2005-01-01

On the Field Equations of Kaluza's Theory

Rossen Ivanov  
*Technological University Dublin*

Emil Prodanov  
*Dublin Institute of Technology*, emil.prodanov@dit.ie

Follow this and additional works at: https://arrow.dit.ie/scschmatart

Part of the Cosmology, Relativity, and Gravity Commons

Recommended Citation

https://arrow.dit.ie/scschmatart/81

This Article is brought to you for free and open access by the School of Mathematics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact yvonne.desmond@dit.ie, arrow.admin@dit.ie, brian.widdis@dit.ie.

This work is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 License.
On the field equations of Kaluza’s theory

Rossen I. Ivanov\textsuperscript{a,b}, Emil M. Prodanov\textsuperscript{c}

\textsuperscript{a} School of Mathematics, Trinity College, University of Dublin, Ireland

\textsuperscript{b} Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, Sofia 1784, Bulgaria

\textsuperscript{c} School of Physical Sciences, Dublin City University, Glasnevin, Ireland

Received 18 April 2005; received in revised form 14 July 2005; accepted 14 July 2005

Available online 21 July 2005

Editor: N. Glover

Abstract

The field equations of the original Kaluza’s theory are analyzed and it is shown that they lead to modification of Einstein’s equations. The appearing extra energy–momentum tensor is studied and an example is given where this extra energy–momentum tensor is shown to allow four-dimensional Schwarzschild geometry to accommodate electrostatics. Such deviation from Reissner–Nordström geometry can account for the interpretation of Schwarzschild geometry as resulting not from mass only, but from the combined effects of mass and electric charge, even electric charge alone.

© 2005 Elsevier B.V. All rights reserved.

PACS: 04.50.+h; 04.20.Cv; 03.50.De; 98.80.-k

1. Introduction

Kaluza–Klein’s theory has been an area of extensive research for almost a century [1]. Kaluza’s original theory [2] seems unattractive because of the apparent lack of gauge invariance (it is Klein’s later modification [3] which proposes the gauge invariant version of the theory). The two theories, however, are intrinsically related and dual—these are the slicing and threading decomposition of a five-dimensional spacetime [4]. Therefore, one would expect that the physics in either of these two pictures should somehow induce the physics in the other one. The physics behind the field equations of the original Kaluza’s theory is compared to that of Klein’s theory, as analyzed by Jordan and Thiry [5]. It is shown that in Kaluza’s model, gauge invariance of the “electromagnetic” fields is actually transferred as a gauge freedom to fix the dilaton field (which models Newton’s constant) as needed by experiments. This is the crucial difference between these two dual theories: Klein’s theory fails to achieve a constant solution for the dilaton field since an unwanted constraint for the Maxwell’s tensor appears; Kaluza’s theory sacrifices the gauge...
2. Kaluza–Klein duality

In this section we give an introduction to the slicing–threading duality [4]. The original Kaluza’s theory [2] and Klein’s subsequent modification [3] are dual theories. Namely, this is a duality between decomposition of the five-dimensional spacetime with dimension four surfaces (slicing) and the decomposition of the five-dimensional spacetime with co-dimension four surfaces (threading).

We will explain the decomposition of a bundle metric in terms of the metrics of the base space and the fibre space—see [4] for the details. Let \( M \) be an \((m + n)\)-dimensional fibre bundle with \( m\)-dimensional base space \( B \) (with local coordinates \( x^1, x^2, \ldots, x^m \)), \( n\)-dimensional fibre \( \mathcal{F} \) (with local coordinates \( y^1, y^2, \ldots, y^n \)) and projection \( \pi_1 : M \rightarrow B \). The projection \( \pi_1 \) defines a local trivialization for each neighborhood \( U \) of the point \( x \in B \), \( \pi_1^{-1}(U) \cong U \times \mathcal{F} \) with local coordinates \( (x^1, y^a) \) (in this section only, the indexes \( i, j \) run from 1 to \( m \), the indexes \( a, b \) run from 1 to \( n \) and Greek indexes run from 1 to \( m + n \)). Let \( G \) be a metric, defined on \( TM \otimes TM \). At each point \( P \) of \( M \) there exists a natural subspace, called the vertical subspace \( V_P \subset T_PM \) with basis \( \partial/\partial y^a \). Let us define a horizontal space \( H_P \), complementing \( V_P : T_PM = V_P \oplus H_P \), such that the two are orthogonal with respect to \( G \), i.e., if \( H_1, H_2, \ldots, H_n \) is the basis of \( H_P \), then:

\[
G(H_i, \frac{\partial}{\partial y^a}) = 0, \tag{1}
\]

where \( H_i = \partial/\partial x^i - \Gamma_i^a \partial/\partial y^a \) is a horizontal lift of \( \partial/\partial x^i \). We will use the following shorthand notations:

\[
G_{ij} \equiv G(\partial/\partial x^i, \partial/\partial x^j), \quad G_{ia} \equiv G(\partial/\partial x^i, \partial/\partial y^a), \\
G_{ab} \equiv G(\partial/\partial y^a, \partial/\partial y^b).
\]

The metric \( g \) on \( TB \otimes TB \) can be defined as:

\[
g\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \equiv G(H_i, H_j) \tag{2}
\]

and, accordingly, we will use the shorthand notation \( g_{ij} \equiv g(\partial/\partial x^i, \partial/\partial x^j) \).

The metric \( h \) on \( T\mathcal{F} \otimes T\mathcal{F} \) will be defined simply as a pull-back metric of \( G \):

\[
h\left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right) \equiv G\left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right) \tag{3}
\]

or \( h_{ab} \equiv G_{ab} \) for short.

Now from (1) and (3), it follows that \( G_{ia} = \Gamma_i^b h_{ab} \) and using (2) and (3), we obtain the decomposition in the described bases:

\[
G_{\mu\nu} = \begin{pmatrix} G_{ij} & G_{ij} \Gamma_i^b h_{ab} \\ G_{ij} & G_{ij} \Gamma_i^b h_{ab} \end{pmatrix} \tag{4}
\]

Example 1 \((m = 1, \text{slicing})\). In this case, \( \mathcal{F} \) is still an \( n\)-dimensional hypersurface, \( B = \mathbb{R}^1 \), \( \Gamma_i^1 \equiv B^a \) is an \( n\)-vector, and \( g = g_{11} \) is a scalar. Defining the new scalar \( \phi \equiv g_{11} + h_{ab} B^a B^b \), we obtain the \((1 + n)\)-decomposition, known as slicing:

\[
G_{\mu\nu} = \begin{pmatrix} \phi & B_a \\ B_a & h_{ab} \end{pmatrix}. \tag{5}
\]

For \( n = 4 \), one can immediately identify here the original Kaluza’s metric [2].

Example 2 \((n = 1, \text{threading})\). In this case, \( B \) is an \( m\)-dimensional hypersurface, \( \mathcal{F} = \mathbb{R}^1 \), \( \Gamma_1^i \equiv A_i \) is an \( m\)-vector, and \( h = h_{11} \) is a scalar which we take as \( V^2 \).

The \((m + 1)\)-decomposition, known as threading, is:

\[
G_{\mu\nu} = \begin{pmatrix} g_{ij} + V^2 A_i A_j & V^2 A_i \\ V^2 A_i & V^2 \end{pmatrix}. \tag{6}
\]

For \( m = 4 \), this reduces to the Klein’s metric [3].

One should also note that the Kaluza’s metric (5) has the same form as the inverse of Klein’s metric (6).
3. Spacetime structure

We would like now to review the spacetime structure of Kaluza–Klein theory and prepare the set-up for our further analysis.

Kaluza’s original theory is a unitary representation of gravitation and electromagnetism. The stage for this theory is a five-dimensional spacetime foliated by a one-parameter family of four-dimensional hypersurfaces, each of which could be interpreted as a “particular” universe. The geometry of the five-dimensional spacetime arises from the geometries of the four-dimensional “slices”. As suggested by Kaluza in his original paper, it is quite natural to demand that the four-dimensional “slices”. As suggested by Kaluza in his original paper, it is quite natural to demand that the four-dimensional world (it appears naturally as a slicing decomposition, or Klein’s theory).

The five-dimensional Kaluza’s metric is simply the $(x$ in Kaluza’s theory). The five-dimensional Kaluza’s original theory is a unitary representation of gravitation and electromagnetism. The stage for this theory is a five-dimensional spacetime foliated by four-dimensional surfaces (slicing decomposition, or the original Kaluza’s theory). The cylinder condition is rather artificial—as argued by Einstein and Bergmann [6], it is not really a step forward to introduce a five-dimensional metric and a vector for which arbitrary restrictions are assumed, instead of having a four-dimensional metric and a four-vector—the real step forward, according to Einstein and Bergmann, was the introduction of the five-dimensional space alone. Additionally, in Kaluza’s original theory, the vector $B_i$ is not defined up to an additive gradient of an arbitrary function, namely, Kaluza’s theory lacks gauge invariance. Einstein and Bergmann generalized Kaluza’s theory as follows [6]. They considered a four-dimensional surface, cutting each of the geodesics to which $B^j$ are tangents (the threads) once and only once. The four coordinates, introduced on this surface are $y^j$ and the extra coordinate $x$ is assumed constant on this surface. As shown by Einstein and Bergmann, this coordinate system is preserved by only two coordinate transformations:

\begin{enumerate}
  \item \textbf{four-transformation:}
    \begin{align}
    y^j & \rightarrow y^j(y^i), \\
    x & \rightarrow x,
    \end{align}
  \item \textbf{cut-transformation:}
    \begin{align}
    y^j & \rightarrow y^j, \\
    x & \rightarrow x + f(y^i).
    \end{align}
\end{enumerate}

The four-transformation (9) is a transformation from one thread to another while staying on the same four-dimensional slice. This transformation is the only natural transformation for a five-dimensional spacetime foliated by four-dimensional surfaces (slicing decomposition, or the original Kaluza’s theory). The cut-transformation (10) is a transformation from one four-dimensional slice to another while staying on the same thread. This is the only natural transformation for a five-dimensional spacetime foliated by surfaces of codimension four, i.e. a congruence of threads (threading decomposition, or Klein’s theory).

Kaluza’s theory imposes another property on the vector field $B^j$—the lines to which $B^j$ are tangents (we will call these threads) must be geodesics [6]. This results in the fact that the norm of the vector $B$ is constant in the whole space: $B^j \nabla_j B_i = 0$.

The slicing lapse function is $N^{-1}$, while the slicing shift vector field is given by $B^i$.

The tensor $h_{ij}$ was identified as the metric of our four-dimensional world (it appears naturally as a slicing metric), and the vector $B_i$—as the electromagnetic potential. The antisymmetrical covariant derivatives of $B_i$ with respect to $y^j$, namely $F_{ij} = \nabla_i B_j - \nabla_j B_i = \partial_i B_j - \partial_j B_i$, were identified as components of the Maxwell tensor. The scalar $\phi$ was left uninterpreted.

The symmetrical covariant derivatives of $B_i$, that is $\Sigma_{ij} = \nabla_i B_j + \nabla_j B_i$, also enter the field equations. For the consistent interpretation of Kaluza’s theory as a five-dimensional theory of gravitation and electromagnetism, one has to make sure that the symmetrical derivatives $\Sigma_{ij} = \nabla_i B_j + \nabla_j B_i$ vanish in the whole space. This is the so-called “cylinder condition”. Mathematically, this condition is achieved by requesting that the five-dimensional interval (7) is invariant under the transformation [6]:

\begin{equation}
    y^\mu \rightarrow y^\mu + \epsilon B^\mu,
\end{equation}

where $y^0 = x$, $B^0 = \phi$ and $\epsilon$ is an infinitesimal parameter (Greek indexes from now on run from 0 to 4).
Under the cut-transformation (9), the four-dimensional metric $h_{ij}$ transforms as a tensor and $B_i$ transforms as a vector. However, this is not the case under the cut-transformation (gauge transformation) (10):

$$B_i \rightarrow B_i - \partial_i f,$$

$$h_{ij} \rightarrow h_{ij} - B_i \partial_j f - B_j \partial_i f + (\partial_i f)(\partial_j f).$$

However,

$$g_{ij} = h_{ij} - B_i B_i$$

are invariant under the cut-transformation [6]. Apparently, even though that $B_i$ does not transform as a four-vector, the quantity $F_{ij} = \nabla_i B_j - \nabla_j B_i = \partial_i B_j - \partial_j B_i$ is invariant under the cut-transformation. In view of the transformation law (11), the vector $B_i$ is defined up to an additive gradient of an arbitrary function—exactly as an electromagnetic potential. The invariant $F_{ij}$ is then Maxwell’s tensor. In view of its gauge invariance, $g_{ij}$ can be identified as a metric tensor. However, a surface whose metric is $g_{ij}$ does not necessarily exist. The tensor $g_{ij}$ gives the proper distance between two threads passing through points $P_1(y^i, x)$ and $P_2(y^i + d y^i, x + d x)$ (the distance between points $P_1$ and $P_2$ is given by the interval $ds$) [4].

In addition, Einstein and Bergmann, in their quest for simplification of the basic geometric assumptions, proposed to drop the artificial cylinder condition and, instead, introduce a periodicity condition: space is periodic in the extra direction [6]. These are the ingredients of the theory generally referred today as Kaluza–Klein theory and, without doubt, its precise mathematical formulation must be attributed to Einstein and Bergmann. We will refer to this picture as Klein’s theory.

There is an extremely important observation due here. If $\psi$ is some scalar, then $\partial_i \psi$ is not invariant under the cut-transformation—the only transformation allowed in Klein’s theory. It is the starry derivative,

$$\partial_i^* = \partial_i - B_i \partial_x,$$

that is invariant under the cut-transformation [6]. To build tensor analysis with respect to the cut-transformation, namely to build tensor analysis in Klein’s theory, one has to invoke the starry derivative. Therefore, the Riemann curvature tensor

$$R^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} - \Gamma^i_{km} \Gamma^m_{jl} + \Gamma^i_{jm} \Gamma^m_{kl},$$

must be replaced by the tensor [6]:

$$Z^i_{jkl} = R^i_{jkl} - B_i \partial_k \Gamma^i_{jl} + B_k \partial_i \Gamma^i_{jl}.$$  

Today, this tensor is referred to as Zelmanov curvature tensor [7].

Replacing the Riemann curvature tensor by the tensor (16) corresponds to a surface forming mechanism—in Klein’s theory the four-dimensional world is not naturally formed as it is in Kaluza’s original theory. In Klein’s theory, the congruence of x-like curves could be interpreted as the “world lines” of a family of observers and the “rest frames” of these observers do not necessarily form a surface. Requesting independence on the extra dimension means that $Z^i_{jkl} = R^i_{jkl}$ and that the hypersurfaces for different $x$ are “smeared” into one (which is not a gross injustice under the condition of periodicity along the extra dimension).

We are now in a position to define the set-up for our analysis. The field equations of Klein’s theory are well known (for the sake of completeness, we dedicate the next section to them). It is interesting to study the field equations of the original Kaluza’s theory in view of the duality between the two models. Of course, the additional conditions imposed, namely cylinder condition versus periodicity condition are not related under the Kaluza–Klein (slicing–threading) duality and we will study the full field equations of Kaluza’s theory without imposing the cylinder condition. Mathematically, imposing the cylinder condition, together with imposing independence on the extra dimension $x$, amounts to requesting zero extrinsic curvature. We will keep the extrinsic curvature and study the field equations as found in [8,9]. We will disregard dependence on the extra dimension. We will show that the resulting field equations are plausible generalizations of Einstein’s and Maxwell’s equations, we will study the transfer of gauge invariance from the fields $B_i$ to the dilaton (which models Newton’s constant) and we will give an illustration with a particular example for constant dilaton.

4. Field equations of Klein’s theory

Klein’s model has the $(4 + 1)$ metric (6). The five-dimensional interval is:
\[ ds^2 = g_{ij} dx^i dx^j + V^2 (A_i dx^i + dy)^2. \]  

Here \( x^i \) are the four-dimensional coordinates and \( y \) labels the extra dimension. The field \( V^2 \) is the threading lapse function, while \( A_i dx^i \) is the threading shift one-form. The fields \( g_{ij} \) are the components of the threading metric.

The field equations \( \mathcal{R}_{\mu \nu}^{(5)} = 0 \) are [5]:

\[
\mathcal{R}_{ij} - \frac{1}{2} g_{ij} \mathcal{R} = \frac{V^2}{2} T_{ij}^{EM} - \frac{1}{V} \left[ D_i (\partial_j V) - g_{ij} \Box V \right],
\]

(18)

\[
D_i \tilde{F}^{ij} = -\frac{3}{V} \tilde{\partial} V \tilde{F}^{ij},
\]

(19)

\[
\Box V = \frac{V^3}{4} \tilde{F}_i \tilde{F}^{ij},
\]

(20)

where \( \mathcal{R}_{ij}^{(5)} \) is the five-dimensional Ricci tensor, \( \mathcal{R}_{ij} \) is the four-dimensional Ricci tensor, \( \mathcal{R} \) is the four-dimensional scalar curvature, \( D_i \) is the four-dimensional covariant derivative, \( \Box = g^{jk} D_i D_k \), \( \tilde{F}^{ij} = \partial_i A_j - \partial_j A_i \) is the Maxwell tensor, and the derivatives are with respect to \( x^i \). In mostly plus metric, the electromagnetic energy–momentum tensor is:

\[
T_{ij}^{EM} = \tilde{F}_{ij} \tilde{F}_j - \frac{1}{4} g_{ij} \tilde{F}_{kl} \tilde{F}^{kl}.
\]

(21)

One notes from Eq. (18) that Newton’s constant \( G_N \) is expressed as a dynamical field:

\[
\frac{V^2}{2} = \kappa = \frac{8 \pi G_N}{c^4}.
\]

(22)

A constant solution for \( V \) (and, consequently) for the Newton’s constant) reduces Eqs. (18) and (19) to the usual Einstein and Maxwell equations.

However, in this case Eq. (20) reduces to \( \tilde{F}^2 = \tilde{F}_{ij} \tilde{F}^{ij} = 2 (c^2 B^2 - E^2) = 0 \), that is the square of the electric field must be equal to \( c \) times the square of the magnetic field. While this is indeed the case for plane electromagnetic waves for instance, there is no physical mechanism which will require \( \tilde{F}^2 = 0 \). In this sense, \( F^2 = 0 \) is an unphysical condition.

5. Field equations of Kaluza’s theory

For Kaluza’s model (7), the lack of appropriate gauge invariance for the fields \( B^i \), which we, nevertheless, will identify with the electromagnetic potentials, is transformed as a gauge degree of freedom for the dilaton \( N \). As a result we end up with a gauge-fixed electrodynamics, but we are free to fix the value of \( N \) as needed by the model, including \( N = \text{const.} \). This corresponds to the electromagnetic gauge freedom in the model of Klein [3] (see also the equations of Jordan–Thiry [5]).

Let us denote the five-dimensional Ricci tensor by \( R_{5\mu \nu} \), the four-dimensional Ricci tensor by \( r_{ij} \) and the four-dimensional scalar curvature by \( r \). As before, we use \( \nabla_i \) for the four-dimensional covariant derivative with respect to \( x^i \) and \( F_{ij} = \partial_i B_j - \partial_j B_i \) for the Maxwell electromagnetic tensor. The extrinsic curvature is \( \pi_{ij} = -(N/2)(\nabla_i B_j + \nabla_j B_i) \). Then the field equations \( R_{5\mu \nu} = 0 \), determined in [8], are (see also [9] for their form in terms of the extrinsic curvature \( \pi_{ij} \)):

\[
r_{ij} - \frac{1}{2} r_{ij} r = \frac{N^2}{2} T_{ij},
\]

(23)

\[
\nabla_i F_{ij} = -2 B_i r_{ij} + \frac{2}{N^2} \left( \pi_{ij} - \pi_i^k h^{kj} \right) \partial_k N,
\]

(24)

\[
\nabla_i \left( B_j \pi^{ij} + \frac{1}{N^2} \nabla^i N \right) = 0.
\]

(25)

Here, \( x \)-dependent terms are dropped out.

The dilaton \( N \) is related to the Newton’s constant \( G_N \) via [8]:

\[
\frac{N^2}{2} = \kappa = \frac{8 \pi G_N}{c^4}.
\]

(26)

The energy–momentum tensor \( T_{ij} \) appearing in Eq. (23) is given by [8]:

\[
T_{ij} = T_{ij}^{EM} + \nabla^k \Psi_{ijk} + \nabla^k \Theta_{ijk} + C_{ij} + D_{ij},
\]

(27)

where:

\[
T_{ij}^{EM} = F_{ik} F_{jl} - \frac{1}{4} h_{ij} F_{kl} F^{kl},
\]

(28)

\[
\Psi_{ijk} = B_k \nabla_i B_l - B_l \nabla_k B_i + B_i F_{jk},
\]

(29)

\[
\Theta_{ijk} = \nabla_i (B_k B_l) + h_{ij} \left( B^l \nabla_k B_l - B_k \nabla^l B^l \right),
\]

(30)

\[
C_{ij} = h_{ij} B^k B^l r_{kl} - 2 h_{ik} B_j r_{ik} - 2 B^k B_l r_{jk},
\]

(31)

\[
D_{ij} = \frac{2}{N} \nabla_i \nabla_j \frac{1}{N} - \frac{2}{N^2} \pi^k \left( B_i \partial_j N + B_j \partial_i N \right)
\]

\[
+ \frac{2}{N^2} \left( -B^l \pi_{ij} + B_i \pi^k_j + B_j \pi^k_i \right)
\]

\[
- h_{ij} \left( B^i \pi^k_j - B^j \pi^k_i \right) \partial_k N.
\]

(32)
Let us now consider the case when the dilaton $N$ is constant, i.e. Newton’s constant is indeed a constant. Kaluza’s equations then reduce to:

$$\left( r_{ij} - \frac{1}{2} h_{ij} r \right) - \frac{N^2}{2} r_{ij}^G = \frac{N^2}{2} T_{ij}^{EM} + \frac{N^2}{2} C_{ij}, \quad (33)$$

$$\nabla_i F^{ij} = -2 B_i r^{ij}, \quad (34)$$

$$\nabla_i \left( B_j \pi^{ij} \right) = 0. \quad (35)$$

Here one can immediately recognize (34) as a generalization of Maxwell’s equations. The right-hand side describes interaction between the electromagnetic fields and gravitation. The first of these equations, (33), is a modified Einstein’s equation. Apart from the Maxwell’s energy–momentum tensor $T_{ij}^{EM}$, on the right-hand side we have the tensor $C_{ij}$ which describes interaction between the electromagnetic fields and gravitation. On the left-hand side we have the modified Einstein’s tensor $r_{ij} - \frac{1}{2} h_{ij} r - (N^2/2) r_{ij}^G$.

We will give interpretation of the extra term $T_{ij}^G$ as a ghost energy–momentum tensor and we will comment on the possible implications of such tensor. The last equation, (35), is an equation for $B_i$, additional to Maxwell’s equations (34) and can be interpreted as a gauge-fixing equation for the electromagnetic potentials (in Kaluza’s theory the electromagnetic potentials $B_i$ are gauge-fixed).

The parameter $\kappa = N^2/2 = 8 \pi G_N/c^4$ is small. Then, from (23), we see that $r_{ij}$ is of order of $\kappa$. Using Landau symbols, $r_{ij} = O(\kappa)$. Thus, $B_i r^{ij} = O(\kappa)$ and $\nabla_i C_{ij} = O(\kappa^2)$. Note that the scale of $B_i r^{ij}$ cannot be increased with a gauge transformation and thus change the order of the interaction terms $B_i r^{ij}$ or $\nabla_i C_{ij}$.

Taking covariant derivative from Eq. (33), and using the contracted Bianchi identities, $\nabla^i (r_{ij} - \frac{1}{2} h_{ij} r - \frac{N^2}{2} r_{ij}^G) = 0$, yields:

$$\nabla^i T_{ij}^{EM} + \nabla^i \nabla^k \Theta_{ijk} + \nabla^i \nabla^k \Psi_{ijk} + \nabla^i C_{ij} = 0. \quad (36)$$

The meaning of the tensors involved is best understood if we consider the leading order of $\kappa$. For the Maxwell energy–momentum tensor we have:

$$\nabla^i T_{ij}^{EM} = F_{ik} \nabla_j F^{jk} = -2 B_j F_{ijk} = O(\kappa), \quad (37)$$

due to (34). Eq. (37) is the conservation law for the energy and momentum resulting from Maxwell’s equations (34).

The tensor $\nabla^k \Theta_{ijk}$ satisfies:

$$\nabla^i \nabla^k \Theta_{ijk} = -\frac{2}{N} \nabla_j \nabla_i (B_k \pi^{jk}) + O(\kappa) = 0 + O(\kappa), \quad (38)$$

in view of (35). We now note that Maxwell’s equations (34) are to $T_{ij}^{EM}$ as the gauge-fixing equation (35) is to $\nabla^k \Theta_{ijk}$ (i.e. each of Eqs. (34) and (35), in turn, guarantees the vanishing of the covariant derivatives of the tensors $T_{ij}^{EM}$ and $\nabla^k \Theta_{ijk}$, respectively). And since (35) is the gauge-fixing equation, we will interpret $\nabla^k \Theta_{ijk}$ as a ghost energy–momentum tensor.

Considering the remaining term, $\nabla^k \Psi_{ijk}$, we see that it satisfies:

$$\nabla^i \nabla^k \Psi_{ijk} = \frac{1}{2} \left( \nabla^i (\nabla^k \nabla^k + \nabla^k \nabla^j) \Psi_{ijk} + \frac{1}{2} \left[ \nabla^j, \nabla^k \right] \Psi_{ijk} \right)$$

$$= 0 + O(\kappa) \quad (39)$$

in view of the antisymmetry $\Psi_{ijk} = -\Psi_{ikj}$. (The second term is of order $O(\kappa^2)$.) Note that, due to (36), $\nabla^i C_{ij}$ exactly compensates the sum of the terms of order $O(\kappa)$, which we put aside in Eqs. (37)–(39). The tensor $\nabla^k \Psi_{ijk}$ does not describe any dynamics (in leading order). Its only purpose is to make the linear combination $\nabla^k \Theta_{ijk} + \nabla^k \Psi_{ijk}$ symmetric under exchange of indexes $i$ and $j$. Therefore, the full ghost energy–momentum tensor is given by the Belinfante tensor:

$$T_{ij}^G = \nabla^k (\Theta_{ijk} + \Psi_{ijk}). \quad (40)$$

From Eqs. (38) and (39) it follows that

$$\nabla^i T_{ij}^G = 0, \quad (41)$$

which represents the ghost conservation law for Eq. (35), analogical to the matter conservation law (37) for Eq. (34).

6. Applications

The observed flat galaxy rotation curves is important evidence for the existence of a large fraction of dark matter in the Universe in the framework of Newton’s and Einstein’s theories of gravitation. Over the last decades however, alternative scenarios which modify Newton’s and Einstein’s theories have gained
momentum. The first attempt in this direction was made by Einstein himself in search for a static solution to Einstein’s equations:

\[ E_{ij} + \Lambda g_{ij} = \kappa T_{ij}, \]  

(42)

where \( E_{ij} = r_{ij} - \frac{1}{2} h_{ij} r \) is the Einstein’s tensor, \( T_{ij} \) is the material energy tensor and \( \Lambda \) is the cosmological constant.

Later on, in 1948, Bondi et al. [10] proposed the steady-state cosmological model by the manual introduction of an extra tensor in Einstein’s equations which continuously generates matter that compensates the Universe’s expansion and makes the Universe look perpetual:

\[ E_{ij} + C_{ij} = \kappa T_{ij}. \]  

(43)

The newly introduced tensor \( C_{ij} \) is defined as the covariant derivative of some constant vector field: \( C_{ij} = \nabla_i C_j \). This tensor plays a role similar to that of the cosmological constant and, as the cosmological constant, substantially changes the physical picture.

Until the discovery of the cosmic microwave background radiation, the steady-state cosmology was a viable alternative to the Big Bang model. Recently, it was revived as quasi-steady-state theory [11]. In the framework of the (quasi-)steady-state theory, the dark matter problem takes on a different complexion—there is no restriction like \( \Omega = 1 \) in this cosmology and so the dark matter component need not be very high [12]. The extent of dark matter has to be estimated from improved observations. Even though that our model has nothing to do with the (quasi-)steady-state theory, exactly the same issue surfaces here—we have shown (rather than introduced) a very similar modification of Einstein’s equations (33) which, again, cannot lead to the restriction \( \Omega = 1 \).

Other reasonably simple modifications of gravity which describe galaxy rotation curves quite well, without requiring the existence of dark matter at all, have been proposed—see the topical review [13] and the references therein.

Therefore, a possible scenario for the avoidance of the dark matter problem (by modification of gravity) is to perform a cosmological gauge transformation of the Einstein’s tensor \( E_{ij} \rightarrow E_{ij} - \frac{N^2}{2} \nabla^k (\Theta_{ijk} + \Psi_{ijk}) \) by an extra (ghost) energy–momentum tensor:

\[ E_{ij} \rightarrow E_{ij} - \frac{N^2}{2} \nabla^k (\Theta_{ijk} + \Psi_{ijk}). \]  

(44)

Such transformation is a continuation along the line of modifications of the type (42) and (43) of Einstein’s theory and, as it substantially changes the physical picture by the introduction of mass and energy, it could account for the description of the missing matter and energy in the Universe.

Interestingly, it was shown by Arkani-Hamed et al. [14] that ghost condensate may contribute to both the dark matter and the dark energy.

Returning to the field equations of Kaluza’s theory, we consider, as an example, the following five-dimensional metric:

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{\alpha}{r}\right) dt^2 - 2\beta \left(1 - \frac{\alpha}{r}\right) dt dx \\
    &\quad + \left(1 - \frac{\alpha}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\psi^2 \\
    &\quad + \left[1 - \left(1 - \frac{\alpha}{r}\right)^2\right] dx^2
\end{align*}
\]  

(45)

which is a solution to the five-dimensional vacuum Einstein’s equation \( R_{\mu\nu}^{(5)} = 0 \). This metric corresponds to \( r_{ij} = 0 \) and \( N = 1 \). One can identify

\[ B_i = -\beta \left(1 - \frac{\alpha}{r}\right) \delta_{i0}, \]  

(46)

as an “electrostatic” potential generated by charge

\[ q = \alpha\beta. \]  

(47)

The metric (45) is also a solution to the following field equations:

\[ T^EM_{ij} + T^G_{ij} = 0, \]  

(48)

\[ \nabla_j F_{ij} = 0, \]  

(49)

\[ \nabla_i (B_j \pi^{ij}) = 0. \]  

(50)

Form these equations we see that a four-dimensional Ricci-flat slice (with \( r_{ij} = 0 \)) can accommodate electrodynamics. The allowance for this comes from the ghost energy–momentum tensor \( T^G_{ij} \) which fully compensates the Maxwell’s energy–momentum tensor.

Consider now the Reissner–Nordström geometry of a charged particle (see for example, [15]):

\[
\begin{align*}
    ds^2_{(4)} &= -\left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right) dt^2 \\
    &\quad + \left(1 - \frac{2\mu}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2
\end{align*}
\]

(46)
Here $\mu$ is an integration constant called Reissner–Nordström geometrical mass—see [16]. It can be related to the physical mass $m$ and the charge $q$ of the particle via: $\mu = (m^2 + q^2)^{1/2}$.

To assure that the four-dimensional part of the solution (45) is compatible to the Reissner–Nordström solution up to and including terms with $1/r$, we must identify $\alpha$ with $2\mu$. Hence:

$$\alpha = 2\sqrt{m^2 + q^2}.$$  \hspace{1cm} (52)

In Eq. (45), $\alpha$ is not the usual Schwarzschild mass but is, instead, the geometrical mass. Then the integration constant $\beta$ is given by:

$$\beta = \frac{q}{2\sqrt{m^2 + q^2}}.$$  \hspace{1cm} (53)

The presented example deviates, at fixed $x$, from the Reissner–Nordström geometry of a charged particle: the term $q^2/r^2$ has dropped out due to the ghost energy–momentum tensor and this results in to Schwarzschild geometry. One can therefore interpret the Schwarzschild geometry as arising not from mass only, but from the combined effects of mass and electric charge; even from electric charge only (for a massless charged center, $\alpha = 2|q|$ and $\beta = (1/2) \text{sign}(q)$).

7. Conclusions

In conclusion, we would like to point out that the condition $N = \text{const}$ is the only “manually” imposed condition in Kaluza’s framework. The motivation for this condition is purely physical—one would expect that Newton’s constant (which is modeled by the dilaton $N$) is indeed a constant. The choice $N = \text{const}$ immediately turns Eq. (25) into a gauge-fixing condition for the “electromagnetic” potentials $B_i$.

Within Kaluza’s framework, Einstein’s equations are modified by the appearance of extra terms. These terms are called ghost since they are related, as we showed, to the gauge-fixing condition (35) and not to Maxwell’s equation (34). The modification of Einstein’s equation (33) is by no means arbitrary—it is dictated by Kaluza’s theory—and substantially changes the physical picture—as we showed, a four-dimensional Ricci-flat Kaluza universe can accommodate electrostatics. It is remarkable that the ghost terms are adynamical; the equations of motion for $T^G_{ij} = \nabla^k(\delta_{ij} + \Psi_{ijk})$ are trivial in view of (38) and (39).

Such modification of Einstein’s equations brings up the possibility for a cosmological gauge transformation of Einstein’s tensor $E_{ij}$ in standard general relativity in principle:

$$E_{ij} \rightarrow E_{ij} + \nabla^k \Xi_{ijk},$$  \hspace{1cm} (54)

where the extra term $\Xi_{ijk}$, like in Kaluza’s theory, is adynamical in leading order of $\kappa$, but not necessarily a ghost term (as it is in Kaluza’s context). This is also a reminiscence of the fact that the matter energy–momentum tensor in flat space is defined modulo an additive divergence of a tensor field [17]: $T^j_i \rightarrow T^j_i + \partial_k \eta_{ijk}$, where $\eta_{ijk} = -\eta_{ikj}$ (we have transferred this extra term on the “cosmological side” of Einstein’s equations). In result, Friedmann’s equations will be altered (as it happens by the introduction of the cosmological constant or in the steady-state cosmology) and issues like dark matter and dark energy will fall into new light—without the restriction $\Omega = 1$. Unlike Kaluza’s theory, where the extra terms appear naturally, in the general case there would be a freedom to model the extra term $\Xi_{ijk}$ as dictated by experiments—a suitable choice of $\Xi_{ijk}$ could even render general relativity free of dark matter and dark energy. This, however, is beyond the scope of the present Letter.

Acknowledgements

We would like to thank Brian Dolan and Vesselin Gueorguiev for very helpful discussions. R.I.I. acknowledges partial support from NFNI, grant 1410.

References