Spectral Theory of Semibounded Sturm-Liouville Operators with Local Interactions on a Discrete Set

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One-Dimensional Schrödinger Operator with \( \delta \)-Interactions*

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Abstract. The one-dimensional Schrödinger operator \( H_{X,\alpha} \) with \( \delta \)-interactions on a discrete set is studied in the framework of the extension theory. Applying the technique of boundary triplets and the corresponding Weyl functions, we establish a connection of these operators with a certain class of Jacobi matrices. The discovered connection enables us to obtain conditions for the self-adjointness, lower semiboundedness, discreteness of the spectrum, and discreteness of the negative part of the spectrum of the operator \( H_{X,\alpha} \).

Key words: Schrödinger operator, point interactions, self-adjointness, lower semiboundedness, discreteness.

1. Introduction. Let \( I = [0,b) \), \( 0 < b \leq +\infty \), and let \( X = \{x_n\}_{n=1}^{\infty} \subset I \) be a strictly increasing sequence such that \( \lim_{n \to \infty} x_n = b \). Let also \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) be a sequence of real numbers.

In \( I = L^2(I) \), with the differential expression

\[
\ell_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n)
\]

one associates the symmetric operator (see [1])

\[
H^0_{X,\alpha} := -\frac{d^2}{dx^2}, \quad D(H^0_{X,\alpha}) = \{ f \in W^{2,2}_{\text{comp}}(I \setminus X) : f'(0) = 0, \text{ } f(x_n+) = f(x_n-) , f'(x_n+) - f'(x_n-) = \alpha_n f(x_n) \}.
\]

Its closure, denoted by \( H_{X,\alpha} \), is interpreted as the Hamiltonian with \( \delta \)-interactions of strength \( \alpha_n \) at the centers \( x_n \).

The spectral properties of the operator \( H_{X,\alpha} \) have been widely studied in the case \( d_\ast := \inf_{n \in \mathbb{N}} d_n > 0 \). (Here \( d_n := x_n - x_{n-1} \) and \( x_0 := 0 \).) It is known that in this case the operator \( H_{X,\alpha} \) is always self-adjoint [6]. Moreover, it is lower semibounded if and only if the sequence \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) (see [3]). Also the condition \( \lim_{n \to \infty} d_n = 0 \) is necessary for the discreteness of the spectrum.

The cases \( d_\ast = 0 \) and \( d_\ast > 0 \) are different in essence. It was shown in [4] that for \( d_\ast = 0 \) the deficiency indices \( n_+(H_{X,\alpha}) \) and \( n_-(H_{X,\alpha}) \) of the operator \( H_{X,\alpha} \) coincide and are not greater than 1, and the Weyl alternative was established. In particular, the latter implies that for \( n_+(H_{X,\alpha}) = 1 \) the spectra of the self-adjoint extensions of the operator \( H_{X,\alpha} \) are discrete. Moreover, Shubin Christ and Stolz [18] showed that \( n_\pm(H_{X,\alpha}) = 1 \) if \( I = \mathbb{R}_+ \), \( d_n = 1/n \), and \( \alpha_n = -2n - 1, n \in \mathbb{N} \).

The main aim of this note is the spectral analysis of the Hamiltonian \( H_{X,\alpha} \) in the case of \( d_\ast = 0 \). We study the operator \( H_{X,\alpha} \) in the framework of extension theory of symmetric operators using the concept of boundary triplets and the corresponding Weyl functions (see [7] and [5]). For \( d_\ast > 0 \), this approach was first applied by Kochubei [12]. Let us mention that the main difficulty in this approach is the construction of an adequate boundary triplet.

In this note, we present the corresponding construction for \( d_\ast = 0 \) and show that the properties like self-adjointness, lower semiboundedness, and discreteness of the spectrum of the operator \( H_{X,\alpha} \) correlate with the corresponding spectral properties of the Jacobi matrix \( B_{X,\alpha} \) of the form (6) (see

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below). This enables us to obtain conditions for the operator $H_{X,\alpha}$ to be self-adjoint and lower semibounded and to have discrete spectrum. It turns out that these conditions significantly depend not only on $\alpha$ but also on $X$ and are completely different from the corresponding conditions for $d_s > 0$. In the forthcoming paper, we exploit the discovered connection in the opposite direction. Namely, using the quadratic form approach we obtain new results on spectral properties of both lower semibounded Hamiltonians $H_{X,\alpha}$ and the corresponding matrices $B_{X,\alpha}$ of the form (6).

Let us also mention that there is a series of papers (see [17]) developing a different approach to the study of Sturm–Liouville operators with distributional potentials $q \in W^{-1}_{2,\text{loc}}(\mathscr{F})$.

A detailed treatment of the results discussed in this note is given in [10].

2. A boundary triplet. In the remaining part of the paper, we assume without loss of generality that $d^* := \sup_{n\in\mathbb{N}} d_n < \infty$. Consider the operator

$$
H_{\min} := -\frac{d^2}{dx^2}, \quad D(H_{\min}) = W^{2,2}(\mathscr{F} \setminus X).
$$

(3)

The operator $H_{\min}$ is symmetric, $n_{\pm}(H_{\min}) = \infty$, and $H^*_{\min}$ is defined by the same differential expression $-d^2/dx^2$ on the domain $D(H^*_{\min}) = W^{2,2}(\mathscr{F} \setminus X)$. It is clear that $H_{\min} = \bigoplus_{n\in\mathbb{N}} H_n$, where $H_n := -d^2/dx^2$, $D(H_n) = W^{2,2}[x_{n-1}, x_n]$. Define mappings $\Gamma_{0}^{(n)}, \Gamma_{1}^{(n)} : D(H^*_n) = W^{2,2}[x_{n-1}, x_n] \to \mathbb{C}^2$ by

$$
\Gamma_{0}^{(n)} f := \begin{pmatrix}
\frac{d^{1/2} f(x_{n-1}+)}{d_n^{1/2} f(x_{n-1})}
\end{pmatrix}, \quad
\Gamma_{1}^{(n)} f := \begin{pmatrix}
\frac{f'(x_{n-1}+)}{d_n^{1/2} f(x_{n-1})} + \frac{f(x_{n-1}+)-f(x_{n-1})}{d_n} \\
\frac{f'(x_{n-1}-)}{d_n^{1/2} f(x_{n-1})} + \frac{f(x_{n-1}-)-f(x_{n-1})}{d_n}
\end{pmatrix}.
$$

(4)

The collection $\Pi_n = \{\mathbb{C}^2, \Gamma_{0}^{(n)}, \Gamma_{1}^{(n)}\}$ forms a boundary triplet for $H^*_n$ in the sense of [7]. Moreover, we have the following

**Theorem 1.** Let $\Gamma_j^{(n)}$ be defined by (4), let $\Gamma_j := \bigoplus_{n\in\mathbb{N}} \Gamma_j^{(n)}$, $j \in \{0, 1\}$, and let $\mathscr{H} = l_2(\mathbb{N}, \mathbb{C}^2)$.

Then the collection $\Pi = \{\mathscr{H}, \Gamma_0, \Gamma_1\} := \bigoplus_{n\in\mathbb{N}} \Pi_n$ is a boundary triplet for the operator $H^*_{\min}$; that is, the mapping $\Gamma := \text{col}(\Gamma_0, \Gamma_1) : D(H^*_{\min}) = W^{2,2}(\mathscr{F} \setminus X) \to \mathscr{H} \oplus \mathscr{H}$ is surjective, and the Green formula

$$
(H^*_{\min} f, g)_\mathscr{H} - (f, H^*_{\min} g)_\mathscr{H} = (\Gamma_1 f, \Gamma_0 g)_\mathscr{H} - (\Gamma_0 f, \Gamma_1 g)_\mathscr{H}
$$

holds for all $f, g \in D(H^*_{\min})$.

A boundary triplet $\Pi$ for $H^*_{\min}$ may have a simpler form if $d_s > 0$. Namely (see [12]), $\Pi = \bigoplus_{n\in\mathbb{N}} \Pi_n = \{\mathscr{H}, \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$, where $\Pi_n = \{\mathbb{C}^2, \tilde{\Gamma}_0^{(n)}, \tilde{\Gamma}_1^{(n)}\}$, $\tilde{\Gamma}_0^{(n)} f := \text{col}(f(x_{n-1}+), -f(x_{n-1}))$, $\tilde{\Gamma}_1^{(n)} f := \text{col}(f'(x_{n-1}+), f'(x_{n-1}-))$, and $\tilde{\Gamma}_j := \bigoplus_{n\in\mathbb{N}} \tilde{\Gamma}_j^{(n)}$, $j \in \{0, 1\}$. However, this triplet is not a boundary triplet for $H^*_n$ if $d_s = 0$. Namely, $D(\tilde{\Gamma}_j) \not\subseteq D(H^*_{\min})$, $j \in \{0, 1\}$, and the mapping $\tilde{\Gamma} = \text{col}(\tilde{\Gamma}_0, \tilde{\Gamma}_1)$ is not surjective.

It is shown in the recent paper [13] that for an arbitrary sequence of symmetric operators $\{S_n\}_{n\in\mathbb{N}}$ satisfying $n_+(S_n) = n_-(S_n)$ there exists a sequence of boundary triplets $\Pi_n$ for $S^*_n$ such that $\Pi = \bigoplus_{n\in\mathbb{N}} \Pi_n$ is a boundary triplet for the operator $S^* = \bigoplus_{n\in\mathbb{N}} S^*_n$. Furthermore, $\{\Pi_n\}_{n\in\mathbb{N}}$ can be constructed as a special regularization of an arbitrary sequence $\{\tilde{\Pi}_n\}_{n\in\mathbb{N}}$ of boundary triplets for $S^*_n$. Developing the construction in [13, §5], we propose a general regularization procedure for $\tilde{\Pi}_n$, which significantly exploits the Weyl functions $\tilde{M}_n(\cdot)$ corresponding to the triplets $\tilde{\Pi}_n$, $n \in \mathbb{N}$.

In the case under consideration, this construction gives Theorem 1.

3. Connection with Jacobi matrices. Consider the semi-infinite Jacobi matrix

$$
B_{X,\alpha} = \begin{pmatrix}
\frac{1}{r_1} (\alpha_1 + \frac{1}{d_1} + \frac{1}{d_2}) \\
\frac{1}{r_1 r_2 d_2} \\
\frac{1}{r_1 r_2 d_3} & \frac{1}{r_2} (\alpha_2 + \frac{1}{d_2} + \frac{1}{d_3}) \\
\frac{1}{r_2 r_3 d_3} & \frac{1}{r_2 r_3 d_4} & \frac{1}{r_3} (\alpha_3 + \frac{1}{d_3} + \frac{1}{d_4}) \\
& \ddots & \ddots & \ddots
\end{pmatrix},
$$

(6)
where \( r_n := \sqrt{d_n + d_{n+1}} \) and \( d_n = x_n - x_{n-1}, n \in \mathbb{N} \). The matrix \( B_{X,\alpha} \) induces the minimal symmetric operator in \( l^2(\mathbb{N}) \), also denoted by \( B_{X,\alpha} \) (see [2] for details).

**Lemma 1.** Let \( \Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\} \) be the boundary triplet constructed in Theorem 1 for \( H^*_{\min} \), and let \( \Theta_{X,\alpha} \) be the following linear relation in \( \mathcal{H} \):

\[
\Theta_{X,\alpha} := \Gamma D(H_{X,\alpha}) := \{ \langle \Gamma_0 f, \Gamma_1 f \rangle : f \in D(H_{X,\alpha}) \}.
\]

Then \( \Theta_{X,\alpha} = \Theta^{\text{op}} \oplus \Theta^{\infty} \), where \( \Theta^{\text{op}} \) and \( \Theta^{\infty} = \{0, f\} : \{0, f\} \in \Theta_{X,\alpha} \) are the operator and pure parts of \( \Theta_{X,\alpha} \). Moreover, \( \Theta^{\text{op}} \) is unitarily equivalent to the minimal operator \( B_{X,\alpha} \).

Theorem 1, Lemma 1, and some results in [5] and [7] provide the following connection between the spectral properties of the operators \( H_{X,\alpha} \) and \( B_{X,\alpha} \).

**Theorem 2.** Let \( \sup_{n \in \mathbb{N}} d_n < \infty \). Then the following assertions hold.

(i) \( n_*(H_{X,\alpha}) = n_*(B_{X,\alpha}) \).

(ii) The operator \( H_{X,\alpha} \) is lower semibounded (nonnegative) if and only if so is \( B_{X,\alpha} \).

(iii) Suppose that \( H_{X,\alpha} = H^{\text{op}}_{X,\alpha} \). Then the spectrum \( \sigma(H_{X,\alpha}) \) is discrete if and only if \( \lim_{n \to \infty} d_n = 0 \) and the spectrum \( \sigma(B_{X,\alpha}) \) is discrete.

(iv) If \( H_{X,\alpha} = H^*_{\min} \), then the negative part of the spectrum \( \sigma(H_{X,\alpha}) \) is discrete (finite) if and only if so is the negative part of \( \sigma(B_{X,\alpha}) \). In particular, \( \sigma_{\text{c}}(H_{X,\alpha}) \subseteq \mathbb{R}_+ \) if and only if \( \sigma_{\text{c}}(B_{X,\alpha}) \subseteq \mathbb{R}_+ \).

**Remark 1.** An analog of Theorem 2(i) for \( d_s > 0 \) was given in [1, Sec. III.2.1, Theorem 2.1.5]. However, the Jacobi matrices in [1] were completely different from the matrices \( B_{X,\alpha} \) defined by (6), and for \( d_s = 0 \) the connection between the Hamiltonians \( H_{X,\alpha} \) and the Jacobi matrices remained unclear.

Another approach to establishing the connection between the operators \( H_{X,\alpha} \) and \( B_{X,\alpha} \) has been brought to our attention by the referee. Namely, let us identify a function \( f \) in the space \( \mathcal{L} \) of functions having compact support and linear on intervals \([x_{k-1}, x_k]\) with the sequence \( y := \{y_k\}, y_k := f(x_k) \). Then the formulas \( (H_{X,\alpha} f, f) = \sum d_k^{-1} |y_k - y_{k-1}|^2 + \alpha_k |y_k|^2 \) and \( (f, f) = \sum d_k (|y_k - y_{k-1}|^2 + |y_k|^2) / 3 \) give rise to a new Hilbert space \( \mathcal{H}' \) and a form \( \mathcal{L}' \) on it. Clearly, \( (f, f)' = \sum d_k (|y_k - y_{k-1}|^2 + |y_k|^2) = \sum (d_k + d_{k+1}) |y_k|^2 \) defines an equivalent norm on \( \mathcal{H}' \). Therefore, \( U : f \to \tilde{y} \), where \( \tilde{y}_k = y_k \sqrt{d_k + d_{k+1}} \), is an isometry from \( \mathcal{L} \) into \( l^2(\mathbb{N}) \), and \( (H_{X,\alpha} f, f)_{L^2} = (B_{X,\alpha} \tilde{y}, \tilde{y})_{l^2} \). This approach allows one to prove Theorem 2(ii).

**Remark 2.** The mapping \( \Gamma^2 \) in [14, Theorem 1] coincides with the mapping \( \Gamma_0 = \bigoplus_{n \in \mathbb{N}} \tilde{\Gamma}_0^{(n)} \). Therefore, the triplet constructed in [14] is *not* a boundary triplet for \( H^*_{\min} \) if \( d_s = 0 \), since \( \Gamma(\tilde{\Gamma}_0) \nsubseteq D_{\min}(\mathcal{H}) \) in this case. Because of this, the parametrization of extensions of the operator \( H_{X,\alpha} \) in [14] and [15], as well as the conclusions on their spectral properties, is not correct.

4. **Self-adjointness.** By applying Carleman’s criterion [2, Theorem VII.1.5] to the matrix \( B_{X,\alpha} \) and by using Theorem 2(i), we obtain

**Proposition 1.** If \( \sum_{n \in \mathbb{N}} d_n^2 = \infty \), then the operator \( H_{X,\alpha} \) is self-adjoint for any real sequence \( \alpha = \{\alpha_n\}_{n \in \mathbb{N}} \).

Note that a similar condition occurs in a paper by R. S. Ismagilov (see [8, Theorem 1]), where operators with continuous potentials are studied. Also, Proposition 1 can be deduced from [16, Theorem 2].

Proposition 1 is sharp. Namely, the following holds.

**Proposition 2.** Let \( \sum_{n \in \mathbb{N}} d_n^2 < \infty \) and \( d_{n-1}d_{n+1} > d_n^2, n \in \mathbb{N} \). If

\[
\sum_{n=1}^{\infty} d_{n+1} \left| \alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}} \right| < \infty, \tag{7}
\]

then the operator \( H_{X,\alpha} \) is symmetric with \( n_\pm(H_{X,\alpha}) = 1 \).
Proposition 2 is inspired by the result of Shubin and Christ and Stolz [18, p. 496], and its proof is based on the recent improvement by Kostyuchenko and Mirzoev [11, Theorem 1] of the well-known Berezanskii condition [2, Theorem VII.1.5].

For \( \{d_n\} \in l_2 \), the question concerning the self-adjointness of the operator \( H_{X,\alpha} \) is quite subtle. For instance, the following holds.

**Proposition 3.** Let \( \mathcal{S} = \mathbb{R}_+ \) and \( d_n = n^{-1}, \ n \in \mathbb{N} \). Then

(i) \( n_+(H_{X,\alpha}) = 0 \) if \( \sum_{n=1}^{\infty} |\alpha_n|n^{-3} = \infty \).

(ii) \( n_-(H_{X,\alpha}) = 0 \) if \( \alpha_n \leq -2(2n+1) + O(n^{-1}) \).

(iii) \( n_{\pm}(H_{X,\alpha}) = 0 \) if \( \alpha_n \geq -Cn^{-1}, \ n \in \mathbb{N}, \ C \equiv \text{const} > 0 \).

(iv) \( n_+(H_{X,\alpha}) = 1 \) if \( \alpha_n = -2n - 1 + O(n^{-\varepsilon}) \) with some \( \varepsilon > 0 \).

(v) \( n_{\pm}(H_{X,\alpha}) = 1 \) if \( \alpha_n = -A(2n+1) + O(n^{-1}), \ A \in (0,2) \).

5. Lower semiboundedness. By (6) and Theorem 2(ii), we obtain

**Proposition 4.** The operator \( H_{X,\alpha} \) is lower semibounded if

\[
\inf_{n \in \mathbb{N}} \alpha_n(d_{n-1} + d_n)^{-1} > -\infty.
\]

A stronger assertion is given in [3, Theorem 3]. If \( d_* > 0 \), then condition (8) is equivalent to \( \inf_{n \in \mathbb{N}} \alpha_n > -\infty \) and is also necessary. It also follows from [3] that \( H_{X,\alpha} \) is not lower semibounded if \( d_n = n^{-1/2} \) and \( \alpha_n = -n^{-\varepsilon}, \ \varepsilon \in (0,1/2) \); however, \( \lim_{n \to \infty} \alpha_n = 0 \).

6. Discreteness of spectrum. Applying Chihara’s condition for the discreteness of spectra of Jacobi matrices (see [19]), we obtain

**Proposition 5.** Let \( X = \{x_n\}_{n=1}^{\infty} \) and \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \) be such that \( H_{X,\alpha} = H_{X,\alpha}^* \), \( \lim_{n \to \infty} d_n = 0 \), and \( \lim_{n \to \infty} d_n/d_{n+1} = 1 \). If

\[
\lim_{n \to \infty} |\alpha_n|d_n^{-1} = \infty \quad \text{and} \quad \lim_{n \to \infty} (\alpha_n d_n)^{-1} > -1/4,
\]

then the spectrum of the operator \( H_{X,\alpha} \) is discrete.

The second condition in (9) is sharp. (Under additional mild assumptions on \( d_n \) and \( \alpha_n \), the spectrum of the matrix \( B_{X,\alpha} \) is absolutely continuous whenever \( \lim_{n \to \infty} (\alpha_n d_n)^{-1} < -1/4 \) [19, Theorem 2.2].) Proposition 5 also enables us to construct lower unbounded operators \( H_{X,\alpha} \) with discrete spectrum. (Another approach can be derived from [9, §2].) Let us illustrate this by considering the following example.

**Example.** Let \( d_n = n^{-1/2} \) and \( \alpha_n = -cn^{1/2} \), where \( c \in \mathbb{R}_+ \setminus \{4\} \). Then \( H_{X,\alpha} \) is self-adjoint and lower unbounded. The spectrum \( \sigma(H_{X,\alpha}) \) is discrete precisely for \( c > 4 \). If \( c \in (0,4) \), then the spectrum \( \sigma(B_{X,\alpha}) \) is absolutely continuous, \( \sigma_{ac}(B_{X,\alpha}) = \mathbb{R} \) (see [19]), and hence, by Theorem 2(iv), the negative spectrum of \( H_{X,\alpha} \) is not discrete.

By setting \( d_n = n^{-1/2} \) and \( \alpha_n = n^{-\varepsilon}, \ \varepsilon \in (0,1/2) \), we obtain the operator \( H_{X,\alpha} \) with discrete spectrum for the case in which \( \lim_{n \to \infty} \alpha_n = 0 \).

References


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