + B_{13}B_{23},$$

obtain:

$$[\theta(123) + \theta(213)]\big\{A_{12}(A_{13} + P_{13}B_{13})(A_{23} + P_{23}B_{23})$$

$$- P_{23}B_{23}A_{13}(A_{12} + P_{12}B_{12}) + P_{12}B_{12}P_{13}B_{13}A_{23} - A_{12}P_{13}B_{13}A_{23}\big\}$$

$$= [\theta(123) + \theta(213)]\big\{A_{12}A_{13}A_{23} + A_{12}P_{13}B_{13}A_{23} + A_{12}A_{13}P_{23}B_{23} + A_{12}P_{13}B_{13}P_{23}B_{23}$$

$$- P_{23}B_{23}A_{13}A_{12} - P_{23}B_{23}A_{13}P_{12}B_{12} + P_{12}B_{12}P_{13}B_{13}A_{23} - A_{12}P_{13}B_{13}A_{23}\big\}$$

$$= [\theta(123) + \theta(213)]\big\{A_{12}A_{13}A_{23} + \underline{\underline{A_{12}B_{31}A_{21}P_{13}}} + \underline{A_{12}A_{13}B_{32}P_{23}} + A_{12}B_{31}P_{13}P_{23}B_{23}$$

$$- \underline{B_{32}A_{12}A_{13}P_{23}} - B_{32}A_{12}B_{31}P_{23}P_{12} + B_{21}B_{32}A_{12}P_{12}P_{13} - \underline{\underline{A_{12}B_{31}A_{21}P_{13}}}\big\}$$

$$= [\theta(123) + \theta(213)]\big\{A_{12}A_{13}A_{23} + A_{12}(B_{31}B_{12} + B_{21}B_{32} - B_{32}B_{21})P_{23}P_{12}\big\}$$

$$= [\theta(123) + \theta(213)]\big\{A_{12}A_{13}A_{23} + A_{12}(-B_{13}B_{12} + B_{12}B_{23} - B_{23}B_{12})P_{23}P_{12}\big\}$$

$$= [\theta(123) + \theta(213)]\big\{A_{12}A_{13}A_{23}\big\},$$

which is the same as for the region $\theta(321)$. Since, as was shown, the set of terms for all regions $\theta(\sigma')$ is the same, (3.28), and equivalently (3.30), reduces to $A_{12}A_{13}A_{23}$, and (3.25), and equivalently (3.27), reduces to

$$A_{12}A_{13}A_{23}(\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}}), \tag{3.31}$$

and (3.8) for coordinate region $\chi^x(123)\,\chi^y(123) = \chi(123)$ therefore simplifies to

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi(123) \tag{3.32}$$
$$= \chi(123)\,[123](x)(\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})\frac{[123](-y)}{\overline{A_{12}A_{13}A_{23}}}.$$

For explicit integration of (3.32) we evaluate the coefficients:

$$(\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})\frac{1}{\overline{A_{12}A_{13}A_{23}}} \tag{3.33}$$

$$= \frac{1}{A_{12}A_{13}A_{23}}\,(A_{12} + B_{12}P_{12})(A_{13} + B_{13}P_{13})(A_{23} + B_{23}P_{23})$$

$$= 1 + \frac{B_{12}}{A_{12}}\,P_{12} + \frac{B_{23}}{A_{23}}\,P_{23} + \frac{B_{12}}{A_{12}}\,\frac{B_{13}}{A_{13}}\,P_{13}P_{12} + \frac{1}{A_{12}A_{13}A_{23}}$$
$$\times\Big(A_{12}B_{13}P_{13}A_{23} + B_{12}P_{12}B_{13}P_{13}B_{23}P_{23} + A_{12}B_{13}P_{13}B_{23}P_{23} + B_{12}P_{12}B_{13}P_{13}A_{23}\Big),$$

where the expression in the brackets is equal to:

$$|A_{12}|^2\,B_{13}P_{13} + B_{12}^2B_{23}P_{13} + A_{12}B_{13}B_{21}P_{13}P_{23} + A_{12}B_{12}B_{23}P_{12}P_{13}$$

$$= \big[(1 - B_{12}^2)B_{13} + B_{12}(B_{12}B_{13} + B_{23}B_{13})\big]P_{13} + A_{12}(B_{12}B_{23} - B_{12}B_{13})P_{13}P_{23}$$

$$= B_{13}(1 + B_{12}B_{13})P_{13} + A_{12}B_{13}B_{23}P_{13}P_{23}.$$

In conclusion,

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi(123) = \chi(123)\,(f_{\mathbf{1}} + f_2 + f_3 + f_4 + f_5 + f_6), \tag{3.34}$$

where

$$f_1 = [123](x-y), \quad f_2 = \frac{B_{12}}{A_{12}}[123](x)[213](-y), \quad f_3 = \frac{B_{23}}{A_{23}}[123](x)[132](-y),$$

$$f_4 = \frac{B_{23}}{A_{23}}\frac{B_{13}}{A_{13}}[123](x)[312](-y), \quad f_5 = \frac{B_{12}}{A_{12}}\frac{B_{13}}{A_{13}}[123](x)[231](-y),$$

$$f_6 = \frac{B_{13}(1+B_{12}B_{23})}{A_{12}A_{13}A_{23}}[123](x)[321](-y),$$

and the corresponding modified combined matrix for the case $\chi(123)$ is

$$\begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ f_2 & f_1 & f_5 & f_6 & f_3 & f_4 \\ f_3 & f_4 & f_1 & f_2 & f_6 & f_5 \\ f_5 & f_6 & f_2 & f_1 & f_4 & f_3 \\ f_4 & f_3 & f_6 & f_5 & f_1 & f_2 \\ f_6 & f_5 & f_4 & f_3 & f_2 & f_1 \end{pmatrix} \quad \begin{matrix} \theta(\beta_1\beta_2\beta_3) \\ \downarrow \\ (123) \\ (213) \\ (132) \\ (231) \\ (312) \\ (321) \end{matrix} .$$

For coordinate regions $\chi^x(\sigma^x)\,\chi^y(\sigma^y) = \chi(\sigma)$, (3.24) similarly reduces to

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi(\sigma) \tag{3.35}$$

$$= \tau_\sigma\,\chi(123)\,[123](x)\,(\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})\,\frac{[123](-y)}{A_{12}A_{13}A_{23}},$$

and all the matrix entries obtained by replacing indices $1,2,3 \mapsto \sigma_1, \sigma_2, \sigma_3$, respectively, in the expressions for the case $\chi(123)$.
For coordinate regions $\chi^x(123)\,\chi^y(\sigma^y)$, with the use of definition for $W(321)$, (3.20) similarly reduces to

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\,\chi^x(123)\,\chi^y(\sigma^y) \tag{3.36}$$

$$= \chi^x(123)\,\chi^y(\sigma^y)\,[123](x)\,\tau_{23}\tau_{13}\tau_{12}\,W(321)Z(\sigma^y)\,\frac{[123](-y)}{A_{12}A_{13}A_{23}}.$$

Using (3.32) and (3.36) we can conclude that for any region with $\chi^y(123)$ the transformation of the y-component of the formula has to be equal to the identity transformation so that:

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\, \chi^x(\sigma^x)\, \chi^y(123)$$

$$= \chi^x(\sigma^x)\, \chi^y(123)\, \tau_{\sigma^x}\, [123](x)\, \tau_{\sigma^x}^{-1} \tau_{23}\tau_{13}\tau_{12}\, W(321)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}}$$

$$= \chi^x(\sigma^x)\, \chi^y(123)\, [123](x)\, \tau_{23}\tau_{13}\tau_{12}\, W(321)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}}. \tag{3.37}$$

It follows that for the general case of $\chi^x(\sigma^x)\, \chi^y(\sigma^y)$, (3.21) reduces to

$$\widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\, \chi^x(\sigma^x)\, \chi^y(\sigma^y)$$
$$= \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, [123](x)\, \tau_{23}\tau_{13}\tau_{12}\, W(321) Z(\sigma^y)\, \frac{[123](-y)}{A_{12}A_{13}A_{23}}. \tag{3.38}$$

3.6 Evaluation of integrals

In the case $\chi^x(\sigma^x)\, \chi^y(\sigma^y) = \chi(\sigma)$ we need to evaluate the integral of (3.35):

$$\frac{1}{(2\pi)^3} \int d^3k\, \widetilde{\psi_{in}}(x_1x_2x_3, y_1y_2y_3 \mid k_1k_2k_3)\, \chi(\sigma) = \tau_\sigma\, \chi(123)\, \frac{1}{(2\pi)^3}$$
$$\times \int d^3k\, [123](x)\, (\overline{A_{23}} + P_{23}\overline{B_{23}})(\overline{A_{13}} + P_{13}\overline{B_{13}})(\overline{A_{12}} + P_{12}\overline{B_{12}})\, \frac{[123](-y)}{A_{12}A_{13}A_{23}}. \tag{3.39}$$

By (3.34), this is equal to

$$\tau_\sigma\, \chi(123) \frac{1}{(2\pi)^3} \int d^3k\, [123](x) \Big[1 + \frac{B_{12}}{A_{12}} P_{12} + \frac{B_{23}}{A_{23}} P_{23} \tag{3.40}$$
$$+ \frac{B_{23}}{A_{23}}\, \frac{B_{13}}{A_{13}} P_{13}P_{23} + \frac{B_{12}}{A_{12}}\, \frac{B_{13}}{A_{13}} P_{13}P_{12} + \frac{B_{13}(1 + B_{12}B_{23})}{A_{12}A_{13}A_{23}}\, P_{13} \Big] [123](-y).$$

The first integral in (3.40) corresponds to the diagonal terms of the matrix for (3.35) and equals

$$\tau_\sigma\, \chi(123) \frac{1}{(2\pi)^3} \int d^3k\, [123](x - y) = \chi(\sigma)\, \delta^3(x - y). \tag{3.41}$$

The second integral in (3.40) reduces to the second integral in (2.10) for the 2-particle case and hence is 0 according to (2.28). Similarly, replacing 23 in the third integral by 12 we conclude that it is also equal to 0. The fifth integral in (3.40) equals (up to a constant multiple)

$$\int d^3k \, \frac{B_{12}}{A_{12}} \, \frac{B_{13}}{A_{13}} \, [123](x)[231](-y)$$

$$= -c^2 \int d^3k \, \frac{\exp\{i[k_1(x_1 - y_3) + k_2(x_2 - y_1) + k_3(x_3 - y_2)]\}}{(k_1 - k_2 + ic)(k_1 - k_3 + ic)}.$$

After changing of variables

$$q_1 = k_1 - k_3, \;\; q_2 = 2k_2, \;\; q_3 = k_1 + k_3$$
$$\Rightarrow k_1 = \tfrac{1}{2}(q_1 + q_3), \;\; k_2 = \tfrac{1}{2}q_2, \;\; k_3 = \tfrac{1}{2}(-q_1 + q_3),$$

and rearrangement of the exponent,

$$k_1(x_1 - y_3) + k_2(x_2 - y_1) + k_3(x_3 - y_2)$$

$$= (1/2)(k_1 - k_3)[(x_1 - x_3) + (y_2 - y_3)]$$
$$+ (1/2)(k_1 + k_3)[(x_1 + x_3) - (y_2 + y_3)] + k_2(x_2 - y_1)$$

$$= (1/2)\{q_1[(x_1 - x_3) + (y_2 - y_3)] + q_3[(x_1 + x_3) - (y_2 + y_3)] + q_2(x_2 - y_1)\},$$

the integral becomes (up to a constant multiple)

$$\int dq_3 dq_2 \, \exp\left\{\frac{i}{2}[q_3(x_1 + x_3 - y_2 - y_3) + q_2(x_2 - y_1)]\right\} \times \int dq_1 \frac{\exp\{\frac{i}{2}q_1[(x_1 - x_3) + (y_2 - y_3)]\}}{(q_1 + q_3 - q_2 + 2ic)(q_1 + ic)}. \tag{3.42}$$

Set $q_1 = q$ and $[(x_1 - x_3) + (y_2 - y_3)] = z$. Then $z > 0$ because $x_1 > x_3$, $y_2 > y_3$ for the region $\chi(123)$, respectively, $x_{\sigma_1} > x_{\sigma_3}$, $y_{\sigma_2} > y_{\sigma_3}$ for the region $\chi(\sigma)$. The internal integral becomes

$$\int \frac{dq \, e^{iqz/2}}{(q + q_3 - q_2 + 2ic)(q + ic)}. \tag{3.43}$$

To evaluate it we note that it is clearly bounded as the denominator is quadratic in q. We use analytical continuation $q \mapsto q + i\kappa$, where $\kappa > 0$. Since there are only two poles, both with negative imaginary parts, namely: $q = -ic$ and $q = q_2 - q_3 - 2ic$, and $\kappa > 0$, a shift of integration contour from the axis $\kappa = 0$ to the upper half-plane does not change the result of integration and

$$\int \frac{dq\, e^{iqz/2}}{(q+q_3-q_2+2ic)(q+ic)} = \int \frac{dq\, e^{i((q+i\kappa)z)/2}}{(q+q_3-q_2+2ic)(q+ic)}$$

$$= e^{-\kappa z/2} \int \frac{dq\, e^{iqz/2}}{(q+q_3-q_2+2ic)(q+ic)},$$

where $0 < e^{-\kappa z/2} < 1$. It follows that

$$\int \frac{dq\, e^{iqz/2}}{(q+q_3-q_2+2ic)(q+ic)} = 0.$$

A similar evaluation, but with the poles $q = -ic$ and $q = q_3 - q_2 - 2ic$ proves that the fourth integral in (3.40) is equal to 0.

The last integral in (3.40) becomes, after applying the same change of variables and rearrangement in the argument of the exponent:

$$k_1(x_1-y_3)+k_2(x_2-y_2)+k_3(x_3-y_1)$$
$$= (1/2)\{q_1[(x_1-x_3)+(y_1-y_3)]+q_3[(x_1+x_3)-(y_1+y_3)]+q_2(x_2-y_2)\},$$

$$\int dq_3 dq_2 \exp\left\{\frac{i}{2}[q_3(x_1+x_3-y_1-y_3)+q_2(x_2-y_1)]\right\} \tag{3.44}$$
$$\times (-ic)\int dq_1 \frac{[(q_1+q_2-q_3)(q_1-q_2+q_3)-4c^2]\exp\{\frac{i}{2}q_1[(x_1-x_3)+(y_1-y_3)]\}}{(q_1+q_2-q_3+2ic)(q_1-q_2+q_3+2ic)(q_1+ic)}.$$

Set $q_1 = q$, $q_2 - q_3 = s$, $[(x_1-x_3)+(y_1-y_3)] = z > 0$, then the internal integral is equal to (up to a constant multiple)

$$\int dq \frac{q^2-s^2-4c^2}{(q+s+2ic)(q-s+2ic)(q+ic)} e^{iqz/2}. \tag{3.45}$$

We need only consider the highest order term in the numerator, since the other terms are cubically convergent and can be shown to be 0 in the same way as the previous integrals as all poles are in the lower half plane. The q^2 term can be written as

$$\int dq \frac{q^2(q^2-s^2-4c^2-4icq)(q-ic)}{[(q+s)^2+4c^2][(q-s)^2+4c^2](q^2+c^2)} e^{iqz/2}. \tag{3.46}$$

The numerator equals

$$q^2[(q^3-s^2q-8c^2q)-i(5cq^2-cs^2-4c^3)].$$

The real part of (3.46) involves
$(q^3 - s^2 q - 8c^2 q)\cos(qz/2) + (5cq^2 - cs^2 - 4c^3)\sin(qz/2)$ and is therefore an integral of an odd function and equals 0. The imaginary part is

$$\Im \int dq \, \frac{q^2(q^2 - s^2 - 4c^2 - 4icq)(q - ic)}{[(q+s)^2 + 4c^2][(q-s)^2 + 4c^2](q^2 + c^2)} e^{iqz/2}$$

$$= \int dq \, \frac{q^2[(q^3 - s^2 q - 8c^2 q)\sin(qz/2) - (5cq^2 - cs^2 - 4c^3)\cos(qz/2)}{[(q+s)^2 + 4c^2][(q-s)^2 + 4c^2](q^2 + c^2)}.$$

The denominator is of order q^6, so all terms in the numerator of order ≤ 4 obviously converge. That leaves the integral

$$\int dq \, \frac{q^5 \sin(qz/2)}{[(q+s)^2 + 4c^2][(q-s)^2 + 4c^2](q^2 + c^2)}. \tag{3.47}$$

Note that $z > 0$ and integrating by parts, we set
$u = q^5/[(q+s)^2 + 4c^2][(q-s)^2 + 4c^2](q^2 + c^2)$, $dv = \sin(qz/2)$, so that $du = dq \, O(q^{-2})$, $v = -(2/z)\cos(qz)$. It follows that

$$-\frac{2}{z} \frac{q^5 \cos(qz)}{[(q+s)^2 + 4c^2][(q-s)^2 + 4c^2](q^2 + c^2)} \Bigg|_{-\infty}^{+\infty} = 0$$

and

$$\frac{2}{z} \int_{-\infty}^{+\infty} dq \, O(q^{-2}) \cos(qz)$$

is bounded, hence integral (3.47) is bounded too. Since all three poles have a negative imaginary part, we can again use analytical continuation $q \mapsto q + i\kappa$, where $\kappa > 0$, and similar to the previous cases prove that (3.47) is equal 0 so that the last integral in (3.40) is equal 0 and therefore (3.2) holds for the case $\chi(\sigma)$.

Next we consider the region $\chi^x(123)\,\chi^y(\sigma^y)$. According to (3.38) and the definitions $W(321)$, $Z(\sigma^y)$, for any y-component region the numerator of the diagonal term contains one, two or three coefficients $\overline{A_{ij}}$ identical to the terms in the denominator. Then these coefficients can be cancelled out leaving either 1 (for the case $\chi^y(\sigma^y) = \chi^y(123)$) or at least one factor of the form $1/\overline{A_{t_1^y t_2^y}}$, where t_1^y, t_2^y defined as: $t_1^y := \tau_{\sigma^y} t_1$, $t_2^y := \tau_{\sigma^y} t_2$, and $t_1^y > t_2^y$ for this particular coordinate region $\chi^y(\sigma^y)$ because initial inequality $t_1 < t_2$ for the region $\chi^y(123)$ is transposed by the operator τ_{σ^y}. In fact, the pairs of indices and the number of such pairs for these factors are in direct correspondence with the pairs of indices $(\sigma^y_{r_1} \sigma^y_{r_2})$ in $\sigma^y = (\sigma^y_1 \sigma^y_2 \sigma^y_3)$ such that $\sigma^y_{r_1} > \sigma^y_{r_2}$ for $r_1 < r_2$. Since for all pairs $(t_1^y t_2^y)$, $1/\overline{A_{t_1^y t_2^y}} = 1 + \left(\overline{B_{t_1^y t_2^y}}/\overline{A_{t_1^y t_2^y}}\right) =$

$1+\left(B_{t_2^y t_1^y}/A_{t_2^y t_1^y}\right)$, it follows that the integral for such case can be expressed in the form:

$$\frac{1}{(2\pi)^3}\int d^3k\left(1+E(\chi^y(\sigma^y))\right)\exp\left\{i[k_{t_1^y}(x_{t_1}-y_{t_1^y})+k_{t_2^y}(x_{t_2}-y_{t_2^y})+k_{r^y}(x_r-y_{r^y})]\right\}, \tag{3.48}$$

where $t_1^y \neq r^y \neq t_2^y$, $x_{t_1} > x_{t_2}$ and $E(\chi^y(\sigma^y))$ is a sum of terms such that each term contains at least one factor of the form $B_{t_2^y t_1^y}/A_{t_2^y t_1^y}$ and after applying operators of the form $P_{t_2^y t_1^y}$ to y-component does not contain such operators any more.
The first integral in (3.48) is equal to $\delta^3(x-y)$. Since each term of the second integral contains at least one factor of the form $B_{t_2^y t_1^y}/A_{t_2^y t_1^y}$ we define

$k_{t_2^y} - k_{t_1^y} =: 2q_1$, $k_{t_2^y} + k_{t_1^y} =: 2q_2$ for it, so that $B_{t_2^y t_1^y}/A_{t_2^y t_1^y} = -ic/(2q_1+ic)$.

Then with use of

$$k_{t_1^y}(x_{t_1}-y_{t_1^y})+k_{t_2^y}(x_{t_2}-y_{t_2^y})$$
$$=\left\{q_1[(x_{t_1}-x_{t_2})+(y_{t_2^y}-y_{t_1^y})]+q_2[(x_{t_1}+x_{t_2})-(y_{t_2^y}+y_{t_1^y})]\right\},$$

the internal integral for any term of $E(\chi^y(\sigma^y))$ is evaluated similar to that for the corresponding term of (3.40) but with additional poles due to additional factors of the form
$1/(\tau_{\sigma^y}\overline{A_{t_1t_2}}) = 1/\overline{A_{t_1^y t_2^y}} = 1/A_{t_2^y t_1^y} = (k_{t_2^y}-k_{t_1^y})/(k_{t_2^y}-k_{t_1^y}+ic)$, where these poles have negative imaginary part and also $z=(x_{t_1}-x_{t_2})+(y_{t_2^y}-y_{t_1^y})>0$ in the region $\chi^x(123)\,\chi^y(\sigma^y)$, because $x_{t_1}>x_{t_2}$ for $\chi^x(123)$ since $t_1<t_2$ and $y_{t_2^y}>y_{t_1^y}$ for $\chi^y(\sigma^y)$ since $t_2^y<t_1^y$, by the definition of coordinate region. It follows that

$$\frac{1}{(2\pi)^3}\int d^3k\,\widetilde{\psi_{in}}(x_1x_2x_3,y_1y_2y_3\mid k_1k_2k_3)\,\chi^x(123)\,\chi^y(\sigma^y)$$
$$=\chi^x(123)\,\chi^y(\sigma^y)\,\delta^3(x-y). \tag{3.49}$$

Since $\chi^y(\sigma^y)$ is chosen arbitrary, then applying transposition operator τ_{σ^x}, for an arbitrary σ^x, to $\chi^x(123)\,\chi^y(\sigma^y)$ in (3.49) we obtain the result for all possible combinations of $\chi^x(\sigma^x)$ and $\chi^y(\sigma^y)$ so that for any arbitrary coordinate region $\chi^x(\sigma^x)\,\chi^y(\sigma^y)$,

$$\frac{1}{(2\pi)^3}\int d^3k\,\widetilde{\psi_{in}}(x_1x_2x_3,y_1y_2y_3\mid k_1k_2k_3)\,\chi^x(\sigma^x)\,\chi^y(\sigma^y)$$
$$=\chi^x(\sigma^x)\,\chi^y(\sigma^y)\,\delta^3(x-y). \tag{3.50}$$

Summation over all coordinate regions $\chi^x(\sigma^x)\,\chi^y(\sigma^y)$ gives (3.2).

Chapter 4

COMPLETENESS OF δ-INTERACTING PARTICLES IN THE n-PARTICLE CASE

4.1 The wave function in terms of transposition operators

We aim to prove the completeness relation

$$\frac{1}{(2\pi)^n}\int \psi_{in}(x_1...x_n \mid k_1...k_n)\overline{\psi_{in}(y_1...y_n \mid k_1...k_n)}\, d^n k = \delta^n(x-y). \quad (4.1)$$

Since every arbitrary transposition of n elements can be expressed as a composition of elementary transpositions of pairs such that only neighbouring elements are allowed to interchange their positions in an elementary transposition, it is useful, similar to the 3-particle case, to define elementary transposition operators for each pair $\sigma'_j\sigma'_{j+1},\ j = 1, ..., n-1$, where $\sigma' \in S_n$, in the following way.
a) Operators of momentum transposition $P_{\sigma'_j\sigma'_{j+1}}$ - indexed by the numbers of transposed momenta,

$$P_{\sigma'_j\sigma'_{j+1}} f(x_{\sigma_1}...x_{\sigma_n}; k_{\sigma'_1}...k_{\sigma'_j}...k_{\sigma'_{j+1}}...k_{\sigma'_n}) = f(x_{\sigma_1}...x_{\sigma_n}; k_{\sigma'_1}...k_{\sigma'_{j+1}}...k_{\sigma'_j}...k_{\sigma'_n})$$

(since elementary transpositions do not depend on the order of the pair of indices, $P_{\sigma'_j\sigma'_{j+1}} = P_{\sigma'_{j+1}\sigma'_j}$).
b) Operators of coordinate transposition $R_{\sigma'_j\sigma'_{j+1}}$ - indexed, not by the positions of particles but by those of the corresponding momenta:

$$R_{\sigma'_j\sigma'_{j+1}} f(x_{\sigma_1}...x_{\sigma_j}...x_{\sigma_{j+1}}...x_{\sigma_n}; k_{\sigma'_1}...k_{\sigma'_n}) = f(x_{\sigma_1}...x_{\sigma_{j+1}}...x_{\sigma_j}...x_{\sigma_n}; k_{\sigma'_1}...k_{\sigma'_n})$$

(again, since elementary transpositions do not depend on the order of the pair of indices, $R_{\sigma'_j\sigma'_{j+1}} = R_{\sigma'_{j+1}\sigma'_j}$).
c) Operators of index transposition $\tau_{\sigma'_j\sigma'_{j+1}}$ - numbered by the numbers of transposed momenta:

$$\tau_{\sigma'_j\sigma'_{j+1}} f(x_{\sigma_1}...x_{\sigma_j}...x_{\sigma_{j+1}}...x_{\sigma_n}; k_{\sigma'_1}...k_{\sigma'_j}...k_{\sigma'_{j+1}}...k_{\sigma'_n})$$
$$= f(x_{\sigma_1}...x_{\sigma_{j+1}}...x_{\sigma_j}...x_{\sigma_n}; k_{\sigma'_1}...k_{\sigma'_{j+1}}...k_{\sigma'_j}...k_{\sigma'_n}).$$

Obviously, $\tau_{\sigma'_j\sigma'_{j+1}} = \tau_{\sigma'_{j+1}\sigma'_j}$.

Moreover, these operators are related as follows:

$$P_{\sigma'_j\sigma'_{j+1}} R_{\sigma'_j\sigma'_{j+1}} = \tau_{\sigma'_j\sigma'_{j+1}}; \;\; \tau_{\sigma'_j\sigma'_{j+1}} P_{\sigma'_j\sigma'_{j+1}} = R_{\sigma'_j\sigma'_{j+1}}; \;\; \tau_{\sigma'_j\sigma'_{j+1}} R_{\sigma'_j\sigma'_{j+1}} = P_{\sigma'_j\sigma'_{j+1}} \text{ etc.}$$

Now, for any arbitrary region $(x_{\sigma_1} > ... > x_{\sigma_n})$, $\sigma \in S_n$ with characteristic function $\chi(x_{\sigma_1} > ... > x_{\sigma_n}) =: \chi(\sigma_1...\sigma_n)$, we define the set of all index pairs which appear in inverted (transposed) order w.r.t. the region $\chi(1...n)$:

$$D(\sigma) = \{\alpha := (\sigma_{r_2}\sigma_{r_1}) : \quad r_1 < r_2, \quad \sigma_{r_1} > \sigma_{r_2}, \quad r_1, r_2 \in \{1, ..., n\}, \quad \sigma \in S_n\}. \tag{4.2}$$

The number of inverted pairs will be denoted $l(\sigma) := \#D(\sigma)$. In other words, the region $\chi(1...n)$ has a direct order of indices in any pair of indices, an arbitrary region $\chi(\sigma_1...\sigma_n)$ has a number $l \leq \binom{n}{2}$ of transposed pairs of indices. A generalisation of (3.3) to the n-particle case reads

$$\psi(x_1...x_n \mid k_1...k_n) = \sum_{\sigma \in S_n} \overrightarrow{\prod_{\alpha \in D(\sigma)}} (A_\alpha + B_\alpha P_\alpha) \left(\overleftarrow{\prod_{\alpha \in D(\sigma)}} \tau_\alpha \right) [1...n](x)\, \chi(1...n), \tag{4.3}$$

where the direction of the arrow above the products corresponds to the order of multiplication. Since in general there are more than one ways in which a permutation σ can be written as a product of transpositions τ_α such that $\sigma = \overleftarrow{\prod_{\alpha \in D(\sigma)}} \tau_\alpha$, we have to show, that every term in the sum in (4.3) is independent of the order of transpositions but depends only on $D(\sigma)$ which already determines the permutation σ uniquely. The only non-trivial case (inverting the order of all three pairs) for $n = 3$ is considered in the previous Chapter 3 in the *Lemma 1.* A generalisation of *Lemma 1* to arbitrary n is due to C. N. Yang but had not been formally proved yet.

Theorem (Yang)
Every permutation σ is uniquely defined by $D(\sigma)$ and can be written as a product of adjacent transpositions $\sigma = \overleftarrow{\prod_{\alpha \in D(\sigma)}} \alpha$. Moreover, (4.3) is independent of the order of the sequence $\alpha_1, ..., \alpha_l$ of transpositions such that $\sigma = \overleftarrow{\prod_{i=1}^{l(\sigma)}} \alpha_i$, where each α_i transposes elements in neighbouring positions.

Proof.

1. It is clear that any permutation σ is uniquely determined by $D(\sigma)$. In fact, $\sigma^{-1}(i) = i + \#\{j > i : (ij) \in D(\sigma)\} - \#\{j < i : (ji) \in D(\sigma)\}$. This is proven by induction on l as follows. Let σ' be permutation corresponding to the composition of l adjacent transpositions, and σ be permutation corresponding to the composition of $l+1$ adjacent transpositions such that the composition of the first l corresponds to σ'. Then $\sigma = (\sigma(r+1)\sigma(r))\sigma' = (\sigma'(r)\sigma'(r+1))\sigma'$ with $\sigma'(r) < \sigma'(r+1)$ for some $r \in \{1, ..., n-1\}$. We have to show that if the statement for $(\sigma')^{-1}(i)$ is true, then the same statement for $\sigma^{-1}(i)$ is true for all possible position of i relative $\sigma'(r)$,$\sigma'(r+1)$. The cases $i < \sigma'(r)$, $i > \sigma'(r+1)$ and $\sigma'(r) < i < \sigma'(r+1)$ are trivial because $D(\sigma) = D(\sigma') \cup (\sigma'(r)\sigma'(r+1))$ hence addition of the pair $(\sigma'(r)\sigma'(r+1))$ with $\sigma'(r) \neq i$ and $\sigma'(r+1) \neq i$ does not change the numbers of pairs in the sets in the right hand side of the statement for $\sigma^{-1}(i)$ relative $(\sigma')^{-1}(i)$. Only non-trivial cases are $i = \sigma'(r) = \sigma(r+1)$ and $i = \sigma'(r+1) = \sigma(r)$.
In the first case $\#\{j > i : (ij) \in D(\sigma)\} = \#\{j > i : (ij) \in D(\sigma')\} + 1$, $\#\{j < i : (ji) \in D(\sigma)\} = \#\{j < i : (ji) \in D(\sigma')\}$ (because $j = \sigma'(r+1) > \sigma'(r) = i$ does not satisfy $j < i$). It follows that $\sigma^{-1}(i) = (\sigma')^{-1}(i) + 1$ but this is equal to assumption $i = \sigma'(r) \Leftrightarrow (\sigma')^{-1}(i) = r \Leftrightarrow (\sigma')^{-1}(i) + 1 = r+1$.
In the second case $\#\{j < i : (ji) \in D(\sigma)\} = \#\{j < i : (ji) \in D(\sigma')\} + 1$, $\#\{j > i : (ij) \in D(\sigma)\} = \#\{j > i : (ij) \in D(\sigma')\}$ (because $j = \sigma'(r) < \sigma'(r+1) = i$ does not satisfy $j > i$). It follows that $\sigma^{-1}(i) = (\sigma')^{-1}(i) - 1$ but this is equal to assumption $i = \sigma'(r+1) \Leftrightarrow (\sigma')^{-1}(i) = r+1 \Leftrightarrow (\sigma')^{-1}(i) - 1 = r$. Hence the statement is true for σ, i.e. for $l+1$.

2. For the purpose of the proof we renumber elementary transpositions α_i of the sequence in inverse order relative to the statement of *Theorem* so that now $\sigma = \alpha_1 ... \alpha_l$. To analyse the case $n > 3$ we first define a *standard* order as follows. In the above procedure of constructing a sequence $\alpha_1, ..., \alpha_l$, we choose at each stage the pair $(r, r+1)$ with maximal $\sigma(r)$ and a corresponding pair $(\sigma(r+1)\sigma(r))$ with $\sigma(r+1) < \sigma(r)$. For example, for the case $n = 6$, $\sigma = (563412)$, the standard order is $(563412) = (36)(46)(16)(26)(35)(45)(15)(25)(14)(24)(13)(23)$.
Given an arbitrary alternative ordering $(\alpha_1, ..., \alpha_l)$ of the $\alpha \in D(\sigma)$ such that $\sigma = \prod_{i=1}^{l(\sigma)} \alpha_i$, we shall move respective α_i's to the front in standard order. Let i_1 be the smallest index such that α_{i_1} contains n, i.e. is of the form

$\alpha_{i_1} = (\sigma(r_1)n)$. If $n = \sigma(r_1 - 1)$ then this is the first element of the standard order, because letting $r = r_1 - 1$ we obtain the definition of the standard order. In that case, there cannot be $j < i_1$ such that α_j contains $\sigma(r_1)$. To show this, we define $\sigma_i := \prod_{j=i}^{l} \alpha_j$, where $l(\sigma) = l$ for fixed σ, and note that $\sigma_{i_1}(r_1 - 1) = n = \sigma(r_1 - 1)$ since α_j does not contain n for $j < i_1$ by assumption, hence $\sigma_{i_1}(r_1) = \sigma(r_1)$ and as a result α_j cannot contain $\sigma(r_1)$ for $j < i_1$.

Next suppose that α_{i_1} is not the first element of the standard order, and denote the subsequent indices i such that α_i contains n, $i_1 < i_2 < ...$ and set $\alpha_{i_p} = (\sigma(r_p)n)$. Let α_{i_k} ($k > 1$) be the first element in the standard order, so $n = \sigma(r_k - 1)$. Suppose that for $i < i_k$, α_i contains $\sigma(r_k)$. First consider the case of $\alpha_i = (\sigma(r_k)\sigma(r))$ for some $r < r_k$, $\sigma(r) > \sigma(r_k)$. Then $\sigma_{i_k}^{-1}(n) = \sigma_{i_k}^{-1}(\sigma(r_k)) - 1$ (because α_{i_k} transposes $\sigma(r_k)$ and n) and $\sigma_i^{-1}(\sigma(r)) = \sigma_i^{-1}(\sigma(r_k)) - 1$ (because α_i transposes $\sigma(r_k)$ and $\sigma(r)$ by assumption). But $\sigma_j^{-1}(n) < \sigma_j^{-1}(\sigma(r_k))$ for $j < i_k$, since α_{i_k} is to the right to the α_j hence already applied before α_j. It follows that $\sigma_i^{-1}(n) < \sigma_i^{-1}(\sigma(r))$. Then there exists some $j < i_k$ (and $j > i$) such that $\alpha_j = (\sigma(r)n)$ contains n, hence $j = i_p < i_k$ for some $p < k$ and $r = r_p < r_k$. But this contradicts our assumption that α_{i_k} is the first element of the standard order, for if $j \leq i$, $\sigma_j^{-1}(n) < \sigma_j^{-1}(\sigma(r_p)) < \sigma_j^{-1}(\sigma(r_k))$. Taking $j = 1$, $\sigma_j = \sigma \Rightarrow \sigma^{-1}(n) < r_p < r_k$ contradicting the assumption $n = \sigma(r_k - 1)$. It follows that for the case $r < r_k$ there is no $i < i_k$ such that $\alpha_i = (\sigma(r_k)\sigma(r))$.

Now suppose that for $i < i_k$, $\alpha_i = (\sigma(r)\sigma(r_k))$ with $r > r_k$ hence $\sigma(r) < \sigma(r_k)$. Then for $j > i$, $\sigma_j^{-1}(\sigma(r)) < \sigma_j^{-1}(\sigma(r_k))$ (because the position of $\sigma(r)$ is to the left of the position of $\sigma(r_k)$ by assumption, until α_i has been applied). But $\sigma_{i_k}^{-1}(n) = \sigma_{i_k}^{-1}(\sigma(r_k)) - 1$ by assumption so $\sigma_{i_k}^{-1}(\sigma(r)) < \sigma_{i_k}^{-1}(n) < \sigma_{i_k}^{-1}(\sigma(r_k))$. On the other hand, for $j = i + 1$, $\sigma_{i+1}^{-1}(\sigma(r)) = \sigma_{i+1}^{-1}(\sigma(r_k)) - 1$ (because position of $\sigma(r)$ is to the left of the position of $\sigma(r_k)$ just before α_i has been applied) so there exists j such that $i < j < i_k$, $\alpha_j = (\sigma(r)n)$. Hence $j = i_p < i_k$ with $p < k$ and $r = r_p$.

Conversely, suppose $r = r_p$ with $p < k$. First let $r_p < r_k$. We have seen that assumption of existence $i < i_k$, such that $\alpha_i = (\sigma(r_k)\sigma(r_p))$, leads to conclusion $\alpha_i \neq (\sigma(r_k)\sigma(r_p))$ for $i < i_k$. Then for $j \leq r_p$, $\sigma_j^{-1}(n) < \sigma_j^{-1}(\sigma(r_p)) < \sigma_j^{-1}(\sigma(r_k))$ because $(\sigma(r_p))$, $(\sigma(r_k))$ did not interchange their positions. Taking $j = 1$ we get $n < r_p < r_k$ contradicting the assumption $n = \sigma(r_k - 1)$. It follows that we cannot have $r_p < r_k$.

Now suppose $r_p > r_k$. Let $\sigma(r_p) > \sigma(r_k)$. Then $\sigma_j^{-1}(\sigma(r_k)) < \sigma_j^{-1}(\sigma(r_p))$ for all j because $\sigma(r_k)$ and $\sigma(r_p)$ cannot interchange their relative position twice. But $\sigma_{i_k}^{-1}(n) = \sigma_{i_k}^{-1}(\sigma(r_k)) - 1$ because n and $\sigma(r_k)$ interchanged by α_{i_k}. It follows that $\sigma_j^{-1}(n) < \sigma_j^{-1}(\sigma(r_k))$ for $j \leq i_k$. But $\sigma_j^{-1}(\sigma(r_k)) < \sigma_j^{-1}(\sigma(r_p)) \Rightarrow \sigma_j^{-1}(n) < \sigma_j^{-1}(\sigma(r_p)) - 1$. At $j = j_p$ this contradicts $\sigma_{i_p}^{-1}(n) = \sigma_{i_p}^{-1}(\sigma(r_p)) - 1$.

Finally, suppose $\sigma(r_p) < \sigma(r_k)$, $r_p > r_k$ and $\alpha_i = (\sigma(r_p)\sigma(r_k))$ for $i > i_p$. Then for j such that $i_p < j \leq i$, $\sigma_j^{-1}(\sigma(r_k)) < \sigma_j^{-1}(\sigma(r_p))$. Since α_{i_p} interchanges $\sigma(r_p)$ and n, then position of n is the next to the right of $\sigma(r_p)$, just before applying α_{i_p}: $\sigma^{-1}_{i_p+1}(n) = \sigma^{-1}_{i_p+1}(\sigma(r_p)) + 1 \Rightarrow \sigma^{-1}_j(n) > \sigma^{-1}_j(\sigma(r_p))$ for $j > j_p$. It follows that $\sigma^{-1}_j(n) > \sigma^{-1}_j(\sigma(r_k))$. But $\sigma^{-1}_j(n) < \sigma^{-1}_j(\sigma(r_k))$ for $j \leq i_k$ by assumption, hence contradiction.

We conclude that the only possibility such that the relative order of non-commuting elements in σ is: $(\sigma(r_p)\sigma(r_k))...(\sigma(r_p)\sigma(n))...(\sigma(r_k)\sigma(n))$. It now follows that we can move the factors $z_{\alpha_{i_p}}$ and $z_{\alpha_{i_k}}$ to the left adjoining $z_{\alpha_i} = z_{(\sigma(r_p)\sigma(r_k))}$. Subsequently, we can apply the identity derived in the 3-particle case to conclude that

$$z_{\alpha_i} z_{\alpha_{i_p}} z_{\alpha_{i_k}} = z_{\alpha_{i_k}} z_{\alpha_{i_p}} z_{\alpha_i}.$$

The resulting expression corresponds to a reordering of the transpositions α_j such that α_{r_k} is now the $k-1$-th transposition containing n which equals the first transposition in the standard order. By induction on k it follows that $z_{\alpha_{i_k}}$ can be moved to the front. The first transposition in the resulting order then equals that in the standard order and we can replace σ by $\sigma' = \alpha_{i_k}\sigma$, for which $l(\sigma') = l(\sigma) - 1$. This completes the induction step w.r.t. l. □

4.2 Construction of $\psi_{in}(x_1...x_n \mid k_1...k_n)$ in terms of transposition operators

We generalise the definition (3.4) of $\psi_{in}(x_1x_2x_3 \mid k_1k_2k_3)$ to the n-particle case:

$$\psi_{in}(x_1...x_n \mid k_1...k_n) = \sum_{\sigma' \in S_n} \frac{\psi(x_{\sigma'_1}...x_{\sigma'_n} \mid k_{\sigma'_1}...k_{\sigma'_n})}{\prod_{r<s}^{r,s=1,...,n} A_{\sigma'_r\sigma'_s}} \theta(\sigma'_1...\sigma'_n), \tag{4.4}$$

where $\theta(\sigma'_1...\sigma'_n) := \theta(k_{\sigma'_1} > ... > k_{\sigma'_n})$ is the characteristic function of a momentum region.

(4.4) transforms the system of wave function for all regions in coordinate space into a system $\psi_{in}(x_1...x_n \mid k_1...k_n)$ of functions in momentum space for all regions of momentum space. Note that the wave function ψ is normalised so that the coefficient of $[1...n]$ is unity in the region $\chi(1...n)$. This corresponds to an incoming wave with wave vectors $k_1, ..., k_n$ such that $k_1 > ... > k_n$, so that particles with coordinates $x_1 > ... > x_n$ do not interact. However, this wave function is not symmetric with respect to interchange of indices. (4.4) achieves this symmetrisation by summing over permuted wave vectors with the same simultaneous permutations of the particles, and with the corresponding normalisation for each term in the form of the product of all A-coefficients with permuted indices.

From now on and until further notice, we will consider $\psi_{in}(x_1...x_n \mid k_1...k_n)$ only in the standard region $\chi(1...n)$ of coordinate space. Arbitrary regions $\chi(\sigma_1...\sigma_n)$ are considered later. In (4.4), ψ_{in} is defined in terms of wave functions for the different momentum regions and this can equivalently be expressed in the form of momentum transpositions, since every momentum region is a momentum transposition of another region, and eventually of $\theta(1...n)$. However, we have only an expression (4.3) for the wave function $\psi(x_1...x_n \mid k_1...k_n)$ in different coordinate regions $\chi(\sigma_1...\sigma_n)$, i.e. in terms of permutations of coordinate regions. Therefore, we wish to express ψ_{in} in the form of the known transpositions of coordinate regions instead of transpositions of momentum regions. This can be done with the use of antisymmetric property of transpositions in momentum and coordinate space as follows. According to (4.4), ψ_{in} for coordinate region $\chi(1...n)$ can be expressed as:

$$\psi_{in}(x_1...x_n \mid k_1...k_n)\,\chi(1...n)$$

$$= \chi(1...n) \sum_{\sigma' \in S_n} \frac{\psi(x_{\sigma'_1}...x_{\sigma'_n} \mid k_{\sigma'_1}...k_{\sigma'_n})}{\prod\limits_{1\le r<s\le n} A_{\sigma'_r\sigma'_s}}\,\theta(\sigma'_1...\sigma'_n)$$

$$= \chi(1...n) \sum_{\sigma' \in S_n} \tau_{\sigma'}\,\theta(1...n)\,\frac{\psi(x_1...x_n \mid k_1...k_n)}{\prod\limits_{1\le r<s\le n} A_{rs}}, \tag{4.5}$$

where $\tau_{\sigma'}$ is defined by

$$\tau_{\sigma'} f(x_1...x_n; k_1...k_n) = f(x_{\sigma'_1}...x_{\sigma'_n}; k_{\sigma'_1}...k_{\sigma'_n}), \tag{4.6}$$

and can be written in terms of a particular composition of elementary transpositions τ_{ij}. Since the sum over σ' does not depend on the order of summation we can change the sum over σ' to a sum over $(\sigma')^{-1}$. We also multiply both sides by $\mathbb{1} = \sum_{\sigma\in S_n} \chi(\sigma_1...\sigma_n)$ and use the equality $\chi(1...n) = \tau_{\sigma'}^{-1}\,\chi(\sigma'_1...\sigma'_n)$. It follows that

$$\psi_{in}(x_1...x_n \mid k_1...k_n)\,\chi(1...n)$$

$$= \chi(1...n) \sum_{\sigma' \in S_n} \tau_{\sigma'}^{-1}\,\theta(1...n)\,\frac{\psi(x_1...x_n \mid k_1...k_n)}{\prod\limits_{1\le r<s\le n} A_{rs}} \sum_{\sigma\in S_n} \chi(\sigma_1...\sigma_n)$$

$$= \sum_{\sigma' \in S_n} \tau_{\sigma'}^{-1}\,\theta(1...n)\,\frac{\psi(x_1...x_n \mid k_1...k_n)}{\prod\limits_{1\le r<s\le n} A_{rs}}\,\chi(\sigma'_1...\sigma'_n) \sum_{\sigma\in S_n} \chi(\sigma_1...\sigma_n)$$

$$= \sum_{\sigma \in S_n} \tau_{\sigma}^{-1}\,\theta(1...n)\,\frac{\psi(x_1...x_n \mid k_1...k_n)}{\prod\limits_{1\le r<s\le n} A_{rs}}\,\chi(\sigma_1...\sigma_n), \tag{4.7}$$

because of the δ-relation for characteristic functions
$\chi(\sigma'_1...\sigma'_n)\,\chi(\sigma_1...\sigma_n) = \chi(\sigma_1...\sigma_n)\,\delta(\sigma'_1...\sigma'_n, \sigma_1...\sigma_n) = \chi(\sigma_1...\sigma_n)\,\delta(\tau_{\sigma'}, \tau_\sigma)$. Since

the latter relation can be expressed equivalently in terms of σ' transpositions, i.e. $\chi(\sigma'_1...\sigma'_n)\,\chi(\sigma_1...\sigma_n) = \chi(\sigma'_1...\sigma'_n)\,\delta(\tau_{\sigma'},\tau_\sigma)$, then inserting (4.3) in (4.7) and using again δ-relation for characteristic functions, we obtain:

$$\psi_{in}(x_1...x_n \mid k_1...k_n)\,\chi^x(1...n) = \chi^x(1...n) \sum_{\sigma' \in S_n} \left(\overrightarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \theta(1...n)$$
$$\times \frac{1}{\prod\limits_{1 \le r < s \le n} A_{rs}} \overrightarrow{\prod_{\beta \in D(\sigma')}} (A_\beta + B_\beta P_\beta) \left(\overleftarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) [1...n](x), \tag{4.8}$$

where the notations $\chi^x(1...n)$ and $[1...n](x)$ specify the coordinate region for the x-component and the phase factor of the x-component, respectively, and β defined by

$$D(\sigma') = \{\beta := (\sigma'_{s_2}\sigma'_{s_1}) \;\; : s_1 < s_2,\;\; \sigma'_{s_1} > \sigma'_{s_2},\;\; s_1, s_2 \in \{1,...,n\},\;\; \sigma' \in S_n\}, \tag{4.9}$$

is an elementary transposition in the set $D(\sigma')$, the order of the product being as in the *Lemma 1* above, i.e.

$$\tau_{\sigma'} = \overleftarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \;\; \text{and hence,} \;\; \tau_{\sigma'}^{-1} = \overrightarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta. \tag{4.10}$$

The number of inverted pairs will be denoted $l(\sigma') := \#D(\sigma')$.

4.3 Modification of the integrand: $\psi_{in}(x \mid k)\overline{\psi_{in}(y \mid k)} \to \widetilde{\psi_{in}}(x, y \mid k)$

Similar to (3.7) of Chapter 3, we combine the expression for $\psi_{in}(x_1...x_n \mid k_1...k_n)$ with that for $\overline{\psi_{in}(y_1...y_n \mid k_1...k_n)}$ to obtain a combined function and then apply a modification to change the phase of the x-factor to $[1...n](x)$. In the common region $\chi^x(1...n)\,\chi^y(1...n) = \chi(1...n)$ the combined function has the form

$$\psi_{in}(x_1...x_n \mid k_1...k_n)\overline{\psi_{in}(y_1...y_n \mid k_1...k_n)}\,\chi(1...n)$$
$$= \chi(1...n) \left\{ \sum_{\sigma' \in S_n} \left(\overrightarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \frac{\theta(1...n)}{\prod\limits_{1 \le r < s \le n} A_{rs}} \overrightarrow{\prod_{\beta \in D(\sigma')}} (A_\beta + B_\beta P_\beta) \left(\overleftarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) [1...n](x) \right\}$$
$$\times \left\{ \sum_{\sigma' \in S_n} \left(\overrightarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \frac{\theta(1...n)}{\prod\limits_{1 \le r < s \le n} \overline{A_{rs}}} \overrightarrow{\prod_{\beta \in D(\sigma')}} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\overleftarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) [1...n](-y) \right\}. \tag{4.11}$$

Here all operators act only on the corresponding x- or y-components of the formula. Applying the δ-relation for the characteristic functions of momentum regions, this simplifies to

$$\begin{aligned}
&\psi_{in}(x_1...x_n \mid k_1...k_n)\overline{\psi_{in}(y_1...y_n \mid k_1...k_n)}\,\chi(1...n) \\
&= \chi(1...n) \sum_{\sigma' \in S_n} \left\{ \left(\prod_{\beta \in D(\sigma')}^{\rightarrow} \tau_\beta \right) \frac{\theta(1...n)}{\prod\limits_{1 \le r < s \le n} A_{rs}} \prod_{\beta \in D(\sigma')}^{\rightarrow} (A_\beta + B_\beta P_\beta) \left(\prod_{\beta \in D(\sigma')}^{\leftarrow} \tau_\beta \right) [1...n](x) \right\} \\
&\times \left\{ \left(\prod_{\beta \in D(\sigma')}^{\rightarrow} \tau_\beta \right) \frac{1}{\prod\limits_{1 \le r < s \le n} \overline{A_{rs}}} \prod_{\beta \in D(\sigma')}^{\rightarrow} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\prod_{\beta \in D(\sigma')}^{\leftarrow} \tau_\beta \right) [1...n](-y) \right\},
\end{aligned} \tag{4.12}$$

where all operators still act only on corresponding x- or y-components of the formula. To affect the modification of the integrand as in the 2- and 3-particle cases, we need to apply appropriate permutation operators to individual terms as before. We expand the product $\prod_{\beta \in D(\sigma')}(A_\beta + B_\beta P_\beta)$ into individual terms given by subsets $D_0 \subset D(\sigma')$ where A_β is selected:

$$\prod_{\beta \in D(\sigma')}^{\rightarrow} (A_\beta + B_\beta P_\beta) = \sum_{D_0 \subset D(\sigma')} \prod_{\beta \in D(\sigma')}^{\rightarrow} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1}),$$

where $D_1 = D(\sigma') \setminus D_0$. To the term corresponding to D_1, we must then apply the operator

$$\left(\prod_{\beta \in D(\sigma')}^{\rightarrow} \tau_\beta \right) \left(\prod_{\beta \in D_1}^{\leftarrow} P_\beta \right) \left(\prod_{\beta \in D(\sigma')}^{\leftarrow} \tau_\beta \right), \tag{4.13}$$

which is in fact equal to identity for any $\beta \in D_0$. This same operator must also be applied to the y-factor, so that the modified integrand becomes

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi(1...n)$$

$$= \chi(1...n) \sum_{\sigma' \in S_n} \sum_{D_0 \subset D(\sigma')} \left\{ \left[\left(\overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \left(\overset{\leftarrow}{\prod_{\beta \in D_1}} P_\beta \right) \left(\overset{\leftarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \right] \left(\overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \right.$$

$$\left. \times\, \theta(1...n) \frac{1}{\prod\limits_{1 \le r < s \le n} A_{rs}} \overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1}) \left(\overset{\leftarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) [1...n](x) \right\}$$

$$\times \left\{ \left[\left(\overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \left(\overset{\leftarrow}{\prod_{\beta \in D_1}} P_\beta \right) \left(\overset{\leftarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \right] \left(\overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \right.$$

$$\left. \times \frac{1}{\prod\limits_{1 \le r < s \le n} \overline{A_{rs}}} \overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\overset{\leftarrow}{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) [1...n](-y) \right\}. \tag{4.14}$$

Obviously, the factors $\left(\overset{\leftarrow}{\prod\limits_{\beta \in D(\sigma')}} \tau_\beta \right) \left(\overset{\rightarrow}{\prod\limits_{\beta \in D(\sigma')}} \tau_\beta \right)$ cancel. We now prove by induction that we can rewrite the x-factor as follows:

Lemma 2
For any function $f(k_1, ..., k_n)$ and any permutation $\sigma' \in S_n$ we have

$$\left(\overset{\leftarrow}{\prod_{\beta \in D_1}} P_\beta \right) f(k_1, ..., k_n) \overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1})$$

$$= \overset{\leftarrow}{\prod_{\beta \in D(\sigma')}} (A_\beta 1_{D_0} + P_\beta B_\beta 1_{D_1})\, f(k_1, ..., k_n) \left(\overset{\rightarrow}{\prod_{\beta \in D_1}} P_\beta \right), \tag{4.15}$$

where $D_1 = D(\sigma') \setminus D_0$.

Proof

The statement is obviously true if $D_1 = \emptyset$. Assuming that it is true for $\# D_1 = m - 1$ we let β_m be the last element of D_1 and set $D_1' = D_1 \setminus \{\beta_m\}$. Moreover, let D' be the set $D(\sigma')$ with the elements β to the right of β_m (inclusive) omitted. We write

$$\left(\overset{\leftarrow}{\prod_{\beta \in D_1}} P_\beta \right) f(k_1, ..., k_n) \overset{\rightarrow}{\prod_{\beta \in D(\sigma')}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1})$$

$$= P_{\beta_m} \left(\overset{\leftarrow}{\prod_{\beta \in D_1'}} P_\beta \right) f(k_1, ..., k_n) \overset{\rightarrow}{\prod_{\beta \in D'}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1'}) B_{\beta_m} P_{\beta_m} \left(\overset{\rightarrow}{\prod_{\beta \in D(\sigma') \setminus (D' \cup \{\beta_m\})}} A_\beta \right).$$

Since
$$P_{\beta_m}\left(\overleftarrow{\prod_{\beta\in D_1'}} P_\beta\right)\left(\overrightarrow{\prod_{\beta\in D'}} P_\beta 1_{D_1'}\right) P_{\beta_m} = \mathbb{1},$$

we can move the last product to the front of whole expression, then also B_{β_m} through the operator

$$\left(\overleftarrow{\prod_{\beta\in D_1'}} P_\beta\right)\left(\overrightarrow{\prod_{\beta\in D'}} P_\beta 1_{D_1'}\right) = \mathbb{1},$$

so that

$$\left(\overleftarrow{\prod_{\beta\in D_1}} P_\beta\right) f(k_1,...,k_n) \overrightarrow{\prod_{\beta\in D(\sigma')}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1})$$
$$= \left(\overleftarrow{\prod_{\beta\in D(\sigma')\backslash(D'\cup\{\beta_m\})}} A_\beta\right) P_{\beta_m} B_{\beta_m}\left\{\left(\overleftarrow{\prod_{\beta\in D_1'}} P_\beta\right) f(k_1,...,k_n) \overrightarrow{\prod_{\beta\in D'}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1'})\right\} P_{\beta_m}.$$

Substituting the statement of *Lemma 2* for $m-1$ we interchange order of the factors in the brackets, then

$$\left(\overleftarrow{\prod_{\beta\in D_1}} P_\beta\right) f(k_1,...,k_n) \overrightarrow{\prod_{\beta\in D(\sigma')}} (A_\beta 1_{D_0} + B_\beta P_\beta 1_{D_1})$$
$$= \left(\overleftarrow{\prod_{\beta\in D(\sigma')\backslash(D'\cup\{\beta_m\})}} A_\beta\right) P_{\beta_m} B_{\beta_m}\left\{\overleftarrow{\prod_{\beta\in D'}} (A_\beta 1_{D_0} + P_\beta B_\beta 1_{D_1'}) f(k_1,...,k_n) \left(\overrightarrow{\prod_{\beta\in D_1'}} P_\beta\right)\right\} P_{\beta_m}$$
$$= \overleftarrow{\prod_{\beta\in D(\sigma')}} (A_\beta 1_{D_0} + P_\beta B_\beta 1_{D_1}) f(k_1,...,k_n) \left(\overrightarrow{\prod_{\beta\in D_1}} P_\beta\right). \qquad \square$$

Applying this to the x-factor in $\widetilde{\psi_{in}}$ of (4.14) we have

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi(1...n)$$
$$= \chi(1...n) \sum_{\sigma'\in S_n} \sum_{D_0\subset D(\sigma')} \left\{\left(\overrightarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta\right) \overleftarrow{\prod_{\beta\in D(\sigma')}} (A_\beta 1_{D_0} + P_\beta B_\beta 1_{D_1})\right.$$
$$\left.\times\ \theta(1...n) \frac{1}{\prod\limits_{1\le r<s\le n} A_{rs}} \left(\overrightarrow{\prod_{\beta\in D_1}} P_\beta\right)\left(\overleftarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta\right) [1...n](x)\right\}$$
$$\times \left\{\left(\overrightarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta\right)\left(\overleftarrow{\prod_{\beta\in D_1}} P_\beta\right) \frac{1}{\prod\limits_{1\le r<s\le n} \overline{A_{rs}}} \overrightarrow{\prod_{\beta\in D(\sigma')}} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\overleftarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta\right) [1...n](-y)\right\}.$$
(4.16)

We now observe that, due to the insertion of the operator (4.13), there is effectively no operator acting on $[1...n](x)$. Therefore this phase factor can be moved to the front. This is in exact correspondence with our aim to modify the full expression in such a way to obtain only $n!$ different phases of waves for $\widetilde{\psi_{in}}(x, y \mid k)$ instead of $n! \times n!$ phases for $\psi_{in}(x \mid k)\overline{\psi_{in}(y \mid k)}$. Moreover, it also follows that we may assume that all operators now act on the full expression to the right, upon which we can cancel the P_β and τ_β products. The result is:

$$
\begin{aligned}
&\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\,\chi(1...n) \\
&= \chi(1...n)\,[1...n](x) \sum_{\sigma' \in S_n} \sum_{D_0 \subset D(\sigma')} \left\{ \left(\prod_{\beta \in D(\sigma')}^{\rightarrow} \tau_\beta \right) \prod_{\beta \in D(\sigma')}^{\leftarrow} (A_\beta 1_{D_0} + P_\beta B_\beta 1_{D_1}) \right. \\
&\left. \times \frac{\theta(1...n)}{\prod\limits_{1 \le r < s \le n} A_{rs}} \frac{1}{\prod\limits_{1 \le r < s \le n} \overline{A_{rs}}} \prod_{\beta \in D(\sigma')}^{\rightarrow} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\prod_{\beta \in D(\sigma')}^{\leftarrow} \tau_\beta \right) \right\} [1...n](-y).
\end{aligned}
\tag{4.17}
$$

Finally, we observe that the factors $\dfrac{1}{\prod\limits_{1 \le r < s \le n} A_{rs}}$ and $\dfrac{1}{\prod\limits_{1 \le r < s \le n} \overline{A_{rs}}}$ can be combined to $\dfrac{1}{\prod\limits_{1 \le r < s \le n} |A_{rs}|^2}$, which is invariant under permutations of indices and can therefore also be moved to the front. Resumming over D_0 we get

$$
\begin{aligned}
&\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\,\chi(1...n) \\
&= \chi(1...n)\,[1...n](x) \frac{1}{\prod\limits_{1 \le r < s \le n} |A_{rs}|^2} \sum_{\sigma' \in S_n} \left\{ \left(\prod_{\beta \in D(\sigma')}^{\rightarrow} \tau_\beta \right) \prod_{\beta \in D(\sigma')}^{\leftarrow} (A_\beta + P_\beta B_\beta) \right. \\
&\left. \times\ \theta(1...n) \prod_{\beta \in D(\sigma')}^{\rightarrow} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\prod_{\beta \in D(\sigma')}^{\leftarrow} \tau_\beta \right) \right\} [1...n](-y).
\end{aligned}
\tag{4.18}
$$

This expression needs to be integrated over $k_1, ..., k_n$. For this, we determine the contributions from the various regions of momentum space. In fact, in the Section 5 we show that the integrand is the same in all regions as was the case for 2 and 3 particles.

4.4 Construction of $\widetilde{\psi_{in}}(x, y \mid k)$ for an arbitrary region $\chi^x(\sigma^x)\chi^y(\sigma^y)$ of coordinate space

Using the definitions of transformation operators for coordinate and momentum regions respectively, given by (3.18) and (3.19) for 3-particle case, we generalise them to n-particle case. However, application of the statement of *Lemma 2* to the x-factor in $\widetilde{\psi_{in}}$ of (4.14) in previous section transformed the expression for $\widetilde{\psi_{in}}$ to the form with the factor $\left[\prod_{1\le r<s\le n} |A_{rs}|^2\right]^{-1}$; and definitions of transformation operators for this particular form of $\widetilde{\psi_{in}}$ are chosen to be conjugated to the similar (3.18) and (3.19), for convenience:

$$Z(\sigma) = \overrightarrow{\prod_{\alpha\in D(\sigma)}} (\overline{A_\alpha} + P_\alpha \overline{B_\alpha}) \left(\overleftarrow{\prod_{\alpha\in D(\sigma)}} \tau_\alpha \right) ,$$

$$W(\sigma') = \left(\overrightarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta \right) \overleftarrow{\prod_{\beta\in D(\sigma')}} (A_\beta + P_\beta B_\beta), \tag{4.19}$$

where α and β are defined by (4.2) and (4.9) respectively. Then generalising the expression given by (3.23), we obtain for n-particle case and arbitrary coordinate regions of x- and y-components:

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_3)\, \chi^x(\sigma^x)\, \chi^y(\sigma^y) = \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, \frac{[1...n](x)}{\prod_{1\le r<s\le n} |A_{rs}|^2}$$

$$\times Z(\sigma^x) \left[\sum_{\sigma'\in S_n} W(\sigma')\, \theta(1...n)\, W^+(\sigma') \right] Z^+(\sigma^y)\, [1...n](-y)$$

$$= \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, \frac{[1...n](x)}{\prod_{1\le r<s\le n} |A_{rs}|^2} \sum_{\sigma'\in S_n} \left\{ \overrightarrow{\prod_{\alpha\in D(\sigma^x)}} (\overline{A_\alpha} + P_\alpha \overline{B_\alpha}) \left(\overleftarrow{\prod_{\alpha\in D(\sigma^x)}} \tau_\alpha \right) \right.$$

$$\times \left[\left(\overrightarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta \right) \overleftarrow{\prod_{\beta\in D(\sigma')}} (A_\beta + P_\beta B_\beta)\, \theta(1...n) \overrightarrow{\prod_{\beta\in D(\sigma')}} (\overline{A_\beta} + \overline{B_\beta} P_\beta) \left(\overleftarrow{\prod_{\beta\in D(\sigma')}} \tau_\beta \right) \right]$$

$$\left. \times \left(\overrightarrow{\prod_{\alpha\in D(\sigma^y)}} \tau_\alpha \right) \overleftarrow{\prod_{\alpha\in D(\sigma^y)}} (A_\alpha + B_\alpha P_\alpha) \right\} [1...n](-y). \tag{4.20}$$

According to (4.20), the transformation of the system of waves between regions of momentum space is separated into the group of transformations within region $\chi(1...n)$ of coordinate space (the central part of (4.20) in square brackets) and group of transformations between regions of coordinate space (outside the brackets). The transformation within given region of coordinate space leads to the redistribution of waves between different regions of momentum space by the sum of operators

$$\sum_{\sigma' \in S_n} W(\sigma') = \sum_{\sigma' \in S_n} \left(\overrightarrow{\prod_{\beta \in D(\sigma')}} \tau_\beta \right) \overleftarrow{\prod_{\beta \in D(\sigma')}} (A_\beta + P_\beta B_\beta) \tag{4.21}$$

acting on region $\theta(1...n)$. All the terms of (4.21) include different combinations of P_β transforming the region $\theta(1...n)$ into generally different momentum regions.

4.5 Rearrangement of the terms of $\widetilde{\psi_{in}}(x, y \mid k)$ according to the regions of momentum space for coordinate region $\chi(1...n)$

In order to determine the contributions for a given region of momentum space we first need to rearrange (4.18) by moving the τ_β-operators to combine with the corresponding z_β operators.
We claim:

$$\left(\overrightarrow{\prod_{i=1}^{l}} \tau_{\beta_i} \right) \overleftarrow{\prod_{i=1}^{l}} (A_{\beta_i} + P_{\beta_i} B_{\beta_i}) = \overleftarrow{\prod_{i=1}^{l}} [\tau_{\gamma_i} (A_{\gamma_i} + P_{\gamma_i} B_{\gamma_i})], \tag{4.22}$$

where

$$\gamma_i = \beta_1 ... \beta_{i-1}(\beta_i).$$

This follows immediately by induction from the relation

$$\beta_1 \beta_2 = \beta_1(\beta_2)\beta_1,$$

where $\beta\big((ij)\big) = (\beta(i)\beta(j))$. Note that although

$$\sigma' = \overleftarrow{\prod_{i=1}^{l}} \gamma_i = \overleftarrow{\prod_{i=1}^{l}} \beta_1 ... \beta_{i-1}(\beta_i),$$

this is no longer a decomposition in elements of $D(\sigma')$. For example, if $\sigma'(1234) = (3421)$, then $D(\sigma') = \{(12), (13), (23), (14), (24)\}$,
$\sigma' = (24)(14)(23)(13)(12)$ and $(\sigma')^{-1} = (12)(13)(23)(14)(24)$ but the transformed version is $(\sigma')^{-1} = (23)(34)(12)(23)(12)$. Thus, in this case, (4.22) reads:

$$\tau_{12}\tau_{13}\tau_{23}\tau_{14}\tau_{24} z_{24} z_{14} z_{23} z_{13} z_{12} = (\tau_{23} z_{23})(\tau_{34} z_{34})(\tau_{12} z_{12})(\tau_{23} z_{23})(\tau_{12} z_{12}).$$

Obviously, we have a similar formula for the right-hand factor by conjugation, and obtain

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\,\chi(1...n) = \chi(1...n)\,\frac{[1...n](x)}{\prod\limits_{1\le r<s\le n} |A_{rs}|^2} \tag{4.23}$$

$$\times \sum_{\sigma' \in S_n} \left\{ \overleftarrow{\prod_{i=1}^{l}} \left(\tau_{\beta_1...\beta_{i-1}(\beta_i)} z_{\beta_1...\beta_{i-1}(\beta_i)}\right) \theta(1...n) \overrightarrow{\prod_{i=1}^{l}} \left(z^+_{\beta_1...\beta_{i-1}(\beta_i)} \tau_{\beta_1...\beta_{i-1}(\beta_i)}\right) [1...n](-y) \right\}.$$

Here $z_\gamma = A_\gamma + P_\gamma B_\gamma$ and $z^+_\gamma = \overline{A_\gamma} + \overline{B_\gamma} P_\gamma$. Clearly, the operators P_β in τ_β change the momentum region and we need to add all contributions for a given region. We now prove that the contributions to all regions are the same. For this, it obviously suffices to show that the contributions of momentum regions that differ by a single transposition are the same. We therefore subdivide the permutations into two groups differing by a transposition.
Given $i < n$, let Γ_i be the set of permutations σ such that $\sigma^{-1}(i) < \sigma^{-1}(i+1)$, i.e. $(i, i+1) \notin D(\sigma)$. There is clearly a 1-to-1 correspondence between Γ_i and $S_n \setminus \Gamma_i = \{\sigma \in S_n : \sigma^{-1}(i) > \sigma^{-1}(i+1)\}$ given by $\sigma \mapsto \sigma((i)(i+1))$. For a given $\sigma \in \Gamma_i$, we now add the corresponding term in (4.23) and the term corresponding to $\sigma((i)(i+1))$:

$$\left\{ \overleftarrow{\prod_{i=1}^{l(\sigma)}} \left(\tau_{\beta_1...\beta_{i-1}(\beta_i)} z_{\beta_1...\beta_{i-1}(\beta_i)}\right) \theta(1...n) \overrightarrow{\prod_{i=1}^{l(\sigma)}} \left(z^+_{\beta_1...\beta_{i-1}(\beta_i)} \tau_{\beta_1...\beta_{i-1}(\beta_i)}\right) [1...n](-y) \right\}$$

$$+ \left\{ \overleftarrow{\prod_{i=1}^{l(\sigma)}} \left(\tau_{\beta_1...\beta_{i-1}(\beta_i)} z_{\beta_1...\beta_{i-1}(\beta_i)}\right) \tau_{i,i+1}(A_{i,i+1} + P_{i,i+1} B_{i,i+1})\,\theta(1...n) \right.$$

$$\left. \times\; (\overline{A_{i,i+1}} + \overline{B_{i,i+1}} P_{i,i+1})\,\tau_{i,i+1} \overrightarrow{\prod_{i=1}^{l(\sigma)}} \left(z^+_{\beta_1...\beta_{i-1}(\beta_i)} \tau_{\beta_1...\beta_{i-1}(\beta_i)}\right) [1...n](-y) \right\}.$$

In the first term we insert the identity

$$\tau_{i,i+1}\,(A_{i,i+1} + B_{i,i+1} P_{i,i+1})(\overline{A_{i,i+1}} + \overline{B_{i,i+1}} P_{i,i+1})\,\tau_{i,i+1}$$

to the right of $\theta(1...n)$ and remark that $\tau_{i,i+1} B_{i,i+1} P_{i,i+1} = -\tau_{i,i+1} P_{i,i+1} B_{i,i+1}$ commutes with $\theta(1...n)$ and cancels $\tau_{i,i+1} P_{i,i+1} B_{i,i+1}$ in the second part of the formula (corresponding to $\sigma((i)(i+1))$). The result is:

$$\overset{\overset{l(\sigma)}{\longleftarrow}}{\prod_{i=1}} \left(\tau_{\beta_1...\beta_{i-1}(\beta_i)} z_{\beta_1...\beta_{i-1}(\beta_i)}\right) \left[\theta(1...n)\, \tau_{i,i+1} A_{i,i+1} + \tau_{i,i+1} A_{i,i+1}\, \theta(1...n)\right]$$

$$\times \left(\overline{A_{i,i+1}} + \overline{B_{i,i+1}} P_{i,i+1}\right) \tau_{i,i+1} \overset{\overset{l(\sigma)}{\longrightarrow}}{\prod_{i=1}} \left(z^{+}_{\beta_1...\beta_{i-1}(\beta_i)} \tau_{\beta_1...\beta_{i-1}(\beta_i)}\right) [1...n](-y)$$

$$= \overset{\overset{l(\sigma)}{\longleftarrow}}{\prod_{i=1}} \left(\tau_{\beta_1...\beta_{i-1}(\beta_i)} z_{\beta_1...\beta_{i-1}(\beta_i)}\right) \left[\theta(1...n) \tau_{i,i+1} + \tau_{i,i+1}\, \theta(1...n)\right] A_{i,i+1}$$

$$\times \left(\overline{A_{i,i+1}} + \overline{B_{i,i+1}} P_{i,i+1}\right) \tau_{i,i+1} \overset{\overset{l(\sigma)}{\longrightarrow}}{\prod_{i=1}} \left(z^{+}_{\beta_1...\beta_{i-1}(\beta_i)} \tau_{\beta_1...\beta_{i-1}(\beta_i)}\right) [1...n](-y).$$

This shows that permutations differing by a transposition of the form $((i)(i+1))$ must have the same contribution. Since all permutations differ by a sequence of such transpositions, all have the same contribution.

We now determine the contribution from $\theta(n...1)$. Since there is only one way of obtaining this permutation, namely by choosing all τ_{rs} $(r < s)$, where $(rs) =: \beta \in D_n$, and $D_n := D(n...1)$, we have, after moving the first product to the right and cancellation with the last product,

$$\overset{\rightarrow}{\prod_{r<s}} \tau_{rs} \prod_{r<s} A_{rs}\, \theta(1...n) \overset{\rightarrow}{\prod_{r<s}} (\overline{A_{rs}} + \overline{B_{rs}} P_{rs}) \overset{\leftarrow}{\prod_{r<s}} \tau_{rs}\, [1...n](-y)$$
$$= \theta(n...1) \prod_{r<s} \overline{A_{rs}} \overset{\rightarrow}{\prod_{r<s}} (A_{rs} + B_{rs} P_{rs})\, [1...n](-y)$$
$$= \theta(n...1) \prod_{\beta \in D_n} \overline{A_\beta} \overset{\rightarrow}{\prod_{\beta \in D_n}} (A_\beta + B_\beta P_\beta)\, [1...n](-y).$$

Since the same contribution is for all momentum regions, then substituting this to (4.23) and summing over all momentum regions we obtain

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi(1...n)$$
$$= \chi(1...n)\, [1...n](x)\, \frac{1}{\prod\limits_{\beta \in D_n} A_\beta} \overset{\rightarrow}{\prod_{\beta \in D_n}} (A_\beta + B_\beta P_\beta)\, [1...n](-y). \qquad (4.24)$$

The product in the numerator of (4.24)

$$\overset{\rightarrow}{\prod_{\beta \in D_n}} (A_\beta + B_\beta P_\beta) = \overset{\rightarrow}{\prod_{\beta \in D_n}} (\mathbb{1} - B_\beta + B_\beta P_\beta) = \overset{\rightarrow}{\prod_{\beta \in D_n}} (\mathbb{1} - B_\beta(\mathbb{1} - P_\beta)) \qquad (4.25)$$

is equal to the sum of terms of the form $\prod\limits_{\beta \in \Gamma \subseteq D_n} [-B_\beta(\mathbb{1} - P_\beta)]$.

For further rearrangement of (4.24) first we need is to prove for the last term in (4.25) the next

Lemma 3
The following identity holds:

$$\prod_{1\leq i<j\leq n}[B_{ij}(\mathbb{1}-P_{ij})]=\left(\prod_{i<j}B_{ij}\right)\sum_{\sigma'\in S_n}(-1)^{|\sigma'|}P_{\sigma'}, \tag{4.26}$$

where $B_{ij}=-B_{ji}, \quad B_{ij}(B_{jk}-B_{ik})=B_{ik}B_{jk}$,
and $P_{\sigma'}$ *is defined by:* $P_{\sigma'}f(x_1...x_n;k_1...k_n)=f(x_1...x_n;k_{\sigma'_1}...k_{\sigma'_n})$.

Proof

For $n=2$ this is trivial. We proceed by induction on n and write

$$\prod_{1\leq i<j\leq n}[B_{ij}(\mathbb{1}-P_{ij})]=\left(\prod_{1\leq i<j\leq n-1}B_{ij}\right)\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\prod_{j=1}^{n-1}B_{jn}(\mathbb{1}-P_{jn}).$$

Since $\left(\prod_{1\leq i<j\leq n}B_{ij}\right)=\left(\prod_{1\leq i<j\leq n-1}B_{ij}\right)\left(\prod_{j=1}^{n-1}B_{jn}\right)$, we need to prove

$$\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\prod_{j=1}^{n-1}B_{jn}(\mathbb{1}-P_{jn})=\left(\prod_{j=1}^{n-1}B_{jn}\right)\sum_{\sigma'\in S_n}(-1)^{|\sigma'|}P_{\sigma'}.$$

Notice that upon expanding the product $\prod_{j=1}^{n-1}B_{jn}(\mathbb{1}-P_{jn})$, the terms where P_{kn} is the first P-operator selected correspond to permutations with $\sigma'(k)=n$. We now combine terms pairwise according to whether $P_{k+1,n}$ is selected or not. Thus

$$\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\prod_{j=1}^{n-1}B_{jn}(\mathbb{1}-P_{jn})$$

$$=\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\left\{\prod_{j=1}^{n-1}B_{jn}-\sum_{k=1}^{n-1}\prod_{j=1}^{k-1}B_{jn}B_{kn}P_{kn}\prod_{j=k+1}^{n-1}B_{jn}(\mathbb{1}-P_{jn})\right\}$$

$$\begin{aligned}=\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\Bigg\{&\prod_{j=1}^{n-1}B_{jn}-\sum_{k=1}^{n-1}\prod_{j=1}^{k-1}B_{jn}B_{kn}P_{kn}\prod_{j>k}B_{jn}\\&+\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1}\prod_{j=1}^{k-1}B_{jn}B_{kn}P_{kn}\prod_{j=k+1}^{l-1}B_{jn}B_{ln}P_{ln}\prod_{j=l+1}^{n-1}B_{jn}(\mathbb{1}-P_{jn})\Bigg\}.\end{aligned} \tag{4.27}$$

The sum of the second and the third terms of (4.27) is

$$\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\left\{\sum_{k=1}^{n-1}\prod_{j=1}^{k-1}B_{jn}B_{kn}P_{kn}\prod_{j>k}B_{jn}\right.$$
$$\left.-\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1}\prod_{j=1}^{k-1}B_{jn}B_{kn}P_{kn}\prod_{j=k+1}^{l-1}B_{jn}B_{ln}P_{ln}\prod_{j>l}B_{jn}(\mathbb{1}-P_{jn})\right\}. \tag{4.28}$$

We replace σ by the composition $\sigma(kl)$. That does not affect the sum because the result of summation over $\sigma\in S_{n-1}$ does not depend on order of summation. After moving of P_{ln} in the last term to the front through P_{kn} it transforms to $P_{lk}=P_{kl}$ and combines with P_σ, then we move P_{kn} in both terms to the right and get

$$\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\left\{\sum_{k=1}^{n-1}\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{kn}\prod_{j>k}B_{jk}P_{kn}\right.$$
$$\left.-\sum_{k=1}^{n-2}\sum_{l=k+1}^{n-1}\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{ln}\prod_{j=k+1}^{l-1}B_{jl}B_{kl}\prod_{j>l}B_{jk}(\mathbb{1}-P_{kj})P_{kn}\right\}. \tag{4.29}$$

For fixed k consider the terms $l=k+1,k+2$.

$$\left(\prod_{j=1}^{k-1}B_{jn}\right)\{B_{k+1,n}B_{k,k+1}B_{k+2,k}(\mathbb{1}-P_{k,k+2})+B_{k+2,n}B_{k+1,k+2}B_{k,k+2}\}$$
$$\times\prod_{j>k+2}B_{jk}(\mathbb{1}-P_{kj})P_{kn}.$$

In the $P_{k,k+2}$-term we replace σ by the composition $\sigma(k,k+2)$ and after moving $P_{k,k+2}$ to the front and changing of the sign for this term due to additional transposition $(k,k+2)$, the last two term become

$$\left(\prod_{j=1}^{k-1}B_{jn}\right)\{B_{k+1,n}B_{k+2,k+1}B_{k,k+2}+B_{k+2,n}B_{k+1,k+2}B_{k,k+2}\}$$
$$\times\prod_{j>k+2}B_{jk}(\mathbb{1}-P_{kj})P_{kn}$$
$$=\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{k+1,n}B_{k+2,n}B_{k,k+2}\prod_{j=k+3}^{n-1}B_{jk}(\mathbb{1}-P_{kj})P_{kn}.$$

Next add the $l=(k+3)$-term to $(k+1,k+3)$-term:

$$\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{k+3,n}B_{k+1,k+3}B_{k+2,k+3}B_{k,k+3}\prod_{j=k+4}^{n-1}B_{jk}(\mathbb{1}-P_{kj})P_{kn}$$
$$-\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{k+1,n}B_{k,k+1}B_{k+2,k}B_{k,k+3}P_{k,k+3}\prod_{j=k+4}^{n-1}B_{jk}(\mathbb{1}-P_{kj})P_{kn}.$$

Replacing σ by $\sigma(k,k+3)$ and moving $P_{k,k+3}$ to the front we get

$$\left(\prod_{j=1}^{k-1} B_{jn}\right)(B_{k+3,n}-B_{k+1,n})B_{k+1,k+3}B_{k+2,k+3}B_{k,k+3}\prod_{j>k+3} B_{jk}(\mathbb{1}-P_{kj})P_{kn}$$
$$=\left(\prod_{j=1}^{k-1} B_{jn}\right) B_{k+1,n}B_{k+3,n}B_{k+2,k+3}B_{k,k+3}\prod_{j>k+3} B_{jk}(\mathbb{1}-P_{kj})P_{kn}.$$

In general, we combine the $(k+1,m)$-term with the $l=m$-term:

$$\left(\prod_{j=1}^{k-1} B_{jn}\right) B_{mn}\left(\prod_{j=k+1}^{m-1} B_{jm}\right) B_{km}\prod_{j=m+1}^{n-1} B_{jk}(\mathbb{1}-P_{jk})P_{kn}$$
$$-\left(\prod_{j=1}^{k-1} B_{jn}\right) B_{k+1,n}B_{k,k+1}B_{k+2,k}\left(\prod_{j=k+3}^{m-1} B_{jk}\right) B_{mk}P_{km}\prod_{j=m+1}^{n-1} B_{jk}(\mathbb{1}-P_{kj})P_{kn},$$

and after moving P_{km} to the front we get

$$\left(\prod_{j=1}^{k-1} B_{jn}\right)(B_{mn}-B_{k+1,n})B_{k+1,m}\left(\prod_{j=k+2}^{m-1} B_{jm}\right) B_{km}\prod_{j=m+1}^{n-1} B_{jk}(\mathbb{1}-P_{kj})P_{kn}$$
$$=\left(\prod_{j=1}^{k-1} B_{jn}\right) B_{k+1,n}B_{mn}\left(\prod_{j=k+2}^{m-1} B_{jm}\right) B_{km}\prod_{j=m+1}^{n-1} B_{jk}(\mathbb{1}-P_{kj})P_{kn}.$$

We now have, summing over all l and k:

$$\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1} B_{jn}\right)\sum_{l=k+1}^{n-1} B_{ln}\left(\prod_{j=k+1}^{l-1} B_{jl}\right) B_{kl}\prod_{j=l+1}^{n-1} B_{jk}(\mathbb{1}-P_{kj})P_{kn}=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1} B_{jn}\right)$$
$$\times\left\{B_{k+1,n}B_{k,k+1}\prod_{j=k+2}^{n-1} B_{jk}+\sum_{l=k+2}^{n-1} B_{k+1,n}B_{ln}\left(\prod_{j=k+2}^{l-1} B_{jl}\right) B_{kl}\prod_{j=l+1}^{n-1} B_{jk}(\mathbb{1}-P_{kj})\right\} P_{kn}. \tag{4.30}$$

The first term in (4.30) combines with the first term in the brackets in (4.29) (except $k=n-1$) to give:

$$\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1} B_{jn}\right)(B_{k+1,n}-B_{kn})B_{k,k+1}\prod_{j=k+2}^{n-1} B_{jk}P_{kn}$$
$$=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1} B_{jn}\right) B_{kn}B_{k+1,n}\prod_{j=k+2}^{n-1} B_{jk}P_{kn}=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k+1} B_{jn}\right)\prod_{j=k+2}^{n-1} B_{jk}P_{kn}.$$

The second term is identical to the second term in (4.29) except that it stands at $l=k+2$ and has a factor $B_{k+1,n}$ instead of $B_{k+1,l}$. We can thus repeat the same procedure with $k+1$ replaced by $k+2$, etc. Hence

$$\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{k+1,n}\sum_{l=k+2}^{n-1}B_{ln}\left(\prod_{j=k+2}^{l-1}B_{jl}\right)B_{kl}\prod_{j=l+1}^{n-1}B_{jk}(\mathbb{1}-P_{kj})P_{kn}$$
$$=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{k+1,n}\left\{B_{k+2,n}B_{k,k+2}\prod_{j=k+3}^{n-1}B_{jk}\right.$$
$$\left.+\sum_{l=k+3}^{n-1}B_{k+2,n}B_{ln}\left(\prod_{j=k+3}^{l-1}B_{jl}\right)B_{kl}\prod_{j=l+1}^{n-1}B_{jk}(\mathbb{1}-P_{jk})\right\}P_{kn}.$$

Again, the first term combines with the reminder of the second term of (4.29) to yield

$$\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k+2}B_{jn}\right)\prod_{j=k+3}^{n-1}B_{jk}P_{kn}.$$

We repeat the procedure for $l=k+3$, etc. After the stage $l=n-2$ we are left with the first term corresponding to $k=l=n-2$ and $\prod_{j=k+1}^{n-1}B_{jk}$ for $j=k+1=n-1$ and the second term corresponding to $\prod_{j=1}^{n-1}B_{jn}$:

$$\sum_{k=1}^{n-2}\left\{\left(\prod_{j=1}^{n-2}B_{jn}\right)\prod_{j=n-1}^{n-1}B_{jk}+\left(\prod_{j=1}^{n-1}B_{jn}\right)\right\}P_{kn}$$
$$=\sum_{k=1}^{n-2}\left\{\left(\prod_{j=1}^{n-2}B_{jn}\right)B_{n-1,k}+\left(\prod_{j=1}^{k-1}B_{jn}\right)\left(\prod_{j=k+1}^{n-1}B_{jn}\right)B_{k,n-1}\right\}P_{kn}$$
$$=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1}B_{jn}\right)\left(\prod_{j=k+1}^{n-2}B_{jn}\right)B_{k,n-1}(B_{n-1,n}-B_{k,n})P_{kn}$$
$$=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{k-1}B_{jn}\right)B_{k,n}\left(\prod_{j=k+1}^{n-2}B_{jn}\right)B_{n-1,n}P_{kn}=\sum_{k=1}^{n-2}\left(\prod_{j=1}^{n-1}B_{jn}\right)P_{kn}.$$

The complete result is thus

$$\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\left\{\prod_{j=1}^{n-1}B_{jn}-\prod_{j=1}^{n-2}B_{jn}B_{n-1,n}P_{n-1,k}-\sum_{k=1}^{n-2}\left(\prod_{j=1}^{n-1}B_{jn}\right)P_{kn}\right\}$$
$$=\left(\prod_{j=1}^{n-1}B_{jn}\right)\sum_{\sigma\in S_{n-1}}(-1)^{|\sigma|}P_\sigma\left\{1-\sum_{k=1}^{n-1}P_{kn}\right\}=\left(\prod_{j=1}^{n-1}B_{jn}\right)\sum_{\sigma'\in S_n}(-1)^{|\sigma'|}P_{\sigma'}.\quad\square$$

After getting the result of *Lemma 3* for the last term of $\overset{\rightarrow}{\prod\limits_{\beta\in D_n}}(\mathbb{1}-B_\beta(\mathbb{1}-P_\beta))$, we need a similar statement for an arbitrary term. This can be achieved by an arbitrary choice of only β such that $\beta\in\Gamma\subseteq D_n$ because any factor corresponding $\beta\in D_n\backslash\Gamma$ is equal $\mathbb{1}$. Now we extend the result of *Lemma 3* to all terms in (4.25) and construct the following generalisation, which we have so far not been able to prove:

Lemma 4
For any $m\leq\binom{n}{2}$,

$$\sum_{\substack{\Gamma\subseteq D_n:\\ |\Gamma|=m}}\overset{\rightarrow}{\prod_{\beta\in\Gamma}}[B_\beta(\mathbb{1}-P_\beta)]=\sum_{\substack{M\subseteq D_n:\\ |M|=m}}\left(\prod_{\beta\in M}B_\beta\right)\sum_{\substack{\sigma'\in S_n:\ \sigma'=\overset{\leftarrow}{\prod}\gamma,\\ \gamma\in M\cap D(\sigma')}}(-1)^{|\sigma'|}P_{\sigma'},\quad(4.31)$$

where the order of adjacent transpositions γ *in the product for* σ' *in the right hand side is independent of the chosen order of* $\beta\in\Gamma$ *in the left hand side.*

We are not able to prove this statement directly. However, it appeared to hold for any particular example for 3- and 4-particle cases and we have to leave it as a conjecture. We illustrate that the statement of *Lemma 4* holds on non-trivial case $n=3$, $m=2$.
For this case, we have to evaluate the sum of operators in the right hand side for each of three possible sets of two index pairs.

$$\begin{aligned}1)\ M=\{(12),(13)\}:\ &\sigma'=\mathbb{1}\Rightarrow D(\sigma')\cap M=\emptyset;\\ &\sigma'=(12)(123)=(213)\Rightarrow D(\sigma')\cap M=\{(12)\};\\ &\sigma'=(13)(12)(123)=(231)\Rightarrow D(\sigma')\cap M=\{(12),(13)\}.\end{aligned}$$

Since any of other combinations of $(12),(13)$, namely (13) and $(12)(13)$ are not adjacent, then the full sum of operators is $\mathbb{1}-P_{12}+P_{13}P_{12}$.

$$\begin{aligned}2)\ M=\{(12),(23)\}:\ &\sigma'=\mathbb{1}\Rightarrow D(\sigma')\cap M=\emptyset;\\ &\sigma'=(12)(123)=(213)\Rightarrow D(\sigma')\cap M=\{(12)\};\\ &\sigma'=(23)(123)=(132)\Rightarrow D(\sigma')\cap M=\{(23)\}.\end{aligned}$$

Since any of other combinations of $(12),(23)$, namely $(12)(23)$ and $(23)(12)$ are not adjacent, then the full sum of operators is $\mathbb{1}-P_{12}-P_{23}$.

$$\begin{aligned}3)\ M=\{(13),(23)\}:\ &\sigma'=\mathbb{1}\Rightarrow D(\sigma')\cap M=\emptyset;\\ &\sigma'=(23)(123)=(132)\Rightarrow D(\sigma')\cap M=\{(23)\};\\ &\sigma'=(13)(23)=(312)\Rightarrow D(\sigma')\cap M=\{(13),(23)\}.\end{aligned}$$

Since any of other combinations of $(13),(23)$, namely (13) and $(23)(13)$ are not adjacent, then the full sum of operators is $\mathbb{1}-P_{23}+P_{13}P_{23}$.

According to (4.31), the left hand side is equal to

$$B_{12}(\mathbb{1}-P_{12})B_{13}(\mathbb{1}-P_{13})+B_{12}(\mathbb{1}-P_{12})B_{23}(\mathbb{1}-P_{23})+B_{13}(\mathbb{1}-P_{13})B_{23}(\mathbb{1}-P_{23}).$$

According to recent evaluations, the right hand side is equal to:

$$B_{12}B_{13}(\mathbb{1}-P_{12}+P_{13}P_{12})+B_{12}B_{23}(\mathbb{1}-P_{12}-P_{23})+B_{13}B_{23}(\mathbb{1}-P_{13}+P_{13}P_{23}).$$

After cancellation of some pairs of the same terms for both sides we deduce the right hand side from the left hand side to get:

$$\big[-B_{12}P_{12}B_{13}(\mathbb{1}-P_{13})-B_{12}B_{13}P_{13}-B_{12}P_{12}B_{23}(\mathbb{1}-P_{23})-B_{13}P_{13}B_{23}(\mathbb{1}-P_{23})\big]$$
$$+\big[B_{12}B_{13}P_{12}-B_{12}B_{13}P_{13}P_{12}+B_{12}B_{23}P_{12}+B_{13}B_{23}P_{13}P_{23}\big].$$

Grouping the terms with the same transposition operators, we rearrange the expression and then use the equalities

$$B_{12}=-B_{21},\;\; B_{12}B_{23}=B_{12}B_{13}+B_{13}B_{23},\;\; P_{12}P_{13}=P_{13}P_{23},\;\; P_{12}P_{23}=P_{13}P_{12}$$

to obtain:

$$(-B_{12}B_{13}P_{13}-B_{13}B_{21}P_{13})+(-B_{12}B_{23}P_{12}-B_{12}B_{13}P_{12}+B_{12}B_{13}P_{12}+B_{12}B_{23}P_{12})$$
$$+(B_{12}B_{23}P_{12}P_{13}+B_{13}B_{21}P_{13}P_{23}-B_{13}B_{23}P_{13}P_{23})+(B_{12}B_{13}P_{12}P_{23}-B_{12}B_{13}P_{13}P_{12})=0.$$

The possible way of proving *Lemma 4* in future is similar to the proof of *Lemma 3* by induction on n.

Using (4.25) we get

Corollary of Lemma 4
The following identity holds:

$$\prod_{\beta\in D_n}^{\rightarrow}(\mathbb{1}-B_\beta(\mathbb{1}-P_\beta))=\sum_{\sigma'\in S_n}\left\{\sum_{\substack{\Gamma\subseteq D_n:\ \sigma'=\overset{\rightarrow}{\prod}\gamma,\\ \gamma\in\Gamma\cap D(\sigma')}}\left(\prod_{\gamma\in\Gamma}B_\gamma\right)\left(\prod_{\alpha\notin D(\sigma')}A_\alpha\right)\right\}P_{\sigma'}. \tag{4.32}$$

We illustrate that the statement of *Corollary of Lemma 4* holds, as follows. According to (3.33), the corresponding expression for $n=3$, reads:

$$\prod_{\beta\in D_3}^{\rightarrow}(A_\beta+B_\beta P_\beta)=A_{12}A_{13}A_{23}+B_{12}A_{13}A_{23}P_{12}+B_{23}A_{13}A_{12}P_{23} \tag{4.33}$$
$$+B_{23}B_{13}A_{12}P_{13}P_{23}+B_{12}B_{13}A_{23}P_{13}P_{12}+B_{13}(1+B_{12}B_{23})P_{13}$$

Comparing this with (4.32) we conclude that (4.32) holds at least for $n = 3$.

The diagonal element of (4.24) corresponds to $\sigma' = \mathbb{1} \Rightarrow \Gamma = \emptyset \Rightarrow D(\sigma') = \emptyset$ in (4.32). It follows for this case $\prod_{\alpha \notin D(\sigma')} A_\alpha = \prod_{\alpha \in D_n} A_\alpha$ and cancels out with the denominator $\prod_{\beta \in D_n} A_\beta$ of (4.24). Now consider coefficient for an arbitrary off-diagonal term of (4.24). According to (4.32) it is equal (up to the sign) to:

$$\frac{\left(\prod\limits_{\gamma \in \Gamma} B_\gamma\right)\left(\prod\limits_{\alpha \notin D(\sigma')} A_\alpha\right)}{\prod\limits_{\beta \in D_n} A_\beta} = \frac{1}{\prod\limits_{\beta \in (D(\sigma') \setminus \Gamma)} A_\beta} \left(\prod_{\gamma \in \Gamma} \frac{B_\gamma}{A_\gamma}\right), \tag{4.34}$$

because all the coefficients in the numerator are different, so that for each pair of indices in the numerator it is the same pair of indices in the denominator. For $D(\sigma') \neq \emptyset$ there is at least one pair of indices $\gamma =: (r_2 r_1) \in \Gamma$, $r_2 < r_1$, in the last product of (4.34), such that also some P_γ and hence $P_{\sigma'}$ transposes that indices in y-wave. Substituting this result in (4.24) we obtain

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi(1...n) = \chi(1...n) \Bigg\{ [1...n](x-y) \tag{4.35}$$

$$+ \sum_{\substack{\sigma' \in S_n: \\ D(\sigma') \neq \emptyset}} \; \sum_{\substack{\Gamma \subseteq D_n:\, \sigma' = \overleftarrow{\prod}\gamma, \\ \gamma \in \Gamma \cap D(\sigma')}} \frac{1}{\prod\limits_{\beta \in (D(\sigma') \setminus \Gamma)} A_\beta} \left(\prod_{\gamma \in \Gamma} \frac{B_\gamma}{A_\gamma}\right) [1...n](x)\, P_{\sigma'}\, [1...n](-y) \Bigg\}.$$

4.6 Rearrangement of the terms of $\widetilde{\psi_{in}}(x, y \mid k)$ according to the regions of momentum space for an arbitrary coordinate region $\chi^x(\sigma^x)\, \chi^y(\sigma^y)$

To extend the result given by (4.24) for $\chi(n...1)$ to an arbitrary coordinate region $\chi^x(\sigma^x)\, \chi^y(\sigma^y)$ we insert it and also (with use of (4.19)) substitute $Z^+(\sigma^y)$ in (4.20):

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi^x(\sigma^x)\, \chi^y(\sigma^y) = \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, [1...n](x)$$

$$\times \frac{1}{\prod\limits_{\beta\in D_n} A_\beta} \prod_{\beta\in D_n}^{\rightarrow} (A_\beta + B_\beta P_\beta) \left(\prod_{\alpha\in D(\sigma^y)}^{\rightarrow} \tau_\alpha \right) \prod_{\alpha\in D(\sigma^y)}^{\leftarrow} (A_\alpha + B_\alpha P_\alpha)\, [1...n](-y)$$

$$= \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, [1...n](x) \frac{1}{\prod\limits_{\beta\in D_n} A_\beta} \prod_{\beta\in D_n}^{\rightarrow} (A_\beta + B_\beta P_\beta) \prod_{\alpha\in D(\sigma^y)}^{\leftarrow} (\overline{A_\alpha} + \overline{B_\alpha} P_\alpha)\, [1...n](-y), \tag{4.36}$$

because the operator $\prod\limits_{\alpha\in D(\sigma^y)}^{\leftarrow} \tau_\alpha$ does not change the phase of y-wave but reverses the order of indices in each pair $\alpha = (\alpha_i\alpha_j)$ and $A_{\alpha_j\alpha_i} = \overline{A_{\alpha_i\alpha_j}} = \overline{A_\alpha}$. For further rearrangement of (4.36) we consider combined transformation over $\beta \in D_n$ and $\alpha \in D(\sigma^y)$ as follows.

1) Transformation from initial region $\chi^y(1...n)$ to any arbitrary region $\chi^y(\sigma^y)$ performed as a sequence of elementary transformations over $\alpha \in D(\sigma^y)$ with $\#D(\sigma^y) = l(\sigma^y)$. For given σ^y we can choose and fix any inverse ordering $l, ..., 1$ of coordinate transpositions $\Gamma(\sigma^y) := \alpha_1...\alpha_l$ within restriction on transpositions of only adjacent coordinate pairs. Applying $\Gamma'(\sigma') := \beta_m...\beta_1$ for momentum transpositions such that $\beta_i = \alpha_i$ up to $m = l(\sigma^y)$, and with the pairs of only adjacent momenta, we perform transformation from initial momentum region $\theta(1...n)$ to some momentum region $\theta(\sigma')$ and then a further transformation $\theta(\sigma') \mapsto \theta(n...1)$. The idea of such choice is that the first l elementary transpositions of β are in inverse order to all l elementary transpositions of α. This choice is always possible due to the choice of $\Gamma'(n...1)$, as it follows from (4.24).

2) We decompose the maximal set of all transposed pairs:
$D_n = D(\sigma^y(n...1)) \cup D(\sigma^y)$, where $\sigma^y = \prod\limits_{\alpha\in D(\sigma^y)}^{\leftarrow} \tau_\alpha$.

For example, if $n = 5$, $\sigma^y = (35)(12)(45)$, $D(\sigma^y) = \{(12), (35), (45)\}$,
$D_n = D(54321) = \{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\}$, then
$D(\sigma^y(n...1)) = D((12)(35)(45)(54321)) = D(43512)$
$= \{(13), (14), (15), (23), (24), (25), (34)\} = D(54321) \backslash D(\sigma^y)$.

As a result of this consideration we replace

$$\prod_{\beta\in D_n}^{\rightarrow} (A_\beta + B_\beta P_\beta) \prod_{\alpha\in D(\sigma^y)}^{\leftarrow} (\overline{A_\alpha} + \overline{B_\alpha} P_\alpha) \quad \text{by} \quad \prod_{\beta\in D(\sigma^y(n...1))}^{\rightarrow} (A_\beta + B_\beta P_\beta),$$

because $(A_\beta + B_\beta P_\beta)(\overline{A_\alpha} + \overline{B_\alpha} P_\alpha) = \mathbb{1}$ for $\beta = \alpha$, so that everything outside the set $D(\sigma^y(n...1))$ is canceled out by this identity.

It follows that

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi^x(\sigma^x)\, \chi^y(\sigma^y)$$

$$= \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, [1...n](x)\, \frac{1}{\prod\limits_{\beta\in D_n} A_\beta} \prod_{\beta\in D(\sigma^y(n...1))}^{\rightarrow} (A_\beta + B_\beta P_\beta)\, [1...n](-y)$$

$$= \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, [1...n](x)\, \frac{1}{\prod\limits_{\beta\in D(\sigma^y)} A_\beta}\, \frac{1}{\prod\limits_{\beta\in D(\sigma^y(n...1))} A_\beta} \prod_{\beta\in D(\sigma^y(n...1))}^{\rightarrow} (A_\beta + B_\beta P_\beta)\, [1...n](-y). \tag{4.37}$$

Since in the case of $\chi(1...n)$, $D(\sigma^y) = \emptyset$, then comparing the result of (4.37) with (4.24) we observe that in the case of arbitrary x- and y-coordinate regions, $\widetilde{\psi_{in}}$ always contains the same factor

$$\frac{1}{\prod\limits_{\beta\in D_1} A_\beta} \prod_{\beta\in D_1}^{\rightarrow} (A_\beta + B_\beta P_\beta), \tag{4.38}$$

where $D_1 := D(\sigma^y(n...1)) = D_n \backslash D(\sigma^y) \Rightarrow D_1 \subseteq D_n$. The numerator of (4.38)

$$\prod_{\beta\in D_1}^{\rightarrow} (A_\beta + B_\beta P_\beta) = \prod_{\beta\in D_1}^{\rightarrow} (\mathbb{1} - B_\beta + B_\beta P_\beta) = \prod_{\beta\in D_1}^{\rightarrow} (\mathbb{1} - B_\beta(\mathbb{1} - P_\beta)) \tag{4.39}$$

is equal to the sum of terms of the form $\prod\limits_{\beta\in\Gamma\subseteq D_1} [-B_\beta(\mathbb{1} - P_\beta)]$, and after substitution to (4.37) we get the expression for the case of arbitrary coordinate region $\chi^x(\sigma^x)\, \chi^y(\sigma^y)$:

$$\widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\, \chi^x(\sigma^x)\, \chi^y(\sigma^y) = \chi^x(\sigma^x)\, \chi^y(\sigma^y)\, [1...n](x) \tag{4.40}$$
$$\times \frac{1}{\prod\limits_{\beta\in D(\sigma^y)} A_\beta} \left\{ \mathbb{1} + \sum_{\substack{\sigma'\in S_n:\\ D(\sigma')\neq\emptyset}} \sum_{\substack{\Gamma\subseteq D_1:\ \sigma'=\overleftarrow{\prod}\gamma,\\ \gamma\in\Gamma\cap D(\sigma')}} \frac{1}{\prod\limits_{\beta\in(D(\sigma')\backslash\Gamma)} A_\beta} \left(\prod_{\gamma\in\Gamma} \frac{B_\gamma}{A_\gamma} \right) P_{\sigma'} \right\} [1...n](-y).$$

As in the case of $\chi(1...n)$, for any off-diagonal term, at least one pair of indices $\gamma =: (r_2 r_1) \in \Gamma$, $r_2 < r_1$, is such that $P_{\sigma'}$ transposes the corresponding pair of momenta in y-wave.

4.7 Evaluation of integrals.

We need evaluate the integral (4.1). For the case restricted by $\chi(1...n)$ after substitution of diagonal element from (4.35) we get

$$\chi(1...n)\int\frac{d^nk}{(2\pi)^n}\,[1...n](x-y) \tag{4.41}$$

$$=\chi(1...n)\int\frac{d^nk}{(2\pi)^n}\,\exp\{i[k_1(x_1-y_1)+...+k_n(x_n-y_n)]\}=\chi(1...n)\,\delta^n(x-y).$$

For evaluation of any integral with off-diagonal element fix any arbitrary $\Gamma\subseteq D_n$. Then for any $\sigma'\neq \mathbb{1}$ there exists at least one pair of indices $\gamma:=(r_2r_1)$, $r_2<r_1$, such that $(r_2r_1)\in\Gamma$,
$P_{r_2r_1}\,[1...r_2...r_1...n](-y)=[1...r_1...r_2...n](-y)$ and also
$P_{\sigma'}\,[1...r_2...r_1...n](-y)=[\sigma'(1)...r_1...r_2...\sigma'(n)](-y)$,
interchanging relative order of r_2,r_1. Then after applying the operator $P_{\sigma'}$ to y-component of the wave, an integral for an arbitrary off-diagonal element can be expressed as

$$\chi(1...n)\int\frac{d^nk}{(2\pi)^n}\,\frac{1}{\prod\limits_{\beta\in(D(\sigma')\setminus\Gamma)}A_\beta}\,\frac{[1...r_2...r_1...n](x)[\gamma_1...r_1...r_2...\gamma_n](-y)}{\exp\left[i\big(k_{r_2}(x_{r_2}-y_{t_2})+k_{r_1}(x_{r_1}-y_{t_1})\big)\right]}$$

$$\times\left[\left(\prod_{(r_2r_1)\neq\gamma\in\Gamma}\frac{B_\gamma}{A_\gamma}\right)\frac{B_{r_2r_1}}{A_{r_2r_1}}\exp\{i[k_{r_2}(x_{r_2}-y_{t_2})+k_{r_1}(x_{r_1}-y_{t_1})]\}\right]. \tag{4.42}$$

Note, that the position numbers of momentum indices r_2 and r_1 are, respectively, r_2 and r_1 for the x-wave but generally there are the numbers t_1 and t_2 of positions for the momentum indices r_1 and r_2 in the y-wave, according to the composition of transpositions $\overleftarrow{\prod\limits_{\gamma\in\Gamma}}P_\gamma$. The exponent in the numerator $[1...r_2...r_1...n](x)[\gamma_1...r_1...r_2...\gamma_n](-y)$, where $r_1=\gamma_{t_1}$ and $r_2=\gamma_{t_2}$, contains, in particular, the same term $k_{r_2}(x_{r_2}-y_{t_2})+k_{r_1}(x_{r_1}-y_{t_1})$ in its argument as the denominator and therefore can be cancelled out, leaving such term only in square brackets. Then the integral can be decomposed into external and internal integration with the change of variables and following rearrangement:

$$k_{r_2}(x_{r_2}-y_{t_2})+k_{r_1}(x_{r_1}-y_{t_1})$$
$$=q_1[(x_{r_2}-x_{r_1})+(y_{t_1}-y_{t_2})]+q_2[(x_{r_2}+x_{r_1})-(y_{t_1}+y_{t_2})],$$

where $2q_1:=k_2-k_1$, $2q_2:=k_2+k_1$. After substituting the value of

$$\frac{B_{r_2r_1}}{A_{r_2r_1}}=\frac{-ic}{k_{r_2}-k_{r_1}}\,\frac{1}{1-(-ic)/(k_{r_2}-k_{r_1})}=\frac{-ic}{k_{r_2}-k_{r_1}+ic}=\frac{-ic}{2q_1+ic}\quad\text{and}$$

$$\frac{B_\gamma}{A_\gamma}=\frac{-ic}{q_\gamma}\,\frac{1}{1-(-ic)/q_\gamma}=\frac{-ic}{q_\gamma+ic},$$

the internal integral in (4.42) becomes equal (up to a constant coefficient) to

$$\int dq_2 \, \exp\left\{i\big[q_2((x_{r_2}+x_{r_1})-(y_{t_1}+y_{t_2}))\big]\right\} \int \left(\prod_{(r_2r_1)\neq\gamma\in\Gamma} \frac{1}{q_\gamma+ic} \right) \frac{dq_1 e^{iqz}}{q_1+ic/2}, \tag{4.43}$$

where each γ adds at most one pole with negative imaginary part.

$z=(x_{r_2}-x_{r_1})+(y_{t_1}-y_{t_2})>0$ in the region $\chi(1...n)$, because

$\chi(1...n)=\chi^x(1...r_2...r_1...n)\,\chi^y(1...t_1...t_2...n)$

$\equiv \chi(x_1>...>x_{r_2}>...>x_{r_1}...>x_n)\,\chi(y_1>...>y_{t_1}>...>y_{t_2}...>y_n) \Rightarrow$

$\Rightarrow \; x_{r_2}>x_{r_1},\; y_{t_1}>y_{t_2} \;\Rightarrow\; (x_{r_2}-x_{r_1})+(y_{t_1}-y_{t_2})>0.$

Set $q_1=q$, $c/2=b$ and the last integral in (4.43) reduces to

$$\int \left(\prod_{(r_2r_1)\neq\gamma\in\Gamma} \frac{1}{q_\gamma+ic} \right) \frac{dq\, e^{iqz}}{q+ib}=0, \tag{4.44}$$

because of only negative imaginary parts for all poles and since

$$\int \frac{dq\, e^{iqz}}{q+ib}=0,$$

as was shown for 2- and 3-particle case. Due to the arbitrary choice for off-diagonal element it follows that

$$\frac{1}{(2\pi)^n}\int d^nk\, \widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n)\,\chi(1...n)=\chi(1...n)\,\delta^n(x-y). \tag{4.45}$$

Next we consider the region $\chi^x(1...n)\,\chi^y(\sigma^y)$. The integral of the first term in (4.40) is equal to

$$\chi^x(1...n)\,\chi^y(\sigma^y)\int \frac{d^nk}{(2\pi)^n}\frac{1}{\prod\limits_{\alpha\in D(\sigma^y)} A_\alpha}[1...n](x-y)$$
$$=\chi^x(1...n)\,\chi^y(\sigma^y)\int \frac{d^nk}{(2\pi)^n}\prod_{\alpha\in D(\sigma^y)}\left(1+\frac{B_\alpha}{A_\alpha}\right)[1...n](x-y). \tag{4.46}$$

The first integral in (4.46) is equal $\delta^n(x-y)$. Any other term of the integrand in (4.46) contains at least one factor of the form B_α/A_α. Let $\alpha=(r_1r_2)$, where $r_1<r_2$ by assumption for index pairs. After substituting the value of

$$\frac{B_{r_1r_2}}{A_{r_1r_2}}=\frac{-ic}{k_{r_1}-k_{r_2}}\,\frac{1}{1-(-ic)/(k_{r_1}-k_{r_2})}=\frac{-ic}{k_{r_1}-k_{r_2}+ic}=\frac{-ic}{2q_1+ic},$$

where $k_{r_1} - k_{r_2} =: 2q_1$, $k_{r_1} + k_{r_2} =: 2q_2$, and using
$k_1(x_{r_1} - y_{r_2}) + k_2(x_{r_2} - y_{r_1}) = \{q_1[(x_{r_1} - x_{r_2}) + (y_{r_2} - y_{r_1})] + q_2[(x_{r_1} + x_{r_2}) - (y_{r_2} + y_{r_1})]\}$, integral of any term in (4.46) containing the product of the factors B_α/A_α for the set $D_\alpha \subseteq D(\sigma^y)$ becomes (up to a constant coefficient)

$$\int dq_2 \exp\left[i\big(q_2((x_{r_1} + x_{r_2}) - (y_{r_1} + y_{r_2}))\big)\right] \int \left(\prod_{\alpha \in D_\alpha \subseteq D(\sigma^y)} \frac{1}{q_\alpha + ic} \right) \frac{dq_1 e^{iqz}}{q_1 + ic/2}, \tag{4.47}$$

where $z = (x_{r_1} - x_{r_2}) + (y_{r_2} - y_{r_1}) > 0$ because for $r_1 < r_2$, $x_{r_1} > x_{r_2}$ in the region $\chi^x(1...n)$ and $y_{r_2} > y_{r_1}$ in the region $\chi^y(\sigma^y)$, since $\chi^y(\sigma^y) = \left(\overleftarrow{\prod_{\alpha \in D(\sigma^y)}} P_\alpha \right) \chi^y(1...n)$.
It follows that internal integral in (4.47) is equal 0, similar to the previous case.

The integral of any off-diagonal term in (4.40) contains, additionally to the case of (4.47), the product $\left(\prod_{\beta \in (D(\sigma') \backslash \Gamma)} A_\beta \right)^{-1} \left(\prod_{\gamma \in \Gamma} (B_\gamma / A_\gamma) \right)$ and a corresponding momentum transposition operator $\overleftarrow{\prod_{\gamma \in \Gamma}} P_\gamma$ which inverts the order of momenta in each momentum pair γ of y-wave. The difference with the case of (4.47) is only with $z = (x_{r_1} - x_{r_2}) + (y_{t_2} - y_{t_1})$, because the operator $\overleftarrow{\prod_{\gamma \in \Gamma}} P_\gamma$ generally changes the position of momentum indices k_{r_2}, k_{r_1}: $(y_{r_2}, y_{r_1}) \mapsto (y_{t_2}, y_{t_1})$, however it does not invert the relative order of these indices, due to the restriction $\gamma \in (D_n \backslash D(\sigma^y))$. Hence $z > 0$, and since, as it shown in previous Section, there exists at least one pair of indices $\gamma := (r_2 r_1)$, $r_2 < r_1$, such that $(r_2 r_1) \in \Gamma$, then there exists a corresponding coefficient $B_\gamma = B_{r_2 r_1}$ in the numerator of the product $\prod_{\gamma \in \Gamma} (B_\gamma / A_\gamma)$. It follows that integral for any off-diagonal term in (4.40) is equal 0. Summing the results of integration over all terms of (4.40) we obtain

$$\frac{1}{(2\pi)^n} \int d^n k \, \widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n) \, \chi^x(1...n) \, \chi^y(\sigma^y) = \chi^x(1...n) \, \chi^y(\sigma^y) \, \delta^n(x - y).$$

Since $\chi^y(\sigma^y)$ is chosen arbitrary, then applying transposition operator τ_{σ^x} for an arbitrary σ^x to $\chi^x(1...n)\, \chi^y(\sigma^y)$ we obtain the result for all possible combinations of $\chi^x(\sigma^x)$ and $\chi^y(\sigma^y)$ so that for any arbitrary coordinate region $\chi^x(\sigma^x)\, \chi^y(\sigma^y)$,

$$\frac{1}{(2\pi)^n} \int d^n k \, \widetilde{\psi_{in}}(x_1...x_n, y_1...y_n \mid k_1...k_n) \, \chi^x(\sigma^x) \, \chi^y(\sigma^y) = \chi^x(\sigma^x) \, \chi^y(\sigma^y) \, \delta^n(x - y).$$

Summation over all coordinate regions $\chi^x(\sigma^x)\, \chi^y(\sigma^y)$ gives (4.1).

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