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Nonaxisymmetric Stokes Flow Between Concentric Cones

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NON-AXISYMMETRIC STOKES FLOW BETWEEN
CONCENTRIC CONES

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Summary
We study the fully three-dimensional Stokes flow within a geometry consisting of two
infinite cones with coincident apices. The Stokes approximation is valid near the apex
and we consider the dominant flow features as it is approached. The cones are assumed
to be stationary and the flow to be driven by an arbitrary far-field disturbance. We
express the flow quantities in terms of eigenfunction expansions and allow for the
first time for non-axisymmetric flow regimes through an azimuthal wavenumber. The
eigenvalue problem is solved numerically for successive wavenumbers. Both real and
complex sequences of eigenvalues are found, their relative dominance affecting the
flow features observed. The implications for the presence of eddy-like structures
(analogous to those found in other corner geometries) are discussed and we find that
these flow features depend, not only upon the internal angles of the two cones, but
also upon the symmetry of the driving mechanism. For an arbitrary disturbance the
dominant flow mode is not axisymmetric but rather is associated with wavenumber
one and, by breaking axisymmetry, eddies can be avoided in this geometry.

1. Introduction
The study of low Reynolds number flows has enjoyed a resurgence over recent years due to
interest in providing technologies on extremely small length scales. The focus on micromixer
technology has meant that the study of flow phenomena associated with low Reynolds
numbers is particularly pertinent today. It is well known that eddies or recirculation
regions are formed in slow viscous flows associated with a diverse range of geometries.
The recirculation cells are a result of geometric effects, rather than instabilities, and often
persist in the Stokes limit. In some situations eddies arise as a secondary motion at higher
orders of Reynolds number, driven by the Stokes flow of another component, but again
are attributable to the geometry. Recirculating regions are of great practical importance,
inhibiting mixing and dictating flow topology.
Moffatt (1) introduced the analysis of eddy formation within two-dimensional corners by a detailed discussion of the flow between two fixed planes. He showed that, no matter what the external driving mechanism far from the apex, for corner angles less than a critical value, the dominant flow feature will be an infinite sequence of eddies. These eddies arise from complex eigenvalues of the Stokes problem. At greater corner angles the dominant pair of complex conjugate eigenvalues becomes real so that eddies are eliminated and the exterior flow fully penetrates the corner. Following Moffatt’s discussion of the two-dimensional wedge, corner eddies have been analysed for a variety of two-dimensional geometries (see, for example, (2)-(4)). In each case eddies require two factors: complex eigenvalues, solely dependent upon the geometry, and that the flow mode associated with the leading eigenvalue dominates over all other flow modes.

Eddies in two dimensions are characterised by regions of closed streamlines about a stagnation point separated by streamlines that intersect the bounding walls. However, three dimensions is topologically far more complex and hence the analysis and discussion of eddies more difficult. The fact that a simple analytic solution for the flow in a lid-driven cubic cavity is unavailable (5) is indicative of some of the difficulties in extending the study of eddies beyond two dimensions. A detailed discussion of this problem and a possible approach is given by (6) where an approximate solution is constructed as a finite sum using an embedding procedure and least squares. However, there is no single natural extension of the two-dimensional wedge geometry to three dimensions. Shankar (7) provides one example, describing the flow in a semi-infinite wedge and illustrates some of the complications arising from extending Moffatt’s work to three dimensions. However, the formation of eddies still requires the dominance of flow modes resulting from complex eigenvalues.

Eddies have been successfully analysed using eigenfunction expansions for the flow within a cylindrical domain by Shankar (8). He shows that two sequences of eigenvalues arise, one real and one complex, and an infinite sequence of eddies occurs for non-axisymmetric Stokes flow. The dominance of homogeneous eddy structures arising from the geometry over forced flow components was demonstrated by Hills (9) for the flow in the semi-infinite cylinder driven by a rotating end wall. Gomilko et al. (10) considered the formation of eddies in a trihedral corner induced by the rotation or translation of a boundary. An overview of the use of eigenfunction expansions and the formation of eddies in slow viscous flows is provided by Shankar (6).

Perhaps one of the most natural extensions of the two-dimensional corner geometry to three dimensions is the flow within a cone. Both Wakiya (11) and Liu & Joseph (12) have studied this geometry and demonstrated the formation of axisymmetric eddies. Malyuga (13) and Shankar (14) have extended these results to consider fully three-dimensional flows. They allow for \( \phi \)-dependence by a Fourier analysis of the linear problem using an azimuthal wavenumber and consider the case of axisymmetry separately. Again, in common with the flow in a cylinder, two sequences of eigenvalues are found only one of which can produce eddies. The complex eigenvalues are dominant for internal cone angles less than a critical value (dependent upon the wavenumber). It is also found that, in general, the dominant flow mode as the apex is approached is that associated with wavenumber one.

The flow in the cavity between two coaxial cones with coincident apices has been considered by a number of authors, (15)-(18). In particular, Hall et al. (17) have discussed the existence of axisymmetric eddies for variations of the geometry and driving mechanism.
These authors map the parameter range for which eddies are expected and determine the flow for the particular driving mechanism of a rotating lid.

In this paper we investigate the formation of eddies between two coaxial cones with coincident apices. We extend the analysis of Hall et al. (17) to allow for non-axisymmetric flow modes and hence fully three-dimensional flows. We incorporate, for the first time in this geometry, azimuthal dependence via a wavenumber and analyse the form of, and competition between, particular eigenmodes for individual wavenumbers and eigenvalues across the parameter range. Of particular interest will be the analogue of the two sequences of eigenvalues found for the cylindrical and single-cone geometries, the effect of the insertion of a slender inner cone upon the flow in a single cone, and the dominance of non-axisymmetric flow modes over those associated with wavenumber zero. Of practical importance will be an analysis of the presence of eddies, which in the fully three-dimensional flow will depend not only upon complex eigenvalues but also upon particle paths lying on closed surfaces. A strong motivation for studying non-axisymmetric flows is that, in most applications, it is difficult to guarantee an axisymmetric flow, and that in manifestations of this geometry (for example in the cone-plate viscometer) some degree of asymmetry is inevitable. Therefore, it is necessary to temper our formal interpretation of eddies (particle paths lying on closed surfaces) with a pragmatic approach to the presence of observable eddy-like structures appropriate for most practical applications.

In the next section we formulate the problem, describing the geometry and solution method using series of similarity solutions to the governing linear Stokes equations. The numerical methods we employ to solve the resulting eigenvalue problem and calculate the eigenmodes for specific geometries are then given. Section 3 describes the nature of the flows arising from the eigenmodes and presents a discussion of the interpretation of eddies in this instance. In section 4 we present the numerical results of our investigation and interpret the results in the context of the dominant flow modes for particular geometries and eddy formation. Flow visualisations for specific geometries are given.

2. Problem formulation and solution method

2.1 Governing equations

We consider the steady, three-dimensional flow of a linearly viscous, incompressible fluid with constant kinematic viscosity $\nu$ within the geometry shown in figure 1, consisting of two coaxial cones with coincident apices. The geometry is well described by spherical polar coordinates $(r, \theta, \phi)$ with the origin at the apex and angles measured about the common $z$-axis. The cones are held stationary and their conical apices are defined by $\theta = \alpha$ and $\theta = \beta$, with $\alpha < \beta$.

There are no physical length or velocity scales for the system when the geometry is open but the standard formulation of the Reynolds number suggests that, for a fluid of constant kinematic viscosity, no matter what the magnitude of the driving flow, as we approach the apex the Stokes flow will become asymptotically dominant. Hence we may legitimately neglect inertial effects and study the linear Stokes problem. Of particular interest are the effects of non-axisymmetry and the generation of recirculation, or eddy-like, regions in the flow.

The cones are stationary and the flow is assumed to be driven by an arbitrary, unknown disturbance at a large distance from the apex. In this case all velocity components arise at
Fig. 1 The open, two-cone geometry indicating the spherical polar co-ordinate system and the internal angles of the two stationary boundaries. The fluid occupies the gap between the two cones $\alpha < \vartheta < \beta$ and is of infinite extent.

order one and (for non-axisymmetric driving modes) their governing equations are coupled, resulting in a truly three-dimensional flow. No simplification in terms of a streamfunction and swirl component is possible. Thus the governing system of equations we consider consists of the incompressible Stokes equations

\[ \nabla p = \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \]
\[ \mathbf{v}(r, \alpha, \varphi) = 0, \quad \mathbf{v}(r, \beta, \varphi) = 0, \]
\[ \mathbf{v}(r, \vartheta, \varphi) \to 0 \quad \text{as} \quad r \to \infty \]

where $\mathbf{v}$ and $p$ represent the velocity and pressure fields respectively and (2.2) expresses the no-slip boundary conditions.

We wish to determine the asymptotically dominant flow near the apex for arbitrary three-dimensional disturbances and, in this context, the precise nature of the driving mechanism is unimportant. The problem is linear and the velocity can be expressed as a superposition of flow modes. We allow for non-axisymmetric flows by incorporating an azimuthal wave number $m$. The Stokes flow analysis determines which of the flow modes dominates as the corner is approached and hence the flow characteristics. Only the coefficients in the linear combination of flow modes will depend upon the external driving, not the flow features.

Writing the velocity and pressure fields as similarity solutions of the second kind given by

\[ \mathbf{v}(r, \vartheta, \varphi) = \frac{1}{\sin \vartheta} r^{-\Lambda} \mathbf{V}(\vartheta) e^{im\varphi}, \quad p(r, \vartheta, \varphi) = r^{\Lambda-1} P(\vartheta) e^{im\varphi}, \]

defines, with equations (2.1) and (2.2), an eigenvalue problem. For each wavenumber $m$ ($m = 0$ representing axisymmetry) it is found that a sequence of eigenvalues $\Lambda_{mj}$ and eigenmodes, given in terms of the unknown functions $\mathbf{V}_{mj}(\vartheta)$, $P_{mj}(\vartheta)$, exists. Thus, for a
general disturbance, we can write
\[ v = \frac{1}{\sin \vartheta} \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} r^{\lambda_{mj}} V_{mj}(\vartheta) e^{im\varphi}, \quad p = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} r^{\lambda_{mj}-1} P_{mj}(\vartheta) e^{im\varphi}. \] (2.4)

Let us formulate the eigenvalue problem. It is convenient to express the system in terms of the scaled variable
\[ x = (\vartheta - \alpha)/\delta, \quad x \in [0, 1] \quad \text{where} \quad \delta = \beta - \alpha. \]
In addition, to ensure that eigenvalues remain of order unit even for small cone openings, we scale the eigenvalues so that
\[ \lambda_{mj} = \delta \Lambda_{mj}. \]
Taking our similarity solution (2.3) and writing \( V(\vartheta) = f(x) = (f(x), g(x), h(x)) \) the governing system then becomes
\[
\begin{align*}
  f'' - \delta \cot \vartheta f' &+ \left[ \lambda(\lambda + \delta) + \delta^2 (\cot^2 \vartheta - m^2 \csc^2 \vartheta - 1) \right] f \\
  -2\delta g' - 2\delta^2 m \csc \vartheta \partial h & = \delta(\lambda - \delta) P, \\
  g'' - \delta \cot \vartheta g' &+ \left[ \lambda(\lambda + \delta) - \delta^2 m^2 \csc^2 \vartheta \right] g + 2\delta f' \\
  -2\delta^2 \cot \vartheta f - 2\delta^2 \cot \vartheta \csc \vartheta \partial h & = \delta P' - \delta^2 \cot \vartheta P, \\
  h'' - \delta \cot \vartheta h' &+ \left[ \lambda(\lambda + \delta) - \delta^2 m^2 \csc^2 \vartheta \right] h + i2m\delta^2 \csc \vartheta \partial f \\
  +2m\delta^2 \cot \vartheta \csc \vartheta \partial g + im\delta^2 \csc \vartheta \partial P & = \lambda \delta \csc \vartheta \partial h = 0,
\end{align*}
\] (2.5)
with boundary conditions
\[ f(0) = f(1) = g(0) = g(1) = h(0) = h(1) = 0, \] (2.6)
where a prime denotes differentiation with respect to \( x \). As above, we abbreviate \( \lambda = \lambda_j = \lambda_{mj} \) in places: the meaning should be clear from the context.

Thus (2.5), (2.6) represent our eigenvalue problem for each wavenumber \( m \). The eigenvalues \( \lambda_j \) can be complex and we order them by increasing real part: \( 0 < \Re \lambda_1 < \Re \lambda_2 < \Re \lambda_3 \cdots \) (away from points of degeneracy). Complex eigenvalues will occur as complex conjugate pairs but for convenience we shall only label the eigenvalues with positive imaginary component. Due attention to the conjugates will be exercised when taking real parts of physical flow variables. It is clear that the case of axisymmetric flow \((m = 0)\) discussed by Hall et al. (17) is special and the system becomes decoupled. Then two sequences of eigenvalues arise and, when the driving is provided by a rotating spherical lid, the eigenvalues for the azimuthal flow are real and the leading order velocity field in the \((r, \vartheta)\)-plane occurs at \( \mathcal{O}(Re) \) in a low Reynolds number expansion. However, in the general non-axisymmetric case the system is coupled and all velocities appear at \( \mathcal{O}(1) \).

Our ordering of the eigenvalues means that, as the apex is approached, for each wavenumber the velocity is dominated by the first term in the series (2.4), i.e. \( \csc \vartheta r^{\lambda_{mj}} V_{mj} e^{im\varphi} \). Thus our discussion of the dominant flow features due to a general disturbance can focus upon the initial eigenvalues for each wavenumber \( m \). However,
whereas in two dimensions the appearance of eddies within a geometry depends purely upon whether the dominant eigenvalue is real or complex, three dimensions is somewhat more complicated. We defer a discussion of the nature of eddies in three dimensions until section 3.

Before we outline the numerical method by which the eigenvalues and eigenfunctions of the system (2.5), (2.6) are found it is worth noting an important symmetry. The equations are unaffected by the transformation

\[(f, g, h; P; r, \vartheta, \phi) \rightarrow (f, g, -h; P; r, \vartheta, -\phi).\]  

(2.7)

This can be thought of as a reversal of helicity and we shall see that this is reflected in the flow modes found.

2.2 The numerical solution scheme

It is clear that the system (2.5) with its associated boundary conditions (2.6) requires a numerical method of solution. We present here the numerical schemes used to determine the eigenvalues \(\lambda\), velocity components \(f, g, h\) and pressure field \(P\). Different approaches were adopted in order to provide a high degree of accuracy and verification of numerical results. Comparison was made with results from independent studies of the single cone geometry ((13), (14)) and the axisymmetric problem ((16), (17), (18)) to provide further corroboration of our results. In section 4 we give a detailed discussion of the results and their physical interpretation.

The principal numerical method used adopts a finite difference approach and solves the resulting linear system of equations by a QZ algorithm. An evenly spaced grid of \(N + 2\) nodes is constructed over the domain \(x \in [0, 1]\) and standard, fourth-order central finite differences used for the derivatives at interior nodes. From the boundary conditions (2.6) the values of \(f, g\) and \(h\) are known at the first and last nodes corresponding to \(x = 0\) and \(x = 1\). Near the boundaries skewed finite difference formulae are used to produce third-order approximations to the derivatives of the unknown variables. For numerical convenience a \(7N \times 1\) vector \(w = (f, g, h, P, \lambda f, \lambda g, \lambda h)^T\) is constructed where, in this context, \(f\) represents the \(N\)-vector of internal node values of \(f\). Discretising the derivatives in (2.5) allows the governing equations to be simply expressed as

\[Aw = \lambda Bw,\]  

(2.8)

where

\[
A = \begin{pmatrix}
A_{11} & -2\delta D_1 & A_{13} & \delta^2 & \delta & 0 & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} & 0 & \delta & 0 \\
A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & \delta \\
2\delta & D_1 & A_{43} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & \ldots & 0 & \delta & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]  

(2.9)
Each block of the matrices $A$, $B$ corresponds to a $N \times N$ matrix which is independent of
the eigenvalue $\lambda$. In particular

\begin{align*}
A_{11} &= D_2 - \delta \cot \vartheta D_1 + \delta^2 \left( \cot^2 \vartheta - m^2 \csc^2 \vartheta - 1 \right), \quad A_{13} = -2\delta^2 m \csc \vartheta, \\
A_{21} &= 2\delta D_1 - 2\delta^2 \cot \vartheta, \quad A_{22} = D_2 - \delta \cot \vartheta D_1 - \delta^2 m^2 \csc^2 \vartheta, \\
A_{23} &= -2\delta^2 m \cot \vartheta \csc \vartheta, \quad A_{24} = -\delta D_1 + \delta^2 \cot \vartheta, \\
A_{31} &= -A_{13}, \quad A_{32} = -A_{23}, \quad A_{33} = A_{22}, \quad A_{34} = -m\delta^2 \csc \vartheta, \\
A_{43} &= \imath m \delta \csc \vartheta, \\
\end{align*}

(2.10)

where $D_1$ and $D_2$ are the fourth-order, discretised differentiation operators $d/dx$ and
\(d^2/dx^2\) and 1 denotes the identity matrix. At this point, it is important to note two
important aspects of the treatment of the pressure. First, to avoid the generation of spurious
eigenmodes, central finite differences which include the centre node are used for the pressure
derivatives. Second, since the pressure is not known at the conical boundaries, it is necessary
to use skewed finite differences which only include internal nodes to calculate the derivatives
of $P$ near the boundaries. Although this can result in singular submatrices, it does not affect
the invertability of the matrix $A$ and, hence, our formulation allows us to avoid reference
to the pressure at boundary nodes without numerical complication. The numerical scheme
applied for the pressure is, as for the velocity, fourth order at internal nodes and third order
near the boundaries.

Using the formulation (2.8) the eigenvalues and eigenfunctions were computed using the
QZ-algorithm. This is a convenient numerical method as it is robust and does not require
initial estimates of eigenvalues or eigenfunctions. Close agreement was demonstrated with
the numerical results from previous studies of the single cone and axisymmetric cases.
As additional verification of our numerical results and the accuracy of our discretised
differential operators, the Matlab boundary value problem solver \texttt{bvp4c} was also applied. By
eliminating the pressure from the azimuthal component of Stokes equations and $f$ from the
equation of continuity, (2.5), (2.6) were rewritten as a system of two third-order, ordinary
differential equations for $g$ and $h$. The continuity equation applied at $x = 0, 1$ together with
the normalisation condition $h'(0) = 1$ supplied the modified boundary conditions required.
Once again there was close agreement with the results obtained from the QZ-algorithm.

2.3 Alternative formulation in terms of special functions

It is possible to construct solutions of the system (2.1), (2.2) in terms of known special
functions. Although this approach has the benefit of producing expressions for the flow fields
in terms of continuous functions rather than discrete values at selected nodes, determining
the eigenvalues still requires a numerical approach and is cumbersome. However, the
construction and resulting form of the fields in this instance is instructive and thus a brief
description of the approach, extending the method used by Shankar (14), is included.

The problem of determining the velocity and pressure fields of (2.1) is equivalent to finding
a vector potential $\mathbf{A}$ and a scalar potential $B$, satisfying the scalar and vector Laplace’s
equations respectively, such that

\begin{align*}
\mathbf{v} &= \nabla (r \cdot A + B) - 2\mathbf{A}, \\
p &= 2\nabla \cdot A.
\end{align*}

(2.11)
In order to satisfy the boundary conditions on the conical boundaries we require six linearly independent potentials. Two linearly independent similarity solutions of the scalar Laplace’s equation are given by

\[ B_1(r, \theta, \varphi; \tau) = r^\tau P_\tau^m(\cos \theta)e^{im\varphi}, \quad B_2(r, \theta, \varphi; \tau) = r^\tau Q_\tau^m(\cos \theta)e^{im\varphi}, \]  

(2.12)

where \( P_\tau^m(\cos \theta) \) and \( Q_\tau^m(\cos \theta) \) are the associated Legendre functions of the first and second kinds respectively and \( \tau \) is the radial exponent eigenvalue. From these scalar potentials we can generate four linearly independent vector potentials

\[
\mathcal{A}_1(r, \theta, \varphi; \tau) = e_\varphi \times \nabla (rB_1) = r^\tau e^{im\varphi} \left\{ -\frac{im}{\sin \theta}P_\tau^m(\cos \theta)e_\theta + \frac{d}{d\theta}P_\tau^m(\cos \theta)e_\phi \right\},
\]

\[
\mathcal{A}_2(r, \theta, \varphi; \tau) = e_\varphi \times \nabla (rB_2) = r^\tau e^{im\varphi} \left\{ -\frac{im}{\sin \theta}Q_\tau^m(\cos \theta)e_\theta + \frac{d}{d\theta}Q_\tau^m(\cos \theta)e_\phi \right\},
\]

\[
\mathcal{A}_3(r, \theta, \varphi; \tau) = B_1 e_\varphi = r^\tau e^{im\varphi}P_\tau^m(\cos \theta) (e_r - \sin \theta e_\theta),
\]

\[
\mathcal{A}_4(r, \theta, \varphi; \tau) = B_2 e_\varphi = r^\tau e^{im\varphi}Q_\tau^m(\cos \theta) (e_r - \sin \theta e_\theta).
\]

Here \( e_\varphi, e_\theta, e_\phi \) are the standard spherical polar and Cartesian basis vectors and by construction these potentials satisfy Laplace’s equation.

Expressing the potentials \( \mathcal{A}, \mathcal{B} \) of (2.11) as linear combinations of these functions

\[
\mathcal{A} = c_1 \mathcal{A}_1(r, \theta, \varphi; \Lambda) + c_2 \mathcal{A}_2(r, \theta, \varphi; \Lambda) + c_3 \mathcal{A}_3(r, \theta, \varphi; \Lambda) + c_4 \mathcal{A}_4(r, \theta, \varphi; \Lambda),
\]

\[
\mathcal{B} = c_5 B_1(r, \theta, \varphi; \Lambda + 1) + c_6 B_2(r, \theta, \varphi; \Lambda + 1),
\]

results in a velocity \( \mathbf{v} \) proportional to \( r^\Lambda \). Applying the boundary conditions (2.2) we obtain a system of six homogeneous linear equations for the complex constants \( c_1, c_2, \ldots, c_6 \). We require that this system has a non-trivial solution and hence that the determinant of the coefficient matrix vanishes. This condition is then used to find the eigenvalues \( \Lambda \).

3. Flow modes & eddies

3.1 Velocity formulations

Before presenting our numerical results in section 4, we set out here a discussion of the structure of the flow modes generated by individual eigenvalues and interpret them in the context of eddy structures. We begin by considering a single complex eigenvalue \( \Lambda = a + ib \).

The complex eigenvalues appear as conjugate pairs and due to the symmetry in \( \varphi \) expressed by (2.7) we obtain four independent velocity fields given by

\[
e^{im\varphi}r^\Lambda \csc \theta \mathbf{f}, \quad e^{-im\varphi}r^\Lambda \csc \theta \tilde{\mathbf{f}},
\]

(3.1)

and their complex conjugates, where \( \mathbf{f} = (f, g, h) \) and \( \tilde{\mathbf{f}} = (f, g, -h) \).

We are concerned with physical flow behaviour and hence shall take real parts of all flow fields. (It is clear that the imaginary part of any particular velocity field can be simply reproduced by a rotation in \( \varphi \) of \( \pi/2m \).) In addition, due to the complex exponent of \( r \), the phase of each of the two remaining solutions \( (m\varphi + \xi) \) and \( -m\varphi - \xi \) respectively, where \( \xi = b \log r \) is dependent upon the radial variable. Thus any \( \varphi \)-phase shift in one solution
relative to the other can be negated by a radial scaling. Therefore we can effectively reduce our four flow solutions (3.1) to the two linearly independent real velocities
\[
\begin{align*}
\mathbf{v}_1 &= r^a \csc \vartheta \left[ \cos(m\varphi + \xi) \Re f - \sin(m\varphi + \xi) \Im f \right], \\
\mathbf{v}_2 &= r^a \csc \vartheta \left[ \cos(m\varphi - \xi) \Re \hat{f} + \sin(m\varphi - \xi) \Im \hat{f} \right].
\end{align*}
\]

(3.2)

A useful, alternative formulation is to construct symmetric and antisymmetric modes from these velocities:
\[
\begin{align*}
\mathbf{v}_s &= \frac{1}{2} (\mathbf{v}_1 + \mathbf{v}_2) = r^a \csc \vartheta \left( \Re \{ fe^{i\xi} \} \cos m\varphi, \Re \{ ge^{i\xi} \} \cos m\varphi, -\Im \{ he^{i\xi} \} \sin m\varphi \right), \\
\mathbf{v}_a &= \frac{1}{2} (\mathbf{v}_2 - \mathbf{v}_1) = r^a \csc \vartheta \left( \Im \{ fe^{i\xi} \} \sin m\varphi, \Im \{ ge^{i\xi} \} \sin m\varphi, -\Re \{ he^{i\xi} \} \cos m\varphi \right).
\end{align*}
\]

(3.3)

The antisymmetric mode corresponds to the symmetric mode under a phase shift of \(-\pi/2m\) in \(\varphi\) and a radial scaling of \(e^{\pi/2b}\) : \(\mathbf{v}_a(r_0 e^{\pi/2b}, \vartheta_0, \varphi_0 - \pi/2m) = e^{-\pi a/2b} \mathbf{v}_s(r_0, \vartheta_0, \varphi_0)\). However, in this case, the general solution to (2.5), (2.6) requires both linearly independent solutions.

Let us now briefly consider the case of a real eigenvalue. The corresponding eigenfunctions \((f, g, h)\) remain complex, but of constant argument: \(f = e^{\eta}(\bar{f}(x), \bar{g}(x), i\bar{h}(x))\) where \(\eta\) is a constant and \(\bar{f}, \bar{g}, \bar{h}\) are real valued functions. Thus (3.1a) corresponds to the complex conjugate of (3.1b) and vice versa. Hence solutions corresponding to real eigenvalues are given by the real and imaginary parts of
\[
e^{im\varphi + \eta r^a} \csc \vartheta (\bar{f}, \bar{g}, i\bar{h}),
\]

(3.4)

which, since the phase of \(\varphi\) is unimportant, reduce to a single mode
\[
\mathbf{v} = r^a \csc \vartheta (\bar{f} \cos m\varphi, \bar{g} \cos m\varphi, -\bar{h} \sin m\varphi).
\]

(3.5)

3.2 Eddy structures

The concept of eddies in two-dimensional geometries and three-dimensional flows possessing axial symmetry is straightforward. For example, a complex radial exponent in a streamfunction (or velocity component) indicates that as the corner is approached it must alternate in sign. Hence regions of recirculation, or eddies, are formed whose size and intensity decays exponentially. Neighbouring eddies circulate in opposite senses and are separated by surfaces that intersect the boundary. Importantly streamlines within each eddy are closed.

In a fully three-dimensional flow a complex radial exponent does not guarantee eddies. Although as a corner is approached on any radial line the flow field will alternate, particle paths need not remain bounded. Due to variations in the other co-ordinate directions streamlines can escape and ‘break up’ the flow.

Shankar (6) states that the concept of an eddy is a useful but difficult one to define. In two-dimensional flows he defines an eddy as “a region of closed streamlines which either contains a stagnation point or contains a fluid region of closed streamlines with such stagnation points”. However, as he concedes, three dimensions is more topologically complex. We shall therefore adopt a common-sense definition of eddies for our three-dimensional flows – **bounded regions of the fluid domain within which fluid particles travel**
on closed surfaces. These subdomains are bounded by surfaces that intersect the solid walls, but they are not necessarily surfaces of zero velocity. Rather, identifying eddies relies upon finding surfaces of zero mass flux. It should be stressed that imposing a formal definition upon an eddy does not invalidate the useful notion of recirculation regions or eddy-like structures. In many practical applications it is possible to observe eddies which, although they do not strictly adhere to our definition, nevertheless create regions of the flow which are predominantly disconnected from the main flow (see (19), (20)). We must make a distinction between the existence of an eddy in a formal sense (fluid particles that are bounded by closed surfaces) and the pragmatic approach of many practical applications. We will draw this distinction later, but first let us continue with our strict interpretation.

Although a complex eigenvalue in our similarity solutions is a necessary condition for eddies, it is not sufficient. From either formulation, (3.2) or (3.3), an alternating flow is generated as the origin is approached along any radial line, due to the sinusoidal dependence upon log \( r \). On any constant \( \phi \)-plane the flow in the \( r, \theta \) plane will resemble eddies. However, a fluid particle will be transported away from this plane by the azimuthal velocity and the form of the velocity field will vary with \( \phi \). Hence fluid particles may change their sense of circulation and move freely throughout the fluid domain.

However, it is possible for our solutions to produce eddies in the strictest sense. Consider a single flow mode. For the purely symmetric or antisymmetric flow modes of (3.3) we find that since the \( e_r \) and \( e_\theta \) components possess the same \( \phi \) dependence the direction of the velocity field in the \( r, \theta \) plane does not vary with \( \phi \). Hence the projections of particle paths onto this plane are closed curves. We conclude that fluid trajectories lie on the surfaces of ‘tubes’ of constant cross-section, rotated about the \( z \)-axis. In addition particles are trapped between planes of zero \( \phi \)-velocity given by \( \phi = k\pi/m \) for symmetric velocities and \( \phi = (2k+1)\pi/2m \) for antisymmetric velocities. These are surfaces across which there is zero mass flux. (On these surfaces closed trajectories are formed analogous to the two-dimensional eddy structure.) Thus, by our definition, eddies are formed (but not necessarily closed particle trajectories).

The above argument also applies to a combination of purely symmetric or purely anti-symmetric flow modes corresponding to the same wavenumber. However, a general linear combination of both symmetric and antisymmetric velocities (e.g. \( v_1, v_2 \)) will destroy the separation surfaces between recirculation regions and ‘tubes’ are no longer formed. In a similar manner combinations of flow modes arising from different wavenumbers will also destroy eddy structures. Thus to ensure closed recirculation regions we must ensure only pure symmetric or anti-symmetric flow modes for a single wavenumber are excited. We return here to the distinction between eddies given by our earlier definition and a practical interpretation of the term. Although when both symmetric and antisymmetric modes are excited, particle paths on closed surfaces will not be present, eddy-like structures may well be observed. The deviation from closure may be small and the time taken for a particle to exit a recirculating region of the flow very long. For many applications a small degree of deviation is unimportant and therefore it may be necessary, when the destruction of eddies is predicted, to assess the departure from closure.

It should be noted that our ordering of the eigenvalues means that, as we approach the apex, the flow will be dominated by the eigenvalue with smallest real part. Any deviation from this eigenmode will exponentially decrease. Hence for a symmetric flow composed of several wavenumbers and eigenmodes the flow features will strongly resemble the dominant
mode as \( r \to 0 \) and if this possesses eddies then, in practice, any particle will take a considerable time to move from one region of the flow to the next. Thus in the same way that a small deviation from pure modes (e.g. (3.3)) may preserve what might be observed as an eddy, near the corner even significant departures from pure modes may allow recirculation regions to be present. We may still refer to the dominant eigenvalue implying the existence of eddy-like structures if they are observed (although strictly, over long time scales, closed subdomains do not exist) but it may not be appropriate in cases when the recirculation regions will exist on a scale that is not practically observable. One may suspect that for most applications we should discard our strict definition of an eddy and rely solely upon a dominant complex eigenvalue. However, for a combination of both symmetric and antisymmetric flows with the same leading eigenvalue, eddies can be totally destroyed and the flow become fully integrated.

4. Numerical results and flow visualisation

4.1 Eigenvalue spectra

In this section we present the results of the numerical methods given in section 2 paying particular attention to the relative dominance of complex eigenvalues which allow the possibility of eddies. Hall et al. (17) provide a full investigation of the axisymmetric \((m = 0)\) case and hence we concentrate here on the first few non-axisymmetric wavenumbers. Comparison is made with the axisymmetric flow and the non-axisymmetric flow within a single cone. We give results, illustrating the evolution and bifurcations of the eigenvalue spectra for variations in geometry.

We begin by considering the first non-zero wavenumber \( m = 1 \). Figure 2 plots the real part of the first five eigenvalues (ordered by increasing real parts) for \( \alpha = 1^\circ \) and increasing values of \( \beta \). A small value of \( \alpha \) is chosen so that we might gauge the effect of the addition of a slender inner cone on the eigenvalue spectra for the same wavenumber in the single cone geometry. The dotted lines indicate that the eigenvalue is real and the crosses indicate a complex eigenvalue. Thus it can be seen that the leading eigenvalue is always real and that the first two complex eigenvalues demonstrate simple bifurcations for larger values of \( \beta \). The dominant real eigenvalue is reproduced for all values of \( \alpha \). Hence for general disturbances of wavenumber one we would never observe eddies in the double-cone geometry. This is in contrast to the single cone geometry where eddies are possible for half angles less than 74.5\(^\circ\). In common with the axisymmetric case (see (17)), the effect of introducing a slender inner cone is therefore marked.

It is worth noting that the leading eigenvalue in the axisymmetric case is also real but in this case eddies may arise. The \( m = 0 \) problem possesses two sequences of eigenvalues: a real sequence corresponding to the swirl velocity in the \( \varepsilon_\phi \) direction, and a second sequence corresponding to the meridional streamfunction. The leading eigenvalue in the axisymmetric case arises in the swirl function and therefore does not affect the presence of eddies. It might be thought that for \( m > 0 \), since the system is coupled, this real azimuthal eigenvalue provides the leading order eigenvalue, now seen in all velocity components. However, the leading order eigenvalue for \( m > 0 \) arises from a redundant eigenvalue \( \lambda_1 = 1 \) when \( m = 0 \). This eigenvalue makes no contribution to the velocity for \( m = 0 \), being purely a pressure eigenmode, and is usually neglected.\(^1\) But it can be seen from tables 1, 2, which list the

\(^1\) The pressure is proportional to \( r^{\lambda - 1} \) and is constant in this case. The governing differential equations

}\( r^{\lambda - 1} \) and is constant in this case. The governing differential equations
Fig. 2  Plot showing the real parts of the leading eigenvalues for \( m = 1 \) with \( \alpha = 1^\circ \) and an increasing cone opening. Real eigenvalues are denoted by dots, complex eigenvalues by crosses.

Table 1  The three leading eigenvalues for \( \alpha = 5^\circ, \beta = 45^\circ \) and wavenumbers \( m = 0, 1, 2, \ldots, 8 \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \Lambda_1 )</th>
<th>( \Lambda_2 )</th>
<th>( \Lambda_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>4.5745</td>
<td>5.8492 + 2.8465i</td>
</tr>
<tr>
<td>1</td>
<td>2.0798</td>
<td>5.3922</td>
<td>6.0925 + 2.2694i</td>
</tr>
<tr>
<td>2</td>
<td>5.0317 + 0.8914i</td>
<td>7.9070 + 0.9700i</td>
<td>11.4086 + 2.3079i</td>
</tr>
<tr>
<td>3</td>
<td>6.4173 + 1.5212i</td>
<td>9.9200</td>
<td>11.0026 + 1.6736i</td>
</tr>
<tr>
<td>4</td>
<td>8.0109 + 1.6827i</td>
<td>11.5032</td>
<td>12.5710 + 2.0564i</td>
</tr>
<tr>
<td>5</td>
<td>9.6000 + 1.7962i</td>
<td>13.1480</td>
<td>14.2461 + 2.1635i</td>
</tr>
<tr>
<td>6</td>
<td>11.1700 + 1.8943i</td>
<td>14.7777</td>
<td>15.9134 + 2.2399i</td>
</tr>
<tr>
<td>7</td>
<td>12.7245 + 1.9822i</td>
<td>16.3909</td>
<td>17.5624 + 2.3086i</td>
</tr>
<tr>
<td>8</td>
<td>14.2669 + 2.0622i</td>
<td>17.9905</td>
<td>19.1953 + 2.3723i</td>
</tr>
</tbody>
</table>

leading three eigenvalues for two specific geometries and wavenumbers from 0 to 8, that it is this eigenvalue which corresponds to the dominant mode for subsequent wavenumbers. The second eigenvalue for \( m = 1 \) corresponds to the leading real azimuthal eigenvalue found for the axisymmetric flow.

Figures 3, 4 show the real parts of the eigenvalue spectra for wavenumbers \( m = 2 (\alpha = 1^\circ, 5^\circ) \) and \( m = 3 (\alpha = 5^\circ) \) respectively. It is found that throughout the parameter range and for greater wavenumbers the dominant eigenvalue is always due to wavenumber \( m = 1 \) and hence real. Thus we can extend our statement above to conclude that for general disturbances that include wavenumber one, eddies will not be observed. It can also be seen

for \( f, g, h \) [(2.5), (2.6)] become homogeneous with homogeneous boundary conditions and thus the velocity is identically zero.
Non-axisymmetric Stokes flow between concentric cones.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>5.6283</td>
<td>7.6359 + 4.1601i</td>
</tr>
<tr>
<td>1</td>
<td>1.4912</td>
<td>5.8186</td>
<td>7.7329 + 4.0973i</td>
</tr>
<tr>
<td>2</td>
<td>2.5342</td>
<td>6.3523</td>
<td>8.0220 + 3.9172i</td>
</tr>
<tr>
<td>3</td>
<td>3.7688</td>
<td>7.1447</td>
<td>8.4970 + 3.6424i</td>
</tr>
<tr>
<td>4</td>
<td>5.0975</td>
<td>8.1099</td>
<td>9.1431 + 3.3033i</td>
</tr>
<tr>
<td>5</td>
<td>6.4920</td>
<td>9.1799</td>
<td>9.9370 + 2.9280i</td>
</tr>
<tr>
<td>6</td>
<td>7.9469</td>
<td>10.3006</td>
<td>10.8503 + 2.5340i</td>
</tr>
<tr>
<td>7</td>
<td>9.4801</td>
<td>11.4079</td>
<td>11.8586 + 2.1192i</td>
</tr>
<tr>
<td>8</td>
<td>11.2170</td>
<td>12.2940</td>
<td>12.9661 + 1.6447i</td>
</tr>
</tbody>
</table>

Table 2. The three leading eigenvalues for $\alpha = 30^\circ$, $\beta = 60^\circ$ and wavenumbers $m = 0, 1, 2, \ldots, 8$.

that for $m = 2, 3$ the two leading eigenvalues are initially real for small $\beta$ but merge at $\beta = \beta_m$ to give a complex conjugate pair of eigenvalues. This eigenvalue later bifurcates at $\beta = \beta_b$ to once again give two real eigenvalues. For greater values of $\beta$ a complex sequence of bifurcations occurs but the leading eigenvalue remains real. Thus it is only for $\beta_m < \beta < \beta_b$ that flows due to these wavenumbers can possess eddies. It is found that $\beta_m$ increases and $\beta_b$ decreases with increasing $\alpha$. For $m = 2$ the intermediate state is eliminated at $\alpha \simeq 6^\circ$ and the leading eigenvalue is always real, see figure 5. This transition occurs for greater values of $\alpha$ as $m$ increases.

In order to highlight the effect of inserting an inner cone in a single-cone geometry we can compare the value at which the initial complex eigenvalue bifurcates into two real eigenvalues ($\beta = \beta_b$) for this configuration with the double cone geometry and $\alpha = 1^\circ$. Table 3 shows this value for the first four wavenumbers. It is clear that, as we observed, the inclusion of an inner cone has a significant impact for wavenumbers $m = 0, 1$. But the
Fig. 4 Plot showing the real parts of the leading eigenvalues for $m = 3$ with $\alpha = 5^\circ$ and an increasing cone opening. Real eigenvalues are denoted by dots, complex eigenvalues by crosses. The points at which two real eigenvalues merge ($\beta_m$) and bifurcate ($\beta_b$) are marked.

Fig. 5 Plot showing the real parts of the leading eigenvalues for $m = 2$ with $\alpha = 10^\circ$ and an increasing cone opening. Real eigenvalues are denoted by dots, complex eigenvalues by crosses.

influence is less marked for $m \geq 2$. This is due to the fact that for these wavenumbers the $z$-axis is a stagnation line for the single-cone geometry.

4.2 Flow visualisations

Finally in order to illustrate the nature of the non-axisymmetric eddy structure and its breakdown we consider particular modes associated with single complex eigenvalues. Figure 6 shows contour plots of the radial component of velocity for the symmetric flow mode ($v_s$) and anti-symmetric flow mode ($v_a$) for $m = 2$, $\alpha = 1^\circ$, $\beta = 45^\circ$ (for which $\lambda_1 = 4.73 + 1.37i$) on the surface $x = 0.5$. Although the radial velocity is zero on $\varphi = (2k + 1)\pi/4$ for the symmetric fields and $\varphi = k\pi/2$ for the anti-symmetric field, these are not the boundaries of the eddies. Remember we require zero mass flux across the bounding surfaces of the
Non-axisymmetric Stokes flow between concentric cones

<table>
<thead>
<tr>
<th>$m$</th>
<th>Single Cone</th>
<th>$\alpha = 1^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>80.9$^\circ$</td>
<td>109$^\circ$</td>
</tr>
<tr>
<td>1</td>
<td>74.5$^\circ$</td>
<td>150$^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>80$^\circ$</td>
<td>78$^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>80.5$^\circ$</td>
<td>80.5$^\circ$</td>
</tr>
</tbody>
</table>

Table 3  Comparison of the bifurcation value $\beta_b$ for the single-cone and double-cone ($\alpha = 1^\circ$) geometries for the dominant complex eigenvalue and wavenumbers $m = 0, 1, 2, 3$.

Fig. 6  Plot showing contours of constant radial velocity for the symmetric flow mode $v_s$ (left) and antisymmetric flow mode $v_a$ (right) on the surface $x = 0.5$ for the leading eigenmode ($\lambda_1$) when $m = 2$. The geometry is given by $\alpha = 1^\circ, \beta = 45^\circ$.

eddies and hence a zero azimuthal component of velocity on $\varphi =$ constant. Nevertheless the eddy structure is clear and from our previous discussion closed eddies are formed. However, in general the dominant flow associated with this wavenumber for this particular geometry will be a linear combination of the antisymmetric and symmetric flow modes for the leading eigenvalue. In figure 7 we plot the contours of constant radial velocity for the linear combination $0.9v_s + 0.1v_a$ on the surface bisecting the two conical boundaries ($x = 0.5$). It can be seen even in this projection that the eddy structure has been broken up.

To better appreciate the break up of the eddy structure we calculate and plot particle trajectories using a Runge-Kutta method to integrate the velocity components. For the remainder of this section we consider the leading eigenvalue $\lambda_1 = 4.60 + 0.51i$ for $m = 2$, $\alpha = 6^\circ$ and $\beta = 50^\circ$. The eigenvalue is complex and hence we expect the purely symmetric and antisymmetric flow modes to exhibit eddies. Figure 8 shows several typical particle trajectories for the symmetric flow field. The eddy structure is obvious and the particle paths are closed – no longer a necessary consequence in three dimensions. However upon the inclusion of a small antisymmetric component in the flow the eddies are no longer present. Figure 9 shows a single particle trajectory for $v = 0.98v_s + 0.02v_a$. It can be seen that now the particle spirals inwards before leaving the circulation and finally moving outward unboundedly.
Fig. 7  Plot showing contours of constant radial velocity on the surface $x = 0.5$ for the leading eigenmode ($\lambda_1$) when $m = 2$ and $v = 0.9v_s + 0.1v_a$. The geometry is given by $\alpha = 1^\circ$, $\beta = 45^\circ$.

Fig. 8  Plot showing four particle trajectories in the symmetric flow field corresponding to the leading eigenvalue for $m = 2$, $\alpha = 6^\circ$, $\beta = 50^\circ$.

5. Conclusions
The Stokes flow between two concentric cones with coincident apices has many interesting features. In particular, in common with the single cone geometry, it is found that for a general disturbance, exciting modes of all wavenumbers, the dominant flow as the apex is approached is due to the leading eigenvalue of the $m = 1$ mode. However, unlike the single-cone geometry, for the double-cone configuration this eigenvalue is invariably real and eddies cannot occur – rather the flow approaches the apex and then recedes to infinity. Eddies are possible if the driving disturbances possess only one wavenumber, $m \neq 1$. They will arise due to complex eigenvalues but it is also necessary that the two linearly independent real velocities that these eigenvalues produce, (3.2), occur with equal amplitudes. By simple rotation about the azimuthal axis the eddy solutions are then equivalent to either the symmetric or antisymmetric velocities given above. Any general, unequal combination of the two velocities will destroy the eddy structure. This conclusion is drawn based upon the
strict definition of eddies given in section 3.2. However, in many practical circumstances a more flexible approach may be appropriate. Approximate recirculation regions may occur in cases where there are more than one wavenumber or when the amplitudes of the two eigenmodes are not equal – in these cases, the fluid particles will not remain in a bounded region of the flow on closed surfaces (our definition of an eddy) but their transition from one region to another may be very slow.

In any practical application of this geometry, for example the cone-plane viscometer, it will be unlikely that either purely axisymmetric flow, or driving due to a single wavenumber can be guaranteed. Hence our findings have important consequences for the presence of genuine eddy structures which in the majority of cases are likely to be broken up. However, the degree to which fluid particles deviate from closed eddy structures may be small and, in cases where a limited amount of mixing is acceptable, further modelling will be necessary to ascertain whether eddies are observable or total breakup has occurred.

In contrast to many other flows the symmetric and antisymmetric flow modes share a common eigenvalue problem and hence common eigenvalues, thus neither dominates as the apex is approached. For wavenumbers $m \geq 2$ it is found that the leading eigenvalue is complex only for a range of $\beta$ which decreases in size as $\alpha$ increases, after which the eigenvalue bifurcates into two real eigenvalues. For sufficiently large $\alpha$ this range is nonexistent and once again the dominant eigenvalue is real for all $\beta$. The addition of a slender inner cone is found to have a significant effect upon the position of the bifurcation angle of the leading complex eigenvalue for $m = 0, 1$ over the single-cone geometry.

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