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Abstract

We investigate $(4+1)$- and $(5+0)$-dimensional gravity coupled to a non-compact scalar field sigma-model in the context of a single-brane-world scenario with separable metric and a bulk fluid. We briefly discuss the standard cosmological solutions and the family of warp factors (which includes both the original Randall–Sundrum [Phys. Rev. Lett. 83 (1999) 3370, hep-ph/9905221; Phys. Rev. Lett. 83 (1999) 4690, hep-th/9906064] solution and the solution of Kachru, Schulz and Silverstein [H.A. Chamblin, H.S. Reall, Nucl. Phys. B 562 (1999) 133, hep-th/9903225; S. Kachru, M. Schulz, E. Silverstein, Phys. Rev. D 62 (2000) 045021, hep-th/0001206]) for the case of a rolling fifth radius [C. Kennedy, E.M. Prodanov, Phys. Lett. B 488 (2000) 11, hep-th/0003299]. We show how this model can be adjusted so that it describes the standard cosmology on a self-tuning domain wall (with static fifth radius) [C. Kennedy, E.M. Prodanov, hep-th/0010202] and we discuss the solutions. Searching for a possible relation to the negative Euclidean stress energy, appearing in the Giddings and Strominger's axion induced topology change in quantum gravity and string theory [S.B. Giddings, A. Strominger, Nucl. Phys. B 306 (1988) 890], we modify the non-compact sigma-model into a single-field model (with a rolling fifth radius, separable metric, and no bulk fluid) for the more general case of a brane with non-zero curvature parameter. We find a solution (with a Kachru–Schulz–Silverstein warp factor [Phys. Rev. D 62 (2000) 045021, hep-th/0001206]), representing a Tolman wormhole for a $\mathbb{R} \times S^3$ brane with Lorentz metric and for a $\mathbb{R} \times AdS_3$ brane with positive definite metric.

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1. Introduction

Theories with extra dimensions where our four-dimensional world is a hypersurface (three-brane) embedded in a higher-dimensional spacetime and at which gravity is localised have been intensely studied since the work of Randall and Sundrum [1]. This Letter investigates a model of a single brane, embedded in a five-dimensional spacetime. In particular, we study cosmological solutions of $(4+1)$- and $(5+0)$-dimensional gravity coupled to a scalar field sigma-model. In much of the current literature it is assumed that such scalars depend only on the fifth dimension and that the target space metric is of Euclidean signature. By contrast, we consider a non-compact sigma-model and allow the scalars to depend on time, as well as the fifth dimension, which we take to be infinite

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in extent. We also include a bulk fluid with energy–momentum tensor $\widehat{T}^A_B(\rho) = \mathrm{diag}(-\rho, p, p, p, P)$ and equation of state $p = \omega\rho$, $P = \tilde{\omega}\rho$ and show that the fluid exists, provided $\omega = \tilde{\omega} = 1$ (i.e., the fluid is isotropic and stiff). A family of warp factors that includes both the original RS solution [1] and the self-tuning solution of Kachru, Schulz and Silverstein [3] is found. Conventional cosmology is also obtained.

Further, we simplify our model by taking a projection in the target space (onto a dilatonic degree of freedom) and by making the fifth radius static. This results in a Kachru–Schulz–Silverstein warp factor [2,3]. We show that the cosmology on this self-tuning brane is standard, but that the pressure in the fifth direction is constrained by the relation $\tilde{\omega} = \frac{3\omega-1}{2}$. In particular, we find that the pressure in the fifth direction vanishes for a radiation-dominated brane with $\omega = 1/3$.

Finally, we again consider the case of a rolling fifth radius and we exclude the fluid from our analysis. By a different projection in the target space, we end up with a model of one classical scalar field minimally coupled to gravity within the same separable metric ansatz. We also introduce a non-zero curvature parameter on the brane: that is, we consider the most general four-dimensional homogeneous and isotropic Robertson–Walker metric [7] (with a time-dependent scale factor), naturally generalized to a separable five-dimensional Randall–Sundrum [1] context. We find that the warp factor in this case is a Kachru–Schulz–Silverstein one [3] and we find that the solution to the Einstein's equations represents a Tolman wormhole [8] for a $\mathbb{R} \times \mathrm{S}^3$ brane with Lorentz metric and for $\mathbb{R} \times AdS_3$ brane with positive-definite metric. This solution represents a collapsing Universe which starts expansion just before encountering a big crunch singularity. The Universe reaches a moment of minimal spatial volume. This minimum volume edgeless achronal spacelike hypersurface is called a *bounce* [9]. The wormhole is time-dependent and the bounce involves the entire Universe. The motivation for this part of our analysis is based on the the possibility for negative kinetic energy, associated with the non-compact sigma-model. In 1988 Giddings and Strominger [6] showed that for a four-manifold M with $n \geqslant 1$ boundaries of arbitrary topology and one boundary, which is topologically S^3, and with a Euclidean signature metric g on M, which is asymptotically Euclidean near the S^3 boundary and has vanishing extrinsic curvature on the other n boundaries, the Ricci tensor of g has negative eigenvalues somewhere in M [6]. In the four-dimensional case, considered there, the minimal coupling to gravity of a time-dependent only free scalar field resulted in negative kinetic energy and instantonic wormholes, associated with it. In the present 'rolling-radius' five-dimensional case we show that there is no negative kinetic energy, associated with this wormhole.

2. Standard cosmology from sigma-model

In this section, following [4], we shall present our calculations in $(4+1)$-dimensional spacetime with flat spatial three-sections on the brane and only quote analogous results for the $5+0$ case. The action for gravity coupled to a scalar field sigma-model is:

$$S = \int d^4x\, dr \big(\mathcal{L}^{(5)}_{\mathrm{MATTER}} + \mathcal{L}^{(5)}_{\mathrm{GRAVITY}}\big), \tag{1}$$

where:

$$\begin{aligned}\mathcal{L}^{(5)}_{\mathrm{MATTER}} &= -\frac{1}{2}\sqrt{-g^{(5)}}\,\nabla^\mu\phi_i\nabla^\nu\phi_j G^{ij}(\phi) g^{(5)}_{\mu\nu} \\ &\quad - \sqrt{-g^{(5)}}\,U(\phi) - \sqrt{-g^{(4)}}\,V(\phi)\delta(r), \\ \mathcal{L}^{(5)}_{\mathrm{GRAVITY}} &= \frac{1}{\kappa^2}\sqrt{-g^{(5)}}\,R.\end{aligned} \tag{2}$$

Here, $g^{(4)}_{\mu\nu}$ is the pull-back of the five-dimensional metric $g^{(5)}_{\mu\nu}$ to the (thin) domain wall taken to be at $r = 0$. The wall is represented by a delta function source with coefficient $V(\phi)$ parametrising its tension. We take $G_{ij} = \mathrm{diag}(1, -1)$. The "correctly-signed" scalar, ϕ^1, may be interpreted as the dilaton and the "wrongly-signed" scalar, ϕ^2, as an axion. (It is possible to consider a non-trivial coupling between the two—for example, $G_{ij} = \mathrm{diag}(1, -e^{\sigma\phi^1})$ is discussed in [10].)

We assume a separable metric of RS type with flat spatial three-sections on the wall and a 'rolling' fifth radius:

$$\begin{aligned}ds^2 = {}& -e^{-A(r)}\,dt^2 + e^{-A(r)}g(t)\big(dx^2 + dy^2 + dz^2\big) \\ & + f(t)\,dr^2.\end{aligned} \tag{3}$$

Given the above ansatz, it is not unreasonable to assume scalars of the form

$$\phi^i(t, r) = a^i\psi(t) + b^i\chi(r). \tag{4}$$

The linear independence of the scalar fields ϕ^i (which are coordinates on the target spacetime) leads to:

$$\det\begin{pmatrix} a^1 & b^1 \\ a^2 & b^2 \end{pmatrix} \neq 0, \tag{5}$$

and, consequently, to the Schwarz inequality $\frac{(a\cdot a)(b\cdot b)}{(a\cdot b)^2} < 1$.

We also make the ansatz that both the potentials U and V are of Liouville type (see, for instance, [2]):

$$U(\phi) = U_0 e^{\alpha_i \phi^i}, \tag{6}$$

$$V(\phi) = V_0 e^{\beta_i \phi^i}. \tag{7}$$

The energy–momentum tensor for the scalar fields is:

$$\begin{aligned} T_{\mu\nu} = {} & \frac{1}{2}\nabla_\mu \phi^i \nabla_\nu \phi^j G_{ij} \\ & - \frac{1}{2} g_{\mu\nu}\left(\frac{1}{2}\nabla_\alpha \phi^i \nabla_\beta \phi^j G_{ij} g^{\alpha\beta} + U(\phi)\right) \\ & - \frac{1}{2}\frac{\sqrt{-g^{(4)}}}{\sqrt{-g^{(5)}}} V(\phi)\delta(r) g^{(4)}_{ab}\delta^a_\mu \delta^b_\nu. \end{aligned} \tag{8}$$

We introduce a bulk fluid via its energy–momentum tensor [11]:

$$\widetilde{T}^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p, P) \tag{9}$$

with ρ the density and p and P the pressures in the three spatial directions on the brane and in the fifth dimension, respectively. We assume that the equations of state are $P = \tilde{\omega}\rho$ and $p = \omega\rho$. The preferred coordinate system (3) is taken as the rest frame of the fluid. The anisotropy can be considered as a result of the mixing of two interacting perfect fluids [12].

Einstein's equations $G_{\mu\nu} = \kappa^2(T_{\mu\nu} + \widetilde{T}_{\mu\nu})$ reduce to [4]:

$$\begin{aligned} & \frac{1}{4}\frac{\dot f}{f}\frac{\dot g}{g} + \frac{\dot g^2}{g^2} + \frac{1}{4}\frac{\dot f^2}{f^2} - \frac{1}{2}\frac{\ddot f}{f} - \frac{\ddot g}{g} - \frac{\kappa^2}{2} a\cdot a \dot\psi^2 \\ & \quad - \kappa^2 e^{-A}(\rho + p) = 0, \end{aligned} \tag{10}$$

$$\frac{3}{4}\frac{\dot f}{f}\frac{\dot g}{g} + \frac{3}{4}\frac{\dot g^2}{g^2} - \frac{\kappa^2}{4} a\cdot a\dot\psi^2 - \kappa^2 e^{-A}\rho = 0, \tag{11}$$

$$\begin{aligned} & \frac{3}{2}\left(A'^2 - A''\right) + \frac{\kappa^2}{4} b\cdot b \chi'^2 \\ & \quad + \frac{\kappa^2}{2} f U + \frac{\kappa^2}{2} f^{1/2} V\delta(r) = 0, \end{aligned} \tag{12}$$

$$\frac{3}{2}\frac{\ddot g}{g} + \frac{\kappa^2}{4} a\cdot a\dot\psi^2 + \kappa^2 e^{-A} P = 0, \tag{13}$$

$$\frac{3}{2}A'^2 - \frac{\kappa^2}{4} b\cdot b\chi'^2 + \frac{\kappa^2}{2} f U = 0, \tag{14}$$

$$\frac{3}{2}A'\frac{\dot f}{f} + \kappa^2 a\cdot b\dot\psi\chi' = 0. \tag{15}$$

In the above equations we have assumed separability and we have set all separation constants equal to zero (thus losing classes of solutions). In the next sections we will analyse cases with non-zero separation constants.

The density and pressures are each of the form[1] $e^{A(r)}$ times a function of t. We are interested in solutions with $\dot f \neq 0$ ('rolling' fifth radius), thus $a\cdot b \neq 0$.

The equations of motion for the scalar fields

$$\nabla^2 \phi^j G_{jk} - \frac{\partial U(\phi)}{\partial \phi^k} - \frac{\sqrt{-g^{(4)}}}{\sqrt{-g^{(5)}}}\frac{\partial V(\phi)}{\partial\phi^k}\delta(r) = 0 \tag{18}$$

result in the following bulk equations [4]:

$$\partial_t\left(f^{1/2} g^{3/2}\dot\psi\right) = 0, \tag{19}$$

$$b_i\left(2A'\chi' - \chi''\right) + f\alpha_i U_0 = 0, \tag{20}$$

and the jump condition [4]:

$$\lim_{\epsilon\to 0^+}\left[b_i\left(\chi'(\epsilon) - \chi'(-\epsilon)\right)\right] = \beta_i f^{1/2} V\left(\phi(t,0)\right). \tag{21}$$

Eq. (15) implies that we can make the following choice for the scalar fields [4]:

$$\kappa\chi'(r) = \sqrt{6}A'(r), \tag{22}$$

$$\kappa\dot\psi(t) = -\frac{\sqrt{6}}{4}\frac{1}{a\cdot b}\frac{\dot f(t)}{f(t)}. \tag{23}$$

Inserting (22) into (14) gives $U(\phi)$ as:

$$U = -\frac{3}{\kappa^2}\frac{1}{f}A'^2(1 - b\cdot b). \tag{24}$$

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[1] In principle, Einstein's equations can handle ρ and p in the form

$$\rho(t,r) = e^{A(r)}\left(\tilde\rho(t) + F(t,r)\right), \tag{16}$$

$$p(t,r) = e^{A(r)}\left(\tilde p(t) - F(t,r)\right), \tag{17}$$

for arbitrary $F(t,r)$.

However, the constant ω in the equations of state is in the range $-1 \leqslant \omega \leqslant 1$. The generic case $\omega \neq -1$ implies that F should be zero. We shall assume this also to be so in the special case $\omega = -1$.

Expressing the domain wall potential as $V_0 f(t)^{-1/2} \times \delta(r)$, we get the following equation

$$A'' - 2b \cdot bA'^2 - \frac{\kappa^2}{3} V_0 \delta(r) = 0 \tag{25}$$

and options for $A(r)$ and V_0 [4]:

(1) If $b \cdot b = 0$, we find $A(r) = 2\sigma k|r|$, where $\sigma = \pm 1$. Then $V_0 = 12\sigma k\kappa^{-2}$. $\sigma = -1$ is the RS1 solution and $\sigma = +1$ is the RS2 solution, as described in [13].
(2) If $b \cdot b \neq 0$, we find $A(r) = \xi \ln(k|r| + 1)$ where $\xi = -1/2b \cdot b$ and $V_0 = -3k\kappa^2/b \cdot b$. If $b \cdot b$ and k are both positive, then this represents the self-tuning solution of Kachru, Schulz and Silverstein [3].

The above forms for U and V are consistent with (6) if $\beta_i = \alpha_i/2 = 2\kappa b_i/\sqrt{6}$ and $U_0 = -\frac{3}{\kappa^2} A'^2(0)(1 - b \cdot b)$. It can now be verified that (20) is equivalent to (25) in the bulk, whilst (21) yields no further information.

The equation of motion (19) implies that [4]:

$$\frac{\dot f(t)}{f(t)^{1/2}} = \mu g(t)^{-3/2}, \tag{26}$$

where $\mu = -4a \cdot b/\sqrt{6}$.

This equation, together with the time-dependent Einstein's equations and the above equations of state, leads to the following three relations [4]:

$$\omega\rho = p = \frac{1}{3}\rho + \frac{2}{3}P = \frac{1}{3}(1 + 2\tilde\omega)\rho, \tag{27}$$

the equation for the density:

$$\rho(t, r) = \frac{3e^A}{4\kappa^2}\left(\frac{\dot f}{f}\frac{\dot g}{g} + \frac{\dot g^2}{g^2} - \frac{a \cdot a}{8(a \cdot b)^2}\frac{\dot f^2}{f^2}\right), \tag{28}$$

and the cosmology defining equation:

$$\tilde\omega\frac{\dot g^2}{g^2} + 2\frac{\ddot g}{g} + \tilde\omega\frac{\dot f}{f}\frac{\dot g}{g} + (1 - \tilde\omega)\frac{a \cdot a}{8(a \cdot b)^2}\frac{\dot f^2}{f^2} = 0. \tag{29}$$

We seek either power law, $f \sim t^q$, or exponential (inflationary), $f \sim e^{\gamma t}$, solutions of (29). The corresponding solutions for $g(t)$ are $g \sim t^{(2-q)/3}$ and $g \sim e^{-\gamma t/3}$, respectively. The exponents q and γ are non-zero but otherwise arbitrary. The density is positive if $a \cdot a/(a \cdot b)^2 < -2$.

It can be shown [4] that the fluid exists if, and only if, $\tilde\omega = 1$. This implies that $\omega = 1$, that is $P = p$. Thus the fluid, if exists, is isotropic (perfect) ($P = p$) and stiff ($\omega = \tilde\omega = 1$). The attribute “stiff” refers to the fact that the velocity of sound in the fluid is equal to the velocity of light.

The only essential difference between the $5+0$ case and the $4+1$ case considered above is that $\widetilde{T}^\mu{}_\nu$ flips sign. This changes the sign of ρ in (28) so that the density is positive if $a \cdot a/(a \cdot b)^2 > -2$.

We note in passing that the scalar field equations of motion, (18), imply that $\nabla^\mu T_{\mu\nu} = 0$ (and conversely off the brane only). This, in turn, implies that the fluid equation of motion $\nabla_\mu \widetilde{T}^\mu{}_\nu = 0$ is automatically satisfied. In this sense, the same results in the bulk can be obtained from Einstein's equations and $\nabla_\mu \widetilde{T}^\mu{}_\nu = 0$.

3. Standard cosmology on a self-tuning domain wall

In this section we will make a projection in the target space onto a dilatonic degree of freedom (i.e., set $a^1 = a^2 = b^2 = 0$, $b^1 = 1$), consider the case of a *static* fifth radius (i.e., set $f(t) = \text{const}$), and, again, take Liouville type potentials:

$$U(\phi) = U_0 e^{\alpha\chi}, \tag{30}$$

$$V(\phi) = V_0 e^{\beta\phi}. \tag{31}$$

(Note that in this section the scalar field $\psi(t)$ will be excluded from the analysis.)

Introducing *non-zero* separation constants N and M on the right-hand-sides of Einstein's equations (11), (12) and (13), (14), respectively,[2] we get the solutions [5]:

$$g \sim \begin{cases} \sinh^{2q}\left(\sqrt{\frac{M}{3q^2}}\, t\right), & M > 0, \\ t^{2q}, & M = 0, \\ \sin^{2q}\left(\sqrt{\frac{|M|}{3q^2}}\, t\right), & M < 0, \end{cases} \tag{32}$$

where $q = 1/(2 + \tilde\omega) = 2/(3(1 + \omega)) = q_{\text{standard}}$.

[2] Einstein's equations lead to (see [5]) $N = 2M$ and $\omega = \frac{1}{3}(1 + 2\tilde\omega)$.

Defining $A(0)=0$, we see that the *brane* density is [5]:

$$\tilde{\rho}(t)=\begin{cases}\kappa^{-2}M\sinh^{-2}\left(\sqrt{\frac{M}{3q^2}}\,t\right), & M>0,\\ \dfrac{3\kappa^{-2}q^2}{t^2}, & M=0,\\ \kappa^{-2}|M|\sin^{-2}\left(\sqrt{\frac{|M|}{3q^2}}\,t\right), & M<0.\end{cases} \tag{33}$$

When $M\geqslant 0$, we also obtain the de-Sitter solutions $g=e^{\pm 2\sqrt{M/3}\,t}$. These solutions have vanishing density $\tilde{\rho}$ and were discussed in [3,14–16].

For the case $M=0$, we obtain conventional cosmology $H=\dot{a}/a\propto\sqrt{\tilde{\rho}}$ on the brane with evolution at the standard rate.

Of particular note is the case of radiation-dominated fluid on the brane ($\omega=1/3$), for which the pressure in the fifth direction vanishes and the stress tensor is given by:

$$\widetilde{T}^{\mu}{}_{\nu}(\rho)=e^{A(r)}\tilde{\rho}(t)\ \mathrm{diag}\left(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},0\right), \tag{34}$$

with $q_{\mathrm{standard}}=1/2$.

It should be noted that the case of a bulk cosmological constant ($\omega=\tilde{\omega}=-1$) is not covered here; however, it corresponds to the choice $U(\chi)=\mathrm{const}$ instead.

The self-tuning domain wall (solution (I) of [3]) is given by

$$U=M=0,\qquad \beta\neq\pm\frac{1}{a}, \tag{35}$$

$$\chi(r)=a\tau\ln(d-cr), \tag{36}$$

$$A(r)=-\frac{1}{2}\ln(d-cr)-e, \tag{37}$$

where $a=\frac{3}{4\kappa}$ and τ is a sign that takes opposite values either side of the brane at $r=0$. The parameters c, d, and e are constants of integration such that:

$$c_+=-\frac{2}{3}\kappa^2 d_+(a\beta\tau_+-1)V_0e^{a\beta\tau_+\log d_+}, \tag{38}$$

$$c_-=-\frac{2}{3}\kappa^2 d_-(a\beta\tau_++1)V_0e^{a\beta\tau_+\log d_+}, \tag{39}$$

$$d_+=\frac{1}{d_-}>0, \tag{40}$$

$$e_+=-\frac{1}{2}\ln d_+, \tag{41}$$

$$e_-=-\frac{1}{2}\ln d_-, \tag{42}$$

with the convention $A(0)=0$ and the $\pm$ subscript denoting the right (left) side of the brane. The solution is self-tuning because given d_+, $\tau_+=\pm 1$ and $\beta\neq\pm 1/a$, there is a Poincaré-invariant four-dimensional domain wall for any value of the brane tension V_0; V_0 does not need to be fine-tuned to find a solution.

Other warp factors are possible both when $M=0$ and when $M\neq 0$. Solution (II) of [3] with $U=0$ and solution (III) of the same reference with $U\neq 0$ are examples of the former case. The solution presented in [14] with $U=0$ provides an example the latter.

To summarize this section, we state that the self-tuning domain wall, with warp factor given by (37), has vanishing separation constant M and therefore expands according to the power law (32) at the standard rate and exhibits conventional cosmology when coupled to a *bulk* anisotropic fluid. The pressure of the fluid in the fifth direction, P, vanishes for a radiation-dominated brane.

4. Bouncing branes

In view of the Giddings–Strominger theorem [6] (stated in the introduction), which allows negative kinetic energy associated with a free time-dependent only scalar field, minimally coupled to gravity, and in view of the possibility of negative kinetic energy, associated with the sigma-model, we will try to find a wormhole solution for our set-up. For this purpose, we will take another projection in our target space, namely, $a^1=b^1=1$, $a^2=b^2=0$. In other words, we will consider only one of the scalar fields:

$$\phi(t,r)=\psi(t)+\chi(r). \tag{43}$$

Again, both the potentials U and V will be of Liouville type:

$$U(\phi)=U_0e^{\alpha\phi}, \tag{44}$$

$$V(\phi)=V_0e^{\beta\phi}. \tag{45}$$

We will exclude (for simplicity) the fluid from the analysis, consider again a 'rolling' fifth radius case, and introduce non-flat spatial three-sections on the brane. That is, the metric will be:

$$ds^2=se^{-A(r)}\,dt^2+\frac{e^{-A(r)}g(t)}{[1+\frac{\epsilon}{4}(x^2+y^2+z^2)]^2}\left(dx^2+dy^2+dz^2\right)$$

$$+ qf(t)\,dr^2. \tag{46}$$

This is a natural generalisation of the most general four-dimensional homogeneous and isotropic Robertson–Walker metric [7] to a five-dimensional Randall–Sundrum context [1]. The scale factor $g(t)$ is a strictly positive function (we are working with a mostly-plus metric) and the function $f(t)$ is strictly positive as well (the metric is never degenerate). The factors s and q are signs ($s^2 = q^2 = 1$). The curvature parameter is $\epsilon = +1$ (for spherical spatial three-sections) or $\epsilon = -1$ (for hyperbolical spatial three-sections).

Einstein's equations for this case, $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, equivalently written in terms of the Ricci tensor as $R_{\mu\nu} = \kappa^2(T_{\mu\nu} - \frac{1}{3} g^{(5)}_{\mu\nu} T^\alpha_\alpha)$, are:

$$\begin{aligned} &-\frac{s}{q}\frac{1}{f}e^{-A}A'^2 + \frac{s}{2q}\frac{1}{f}e^{-A}A'' \\ &\quad + \frac{1}{4}\frac{\dot f^2}{f^2} - \frac{1}{2}\frac{\ddot f}{f} + \frac{3}{4}\frac{\dot g^2}{g^2} - \frac{3}{2}\frac{\ddot g}{g} \\ &\quad = \frac{\kappa^2}{2}\dot\psi^2 + \frac{\kappa^2}{3}se^{-A}U + \frac{\kappa^2}{6}\frac{s}{f^{1/2}}e^{-A}V\delta(r), \end{aligned} \tag{47}$$

$$\begin{aligned} &2\epsilon - \frac{1}{q}\frac{g}{f}e^{-A}A'^2 + \frac{1}{2q}\frac{g}{f}e^{-A}A'' \\ &\quad - \frac{1}{4s}\frac{\dot f}{f}\dot g - \frac{1}{4s}\frac{\dot g^2}{g} - \frac{1}{2s}\ddot g \\ &\quad = \frac{\kappa^2}{3}e^{-A}gU + \frac{\kappa^2}{6}e^{-A}\frac{g}{f^{1/2}}V\delta(r), \end{aligned} \tag{48}$$

$$\begin{aligned} &-A'^2 + 2A'' + \frac{q}{4s}\frac{\dot f^2}{f}e^A - \frac{q}{2s}\ddot f e^A - \frac{3q}{4s}\dot f\frac{\dot g}{g}e^A \\ &\quad = \frac{\kappa^2}{2}\chi'^2 + \frac{\kappa^2}{3}qfU + \frac{2\kappa^2}{3}qf^{1/2}V\delta(r), \end{aligned} \tag{49}$$

$$-\frac{3}{4}A'\frac{\dot f}{f} = \frac{\kappa^2}{2}\dot\psi\chi'. \tag{50}$$

Similarly to the sigma-model case, the tr-equation, (50), implies:

$$\kappa\dot\psi(t) = -\sqrt{3}\,\frac{\dot f(t)}{f(t)}, \tag{51}$$

$$\kappa\chi'(r) = \frac{\sqrt{3}}{2}A'(r). \tag{52}$$

When $\beta = \frac{\kappa}{2\sqrt{3}}$, the potential $V(\phi)$ can be written in the form:

$$V\big(\phi(t,r)\big) = \frac{1}{\kappa^2}\frac{1}{qf^{1/2}}W, \tag{53}$$

where W is a constant.

Let us assume that the potential $U(\phi)$ can be written as a function of t and r in the separable form:

$$U\big(\phi(t,r)\big) = \frac{1}{\kappa^2}\left[\frac{1}{qf}U_1(r) + e^A U_2(t)\right]. \tag{54}$$

At the end we will recast the potential U back into the original exponential form (44).

Einstein's equations then reduce to:

$$\frac{1}{4}\frac{\dot f^2}{f^2} - \frac{1}{2}\frac{\ddot f}{f} + \frac{1}{4}\frac{\dot f}{f}\frac{\dot g}{g} + \frac{\dot g^2}{g^2} - \frac{\ddot g}{g} - \frac{2\epsilon s}{g} - \frac{3}{2}\frac{\dot f^2}{f^2} = 0, \tag{55}$$

$$-\frac{1}{4s}\frac{\dot f}{f}\frac{\dot g}{g} - \frac{1}{4s}\frac{\dot g^2}{g^2} - \frac{1}{2s}\frac{\ddot g}{g} + \frac{2\epsilon}{g} - \frac{1}{3}U_2 = \frac{C}{qf}, \tag{56}$$

$$\frac{1}{2}\frac{\dot f^2}{f^2} - \frac{3}{2}\frac{\dot f}{f}\frac{\dot g}{g} - \frac{\ddot f}{f} - \frac{2s}{3}U_2 = \frac{2sD}{qf}, \tag{57}$$

$$A'^2 - \frac{1}{2}A'' + \frac{1}{3}U_1 + \frac{1}{6}W\delta(r) = Ce^A, \tag{58}$$

$$\frac{11}{8}A'^2 - 2A'' + \frac{1}{3}U_1 + \frac{2}{3}W\delta(r) = De^A, \tag{59}$$

where C and D are separation constants.

We will be looking for a bounce solution in the form:

$$g(t) = f(t) = Bt^2 + h > 0, \tag{60}$$

where B is a positive constant, not equal to 1 (so that the pull-back of the metric to 4 dimensions is flat only asymptotically[3]), and h is another strictly positive constant.

Upon substitution of the solutions (60) into the Einstein's equations, (55) gives:

$$B = -\frac{2\epsilon s}{3}. \tag{61}$$

On the otherhand, B must be positive. Therefore, $B = 2/3$ and ϵ and s must have opposite signs. Thus the solution is either a brane with spherical three-sections and Lorentz metric or a brane with hyperbolical three-sections and positive definite metric. Clearly, these

[3] The curvature of the brane is $R = \frac{6-6B}{Bt^2+h}$.

two solutions can also be related by a Wick rotation (time t is changed to it and the positive-curvature spacetime becomes a negative-curvature spacetime).

The last term in (55) is the kinetic energy of the scalar field ψ. One can easily see from (55) that it is strictly positive, unlike the Giddings–Strominger case [6].

Einstein's equations (56) and (57) are consistent if we choose

$$U_2(t) = \sigma \frac{\dot{g}^2(t)}{g^2(t)}, \tag{62}$$

where σ is a constant.

The next Einstein's equation, (56), yields that the separation constant C is $\frac{8\epsilon q}{3}$ and that $\sigma = -\frac{3}{2s}$.

The remaining time-dependent Einstein's equation, (57), gives $D = \frac{C}{4} = \frac{2\epsilon q}{3}$.

The r-dependent Einstein's equations (58) and (59) yield:

$$U_1(r) = 10\epsilon q e^{A(r)} - \frac{21}{8} A'(r)^2 \tag{63}$$

and these two equations reduce to a single equation:

$$\frac{1}{8} A'^2 - \frac{1}{2} A'' + \frac{1}{6} W \delta(r) = -\frac{2\epsilon q}{3} e^A. \tag{64}$$

A solution of this equation is of KSS [3] type:

$$A(r) = \ln \frac{1}{(k|r|+1)^2}, \tag{65}$$

where k is a constant, such that $k^2 = \frac{4\epsilon q}{3}$. Therefore, ϵ and q must have the same signs ($k^2 = 4/3$). The constant W in the brane tension V is $-12k$.

The equation of motion for the scalar field:

$$\nabla^2 \phi - \frac{\delta U(\phi)}{\delta \phi} - \frac{\sqrt{|g^{(4)}|}}{\sqrt{|g^{(5)}|}} \frac{\delta V(\phi)}{\delta \phi} \delta(r) = 0, \tag{66}$$

after integration over the fifth dimension in an infinitesimal interval, gives a jump condition accross the brane:

$$A'(+0) - A'(-0) = -4k. \tag{67}$$

Let us now write the potential

$$U\big(\phi(t,r)\big) = \frac{1}{\kappa^2} \left[\frac{1}{qf} U_1(r) + e^A U_2(t) \right]$$

back in the exponential form $U = U_0 e^{\alpha\phi}$. Substituting the solution (65) for $A(r)$ into (63) gives:

$$U_1(r) = -4\epsilon q e^{A(r)}. \tag{68}$$

Using this form of $U_1(r)$, together with (62) for $U_2(t)$ and the value of B, we easily find that:

$$U\big(\phi(t,r)\big) = -\frac{4\epsilon h}{\kappa^2} \frac{e^A}{g^2} = U_0 e^{\frac{2\kappa}{\sqrt{3}}\phi}. \tag{69}$$

For a realistic model, one could choose h sufficiently small.

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