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Reissner–Nordström expansion

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Abstract

We propose a classical mechanism for the cosmic expansion during the radiation-dominated era. This mechanism assumes that the Universe is a two-component gas. The first component is a gas of ultra-relativistic ‘‘normal’’ particles described by an equation of state of an ideal quantum gas of massless particles. The second component consist of ‘‘unusual’’ charged particles (namely, either with ultra-high charge or with ultra-high mass) that provide the important mechanism of expansion due to their interaction with the ‘‘normal’’ component of the gas. This interaction is described by the Reissner–Nordström metric purely geometrically – the ‘‘unusual’’ particles are modeled as zero-dimensional naked singularities inside spheres of gravitational repulsion. The radius of a repulsive sphere is inversely proportional to the energy of an incoming particle or the temperature. The expansion mechanism is based on the inflating of the ‘‘unusual’’ particles (of charge Q) with the drop of the temperature – this drives apart all neutral particles and particles of specific charge q/m such that $\mathrm{sign}(Q)q/m \geqslant -1$. The Reissner–Nordström expansion naturally ends at recombination. We discuss the range of model parameters within which the proposed expansion mechanism is consistent with the restrictions regarding quantum effects.

Keywords: Expansion of the Universe; Radiation-dominated era; Reissner–Nordström; Naked singularity

We propose a classical mechanism for the expansion of the Universe during the radiation-dominated era. Various mechanisms have been proposed for the cosmic expansion. It is generally accepted that a scalar field drove the inflation of the Universe (see, for example [1] and the references therein) and that the expansion during the inflation was exponential or power law [2].

Brisudova et al. [3] considered a cosmological model with a complex scalar field, minimally coupled to a $U(1)$ gauge field. The expansion of the Universe was generated by a long-range repulsive force resulting from the endowment of the photon with a mass depending on the scalar field.

Our model is based on the assumption that the expanding Universe is a two-component gas. The first component is a gas of ultra-relativistic ‘‘normal’’ particles described by an equation of state of an ideal quantum gas of massless particles. We show that the expansion of the Universe could be due to the second fraction of the gas – a component consisting of ‘‘unusual’’ charged particles (namely, either with ultra-high charge or with ultra-high mass) that provide the important mechanism of expansion due to their interaction with the ‘‘normal’’ component of the gas. The ‘‘unusual’’ particles are naked singularities and the interaction mechanism is the gravitational repulsion of the naked singularities. Naked singularities are particles of charge Q greater than their mass M (in geometrized units

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$G = 1 = c$) and we model them as zero-dimensional Reissner–Nordström [4,5] gravitational entities surrounded by spheres of gravitational repulsion [6]. In our picture, the Universe has local Reissner–Nordström geometry, but globally, the geometry is that of Robertson–Walker [5,7]. Namely, we confine our attention to the local spherical neighbourhood of a single naked singularity and consider the Universe as multiple copies (fluid) of such neighbourhoods. We show that the radii of the repulsive spheres grow in inverse proportion with the temperature – the "unusual" particles "grow" as the temperature drops and drive away the "normal" fraction of the Universe. This repulsion results in power law expansion with scale factor $a(\tau) \sim \sqrt{\tau}$, corresponding to the expansion during the radiation-dominated era. The gravitational repulsion is not powerful enough to achieve accelerated expansion that solves the horizon problem and thus account for the inflation of the Universe [$a(\tau) \sim e^{H\tau}$ or $a(\tau) \sim \tau^n$, with $n > 1$], unless charge non-conservation is involved.

We determine a particle's "radius" by calculating the turning radius of a radially moving incoming (charged) test particle of ultra-high energy $kT \gg mc^2$ (where m is the test particle's rest mass). The proposed model is simplified significantly by considering the incoming particles as collisionless probes rather than involving their own gravitational fields and by not considering the more general and physically more relevant Kerr–Newman geometry [5,8,9], thus ignoring magnetic effects caused by rotation of the centre, which drags the inertial reference frames, and the particles' spins.

In 1971, Hawking suggested [10] that a large number of gravitationally collapsed charged objects of very low mass (of the order of Planck's mass) were formed in the early Universe. Hawking argues that gravitational collapse is, essentially, a classical process and a black hole cannot form if its Schwarzschild radius is smaller than the Planck length $(Ghc^{-3})^{1/2} \sim 10^{-35}$ m (or its mass – smaller than 10^{-8} kg), since, at Planck lengths, quantum fluctuations of the metric become relevant. However, for lengths larger than 10^{-35} m, it is legitimate to ignore quantum gravitational effects and treat the metric classically. One would expect that a collapsed object could form if the Schwarzschild radius is greater than the Compton wavelength $h/(mc)$ of one of the elementary particles which formed it. This corresponds to a minimum mass of 10^{11} kg. However, Hawking argues further, this is not the case, since the Compton wavelength of a photon is infinite, yet a sufficient concentration of electromagnetic radiation can cause gravitational collapse. Hawking suggests that the wavelength to be considered should not be the wavelength at rest, but hc/E, where E is the typical energy of the particle. For ultra-relativistic particles, $E \sim kT \gg mc^2$.

We will show that the "radius" of the "unusual" particles is inversely proportional to the temperature of the Universe: $r_0(T) = Q(q + m)/(kT)$ (where Q is the charge of the "unusual" particles, m and q are the mass and charge of the "normal" particles, respectively). This is the characteristic length that is to be considered, instead of a Schwarzschild radius, and compared to the wavelength $\lambda(T) \sim hc/(kT)$. Our classical model is applicable only when quantum gravitational effects do not take place between the "unusual" particles and the "normal" particles. Therefore, the range of validity of the model is determined by the condition that the "radius" $r_0(T)$ of an "unusual" particle exceeds the wavelength $\lambda(T)$. Thus, for a "normal" particle of typical charge e (the electron's charge), the repulsive centre must have charge Q higher than h/e (in geometrized units). If quantum gravitational effects between the "unusual" particles and neutral "normal" particles are to be negligible, the repulsive centre must have charge Q higher than h/m (again, in geometrized units). It is, understandably, hard to comprehend the conditions under which particles of such ultra-high charge could have formed since, for a gravitational collapse, huge gravitational energy will be needed to overcome the electrical repulsion. The formation of particles of such ultra-high charge is an open issue for the model.

Fortunately, there is an alternative to particles of ultra-high charge. These will be particles of ultra-high mass and charge which is comparable to the electron charge e. Of course, the mass of such particle must not exceed its charge so that a naked singularity treatment exists. This puts an upper limit on the mass M of these particles at 10^{21} electron masses or 10^{-9} kg. As the rest mass of these particles is so huge and comparable to kT for quite high temperature ($\sim 10^{31}$ K), the characteristic length that is to be considered for quantum effects is $\lambda \sim h/M$ (or h/e, as M is of the order of e, without exceeding it). One can immediately determine the temperature below which the expansion mechanism with "unusual" particles of mass $M \sim 10^{-9}$ kg is valid: $T < e^3/(hk) \sim 10^{29}$ K [for wavelength $\lambda \sim h/e$ not larger than the particle's "radius" $r_0(T) \sim e^2/(kT)$].

Since the early Universe was very dense and ultra-relativistic, we may speculate that not two-body, but many-body collisions of "normal" particles led to the production of particles with $Q > M$. This is hardly a quantitative explanation, however, the existence of such ultra-heavy particles has been studied by many authors. Of particular importance is the work of de Rujula et al. [11]. The authors consider that very heavy charged particles (CHAMPS), which have survived annihilation, were produced in the early Universe. These CHAMPS are even viewed as dark matter candidates. Secondly, Preskill has shown [12] that ultra-heavy magnetic monopoles were created so copiously in the early Universe that they outweighed everything else in the Universe by a factor of 10^{12}. Again, these ultra-heavy monopoles can serve as the "unusual" fraction of the two-component gas. This time Q will be the magnetic charge. The other fraction of the gas will then consist of magnetically neutral particles.

Not all particles are repelled by naked singularities. If the specific charge q/m of a probe is such that $\mathrm{sign}(Q)q/m \leqslant -1$, the probe will reach the singularity [6]. In result, the absolute value of the charge Q of the naked singularity will diminish, while the mass M of the naked singularity will increase. If this process is repeated a sufficient number of times, the naked singularity will annihilate – it will pick up a horizon and turn into a black hole. We assume that the "unusual" particles have survived such annihilation or have annihilated at a very slow rate. Thus, as the "radius" of an "unusual" particle grows in inverse proportion with the temperature, the Reissner–Nordström expansion would continue forever. This is not the case however – the Reissner–Nordström expansion ends naturally at recombination. For a neutral "normal" particle, at a certain distance from the "unusual" particle, the gravitational repulsion turns into gravitational attraction, while the gravitational repulsion of a charged "normal" particle extends to infinity (the gravitational attraction is not sufficiently strong to overcome the electrical repulsion) [6]. At recombination, charged particles – which have so far been being repelled by the naked singularities – combine to form neutral particles. These neutral particles are formed sufficiently far from the naked singularities – beyond the region of gravitational repulsion – and this stops the Reissner–Nordström expansion mechanism.

We consider the general motion of a particle in Kerr–Newman geometry [8,9,5]. The Kerr–Newman metric in Boyer–Lindquist coordinates [13] and geometrized units is given by

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 + \frac{\sin^2\theta}{\rho^2}\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2, \quad (1)$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \quad (2)$$

$$\rho^2 = r^2 + a^2\cos^2\theta. \quad (3)$$

In the above, M is the mass of the centre, $a > 0$ – the specific angular momentum of the centre (i.e. angular momentum per unit mass) and Q – the charge of the centre. The motion of a particle of mass m and charge q in gravitational and electromagnetic fields is governed by the Lagrangian:

$$L = \frac{1}{2}g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} - \frac{q}{m}A_i\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}. \quad (4)$$

In the above, λ is the proper time τ per unit mass m: $\lambda = \tau/m$ and A is the vector electromagnetic potential, determined by the charge Q and specific angular momentum a of the centre:

$$A_i\mathrm{d}x^i = -\frac{Qr}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi). \quad (5)$$

(The magnetic field is due to the dragging of the inertial reference frames into rotation around the centre.)

The equations of motion for the particle are

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = \frac{q}{m}F^i_{\ j}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}, \quad (6)$$

where $F = \mathrm{d}A$ is Maxwell's electromagnetic tensor and Γ^i_{jk} are the Christoffel symbols. For Kerr–Newman geometry, the geodesic equations (6) can be written as [14] (see also [15])

$$\rho^2\frac{\mathrm{d}t}{\mathrm{d}\lambda} = -a^2E\sin^2\theta + aJ + \frac{r^2+a^2}{\Delta}\left[E(r^2+a^2) - Ja - qQr\right], \quad (7)$$

$$\rho^2\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \pm\sqrt{\left[E(r^2+a^2) - Ja - qQr\right]^2 - \Delta\left[m^2r^2 + (J-aE)^2 + K\right]}, \quad (8)$$

$$\rho^2\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = \pm\sqrt{K - \cos^2\theta\left[a^2(m^2-E^2) + \frac{1}{\sin^2\theta}J^2\right]}, \quad (9)$$

$$\rho^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = -aE + \frac{J}{\sin^2\theta} + \frac{a}{\Delta}\left[E(r^2+a^2) - Ja - qQr\right], \quad (10)$$

where $E = (1/m)\partial L/\partial\dot{t}$ is the conserved energy of the particle, $J = (1/m)\partial L/\partial\dot{\phi}$ is the conserved projection of the particle's angular momentum on the axis of the centre's rotation (dots denote differentiation with respect to τ). K is another conserved quantity given by

$$K = p_\theta^2 + \cos^2\theta\left[a^2(m^2-E^2) + \frac{1}{\sin^2\theta}J^2\right]. \quad (11)$$

Here $p_\theta = (1/m)\partial L/\partial\dot{\theta}$ is the θ-component of the particle's four-momentum.

For simplicity, we will confine our attention to the radial motion of a charged test particle in Reissner–Nordström [4] geometry (see [6] for a very thorough analysis). In other words, we will request $\dot{\theta} = 0 = \dot{\phi}$ and also set $a = 0$.

Eq. (8) then reduces to

$$\dot{r}^2 = r^{-2}\left[(\epsilon^2-1)r^2 + 2\left(1 - \frac{q}{m}\frac{Q}{M}\epsilon\right)Mr + \left(\frac{q^2}{m^2} - 1\right)Q^2\right], \quad (12)$$

where $\epsilon = E/m$ is the specific energy of the particle. Motion is possible only if $\dot{r}^2$ is non-negative. This implies that the radial coordinate of the test particle must necessarily be outside the region (r_-, r_+) where the turning radii $r_\pm$ are given by [6]

$$r_\pm = \frac{M}{\epsilon^2-1}\left[\epsilon\frac{q}{m}\frac{Q}{M} - 1 \pm\sqrt{\left(\epsilon\frac{q}{m}\frac{Q}{M} - 1\right)^2 - (1-\epsilon^2)\left(1 - \frac{q^2}{m^2}\right)\frac{Q^2}{M^2}}\right]. \quad (13)$$

The loci of the event horizon and the Cauchy horizon for the Reissner–Nordström geometry are

$$R_\pm = M\left(1 \pm \sqrt{1 - \frac{Q^2}{M^2}}\right), \quad (14)$$

respectively. For a particle as the electron, $Q/M \sim 10^{21}$. Such solution does not have any horizons and is called a naked singularity. Naked singularities exhibit gravitational repulsion and this explains the existence of turning radii. The centre $r = 0$, however, can be reached (see [6] for the analysis) by a suitably charged incoming particle – when $\text{sign}(Q)q/m \leqslant -1$. For a positively charged center for instance ($Q > 0$), a suitably charged particle (with $q/m \leqslant -1$) will hit the naked singularity and, as shown by [6], a naked singularity can be destroyed if sufficient amount of mass and opposite charge are fed into it. While in existence, this positively charged naked singularity will never be reached by incoming particles of small negative charge (i.e. $-1 < q/m < 0$), neutral particles and all positively charged particles (i.e. $q \geqslant 0$). For these particles the centre is surrounded by an impenetrable sphere of radius $r_0(T) = r_+$ given by Eq. (13). The radius of that impenetrable sphere depends on the energy ϵ of the incoming particles. For very high energies ($\epsilon \to \infty$), the particle's "radius" can be written as

$$r_0(T) = \frac{Q}{\epsilon}\left(\frac{q}{m} + 1\right) = \frac{Q(q+m)}{kT}. \tag{15}$$

The expansion mechanism is based on the identification of the "unusual" particles with naked singularities. Thus, the "unusual" particles will have "radii" $r_0(T)$, as determined in Eq. (15). When the temperature starts to drop, the "unusual" particles rapidly increase their "size" in inverse proportion with the temperature and drive apart the "normal" fraction of the gas. Thus the Universe will increase its size in inverse proportion with the temperature: $a(\tau) \sim r_0(\tau) \sim 1/T(\tau)$, where $a(\tau)$ is the scale factor of the Universe. We therefore get the usual relation:

$$aT = \text{const} \tag{16}$$

or

$$H \equiv \frac{\dot{a}}{a} = \frac{\dot{r}_0}{r_0} = -\frac{\dot{T}}{T}. \tag{17}$$

Let us consider the expansion rate equation (see, for example, [16]) without cosmological constant:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{K}{a^2}. \tag{18}$$

In this equation, K is an integration constant related to the spacetime curvature, while the density ρ includes all forms. At the present epoch, the main contribution in ρ comes from ordinary matter. In our model, the main contribution in ρ comes from the electrostatic field of the "unusual" particles. To estimate this contribution, let us consider a probe which is being driven away by an expanding "unusual" particle. That is, the distance between the "unusual" particle and the probe is equal to $r_0(T)$. The intensity of the electric field of the "unusual" particle at this distance is proportional to $Q/r_0^2(T)$. The energy density of the electric field is proportional to the square of the intensity of the electric field, that is, the main contribution to ρ comes from a term proportional to $Q^2/r_0^4(T)$, namely, a term proportional to the fourth power of the temperature (since $r_0(T)$ is inversely proportional to the temperature). Therefore

$$\frac{\dot{a}}{a} \sim T^2. \tag{19}$$

Substituting (17) into this equation, one gets:

$$\frac{\dot{T}}{T} \sim -T^2. \tag{20}$$

The solution to this equation is $T \sim \sqrt{\frac{1}{\tau}}$ and, therefore, $a(\tau) \sim \sqrt{\tau}$ – behaviour corresponding to the expansion during the radiation-dominated era. This is not surprising as the expansion law was derived for a gas of ultra-relativistic "normal" particles (which, essentially, are pure radiation in view of the fact that their energies are much higher than their rest masses – as it is for massless particles), in a fluid of "unusual" particles. (It would be interesting to study the effects which the "growing" naked singularities have on the photons.) One should also note that in the derivation of the expansion law $a(\tau) \sim \sqrt{\tau}$, the charge Q was considered constant (i.e. not changing with time or the temperature).

It is very plausible that the "unusual" particles are not stable and that there are various possible annihilation mechanisms for them. Apart from annihilation into black holes by "capturing" oppositely charged "normal" particles (as proposed by Cohen and Gautreau in [6]), these superheavy charged particles can annihilate through different competing mechanisms: they can themselves recombine into neutral particles or decay before or after that. Of particular importance for such possibilities is the work of Ellis et al. [17] on the astrophysical constraints on massive unstable neutral relic particles and also the work of Gondolo et al. [18] on the constraints of the relic abundance of a dark matter candidate – a generic particle of mass in the range of 1–10^{14} TeV, lifetime greater than 10^{14}–10^{18} years, decaying into neutrinos.

Our assumption however is that the superheavy charged particles have survived annihilation long enough so as to drive the expansion during the radiation-dominated epoch and, possibly, beyond. To estimate the number density of the "unusual" particles during the radiation-dominated epoch, let us recall our assumption that the Universe, even though having globally Robertson–Walker geometry, has local Reissner–Nordström geometry – namely, it is a fluid of naked singularities. It is plausible to assume that these naked singularities are densely packed spheres that fill the entire Universe and drive its expansion. Thus, the volume V of the Universe, at any moment during the Reissner–Nordström expansion, would be of the order of the number N of these particles, times the "volume" of the repulsive sphere of an "unusual" particle: $V \sim N r_0^3(T)$. Therefore, the number density of the "unusual" particles is of the order of $r_0^{-3}(T)$.

Even though our analysis is focused on a purely classical description (in view of the ultra-high masses of the "unusual" particles), one should not be left with the impression that there are no quantum effects in our model. On the contrary – quantum effects are present throughout the entire process – the "normal" fraction is an ideal quantum gas of massless particles. The laws of quantum theory are present in the neighbourhood of the expanding classical objects of ultra-high mass. Of course, one would also expect quantum interactions between the two fractions and between the "unusual" particles themselves. Our point however, is that these quantum interactions would not compete with and prevent the expansion which is due to classical gravitational effects. Study of quantum interactions between all of the types of particles involved and study of the quantum properties of naked singularities in particular is yet to gain momentum. This will, undoubtedly, broaden the range of validity of this expansion model. Certain aspects, for example, of the classical and quantum properties of repulsive singularities have been analyzed in the context of four-dimensional extended supergravity by Gaida et al. – see [19] and the references therein.

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