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Robust controller design for time delay systems using H-infinity techniques

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Robust Controller Design for Time Delay Systems Using H_∞ Techniques

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Abstract

Using H_∞ control, the design problem is formulated in terms of user defined weighting polynomials on the process closed-loop Sensitivity functions to achieve desired closed-loop performance and robust stability in the presence of process modelling error. In this paper stability conditions , in terms of the process sensitivity functions, are derived for processes containing a pure time delay for the following three design scenarios i) the time delay is neglected in the control design and is considered as part of the unmodelled process dynamics ii) A Pade approximation of the delay is included in the controller design and the effect of the residual time-delay modelling error on stability is considered iii) stability conditions for time delay mismatch using the Smith Predictor are derived.

Keywords: H_∞ , time delay systems.

1. Introduction

The design of controllers for systems which are subject to uncertain time delays is challenging due to the potential for closed-loop instability contributed by the uncertainty in the process phase. The Smith predictor compensator which attempts to cancel the effect of the delay from the closed loop can likewise suffer from stability problems due to uncertainty in the time delay (1). H_∞ control which emerged in the 1980's was developed to provide guaranteed stability properties for systems subject to uncertainty (2,3,4). Stability is achieved by specifying user defined weighting functions to shape the closed-loop Sensitivity and Complementary Sensitivity (or Control Sensitivity) functions. In this paper the approach taken is to consider the time delay as part of the phase uncertainty and derive stability conditions on the Complementary Sensitivity function. A relatively low order H_∞ controller can then be designed using standard methods by appropriate selection of the cost function weighting polynomials. The process model with time delay is defined in section 2 The requirements for guaranteed closed-loop stability are derived in section 3. The H_∞ design algorithm employed is presented and the selection of appropriate H_∞ design weightings for robust stability is considered in section 4. Simulation results are discussed in section 5.

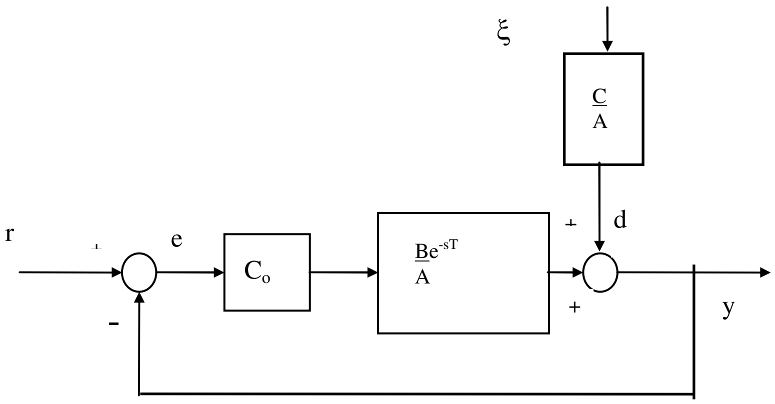

Figure 1: Closed -loop System

2. Process Model

The closed-loop system is as shown in Figure 1 where A(s), B(s) and C(s) are polynomials in the Laplace operator, s. The signals u(t) and y(t) are the process actuating and output signals respectively. The subsystem G_o=B(s)/A(s) represents the nominal (delay free) process dynamics and the process disturbance, d(t), is approximated by passing the white Gaussian noise signal, ξ(t), through the colouring filter C(s)/A(s). The process time delay is T, hence the process model, $G = B(s)/A(s)e^{-sT}$.

3. Robust Design Analysis for Time Delay Systems

The process is represented by the nominal model, $G_o = B/A$. The closed-loop control system is as shown in Figure 1 (where r(t) and e(t) are the process reference and tracking error signals respectively) . Define the closed-loop Sensitivity Functions:

Sensitivity $S = (1 + G_oC_o)^{-1}$

Complementary Sensitivity $T = G_oC_o(1 + G_oC_o)^{-1} = 1 - S$

Control Sensitivity $M = C_o(1 + G_oC_o)^{-1} = C_oS$

where the s dependence of the polynomials is omitted for notational simplicity. In the following analysis, the real process is as previously described, i.e. $G = B/Ae^{-sT}$.

3.1 Robust Stability Requirement for Pure Time Delay

Define the following open-loop transfer functions:

Nominal $L_o(s) = G_o(s)C_o(s)$

Real
$$L(s) = G(s)C_o(s) = G_o(s)e^{-sT}C_o(s) = L_o(s)e^{-sT}$$

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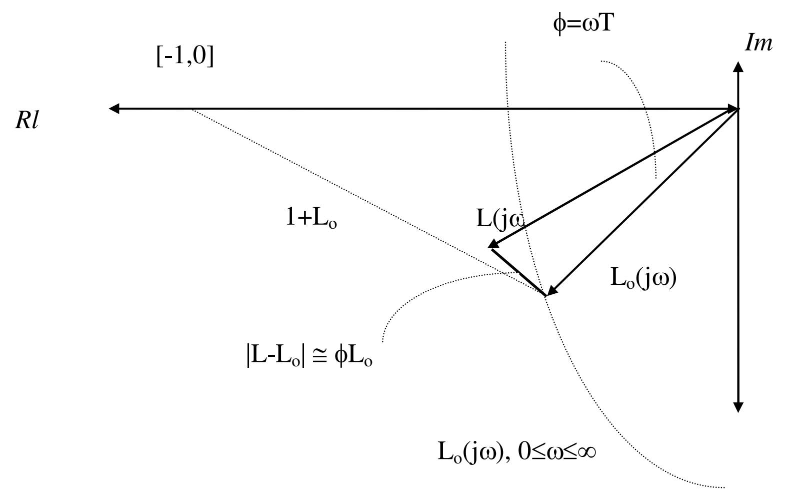

Figure 2: Nyquist Geometry

The controller $C_o(s)$ is designed using H_∞ techniques to guarantee a closed-loop stable system for the nominal plant, $G_o(s)$. To examine the closed-loop stability of the real process consider the Nyquist geometry of the system by setting s = jω (see Figure 2). Consider the magnitude and phase of L(jω). For the purposes of this analysis, the uncertainty is assumed to be only in the pure time delay, hence:

Magnitude: $|L| = |L_o|$

Phase: $arg(L) = arg(L_o) + arg(e^{-sT})$

$= arg(L_o) - \omega T$

$arg(L) \leq arg(L_o) - \omega K$

where K is an upper bound on the unknown time delay, T. Hence the angle φ shown in Figure 2 will always be less than ωK. The length $|L-L_o|$, assuming φ is a small angle, can be approximated of the arc of the circle cut out by φ, i.e., $|L-L_o| \cong \phi L_o$. (The assumption that φ is small can be relaxed to be required at low frequency, particularly in the frequency range near the critical stability point - at higher frequencies the controller roll-off would generally ensure that $|L_o|$ does not encircle [-1, 0]). For guaranteed closed-loop stability, the Nyquist curve L(jω), 0≤ω≤∞, must not enclose the [-1, 0] point. Therefore, the distance from the point represented by $L_o(j\omega)$ to the [-1,0] point must be greater than the length $|L-L_o|$, i.e. $|1+L_o| > |L-L_o|$. Hence:

$$|1+L_o| > \phi\, |L_o|$$

$$|1+L_o| > \omega K\, |L_o|$$

Or: $$|L_o|/|1+L_o| < 1/\omega K$$

Referring to the definition of the Complementary Sensitivity Function, the criterion for robust stability can thus be defined as:

$$|T(j\omega)| < 1/\omega K \text{ for all } \omega \text{ with } 0 \leq \omega \leq \infty$$

3.2 Robust Stability Requirement for First Order Pade approximation of Delay

Using a first order Pade approximation the pure process delay e^{-sT} is approximated by:

$$e^{-sT} \cong (-(T/2)s + 1)/((T/2)s+1)$$

Proceeding with the analysis as in section 3.1 define:

$$\begin{aligned} L_o(s) &= G_o(s)(-(T/2)s + 1)/((T/2)s+1)C_o(s) \\ L(s) &= G(s)C_o(s) \\ &= G_o(s)e^{-sT}C_o(s) \end{aligned}$$

Comparing the magnitude and phase of $L_o(s)$ and $L(s)$:

$$\begin{aligned} | L_o(s) | &= | L(s)| \\ \arg(L(s)) - \arg(L_o(s)) &= -\omega T - \arg(-jT\omega/2+1) + \arg(jT\omega/2+1) \quad (\text{using } s = j\omega) \\ &= -\omega T + 2\text{Tan}^{-1}(T\omega/2) \end{aligned}$$

Use the approximations $\text{Tan}^{-1}(\theta) \cong \theta$, (θ small) and $\text{Tan}^{-1}(\theta) \cong \pi/2$, ($\theta$ large) to find approximate bounds on the phase modelling error due to the Pade approximation of the delay, T, gives:

$$\begin{aligned} \arg(L(s)) - \arg(L_o(s)) &\cong -\omega T + 2.(\omega T)/2 \\ &\cong 0 \quad \text{when } T\omega/2 \text{ is relatively small} \\ \arg(L(s)) - \arg(L_o(s)) &\cong -\omega T + \pi \quad \text{when } T\omega/2 \text{ is relatively large} \end{aligned}$$

Considering the above breakpoints, define a bounding function $K(\omega)$ of the form $k_1 + k_2\omega$ such that $\arg(L(j\omega)) - \arg(L_o(j\omega)) \leq K(\omega)$, for all ω with $0 \leq \omega \leq \infty$. The requirement for closed-loop stability is as before (assuming $\arg(L(s)) - \arg(L_o(s))$ small), i.e.:

$$|1+L_o| > |L-L_o|$$

Hence: $$|1+L_o| > \phi\, |L_o| \quad \text{where } \phi = \arg(L(s)) - \arg(L_o(s))$$

$$|1+L_o| > K(\omega)\, |L_o|$$

Or: $$\boxed{|T| < 1/K(\omega) < 1/ k_1 + k_2\omega}$$

3.3 Robust Stability Requirement for Smith Predictor with Time Delay Mismatch

The Smith predictor structure considered takes the form shown in Figure3 where it is assumed that the process dynamics are known but there is mismatch between the real and nominal process delay (T_1 and T_2 respectively). The requirement is to find a bound on T, the complementary sensitivity function such that closed-loop stability is guaranteed for time delay mismatch between the real process delay (T_1) and the process model delay (T_2) . The controller is designed to stabilise the nominal delay-free process dynamics, G_o. Define:

$$L_o(s) = G_o(s)C_o(s)$$

For the Smith predictor: $$\begin{aligned} L(s) &= G_o(s)(e^{-sT1} - e^{-sT2} + 1).C_o(s) \\ &= L_o(s)\, (e^{-sT1} - e^{-sT2} + 1) \end{aligned}$$

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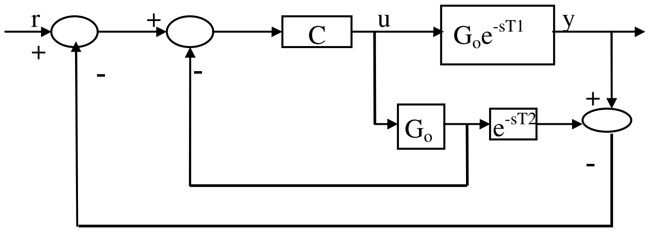

Figure 3: Smith Predictor Compensator

Next consider the Nyquist geometry of the system by setting s = jω (Figure 4):

$$\varphi = \arg(L(j\omega)) - \arg(L_o(j\omega)) = -\omega(T_1-T_2)$$
$$|L(j\omega)| = |L_o(j\omega)(1 + e^{-sT1} - e^{-sT2})|$$
$$\leq |L_o(j\omega)| \,.\, |1+(e^{-sT1} - e^{-sT2})|$$

It can be shown that $e^{-sT1} - e^{-sT2} = 2j\sin(\omega(T_1-T_2)/2)e^{-j(\omega(T1+T2)/2}$, hence:

$$|L(j\omega)| \leq |L_o(j\omega)| \,.\, |1+2j\sin(\omega(T_1-T_2)/2)e^{-j(\omega(T1+T2)/2}|$$
$$\leq |L_o(j\omega)| + |L_o(j\omega)| \,.\, |2\sin(\omega(T_1-T_2)/2)e^{-j(\omega(T1+T2)/2}|$$

Assuming the mismatch between the real and nominal process time delays, (T_1-T_2), is small, bounded by $T_1-T_2 < K$ (hence , at least in the frequency range near the critical point $\omega(T_1-T_2)/2$ is assumed to be small) and using $\sin(\theta) \cong \theta$, θ small:

$$|L(j\omega)| \leq |L_o(j\omega)|(1 + \omega(T_1-T_2))$$
$$\leq |L_o(j\omega)|(1 + \omega K)$$

or: $$|L(j\omega)| - |L_o(j\omega)| \leq |L_o(j\omega)|\,\omega K$$

Using Nyquist stability as before condition for robust stability is:

$$|L(j\omega)| - |L_o(j\omega)| \leq |1 + L_o(j\omega)|$$

which is satisfied if : $|L_o(j\omega)|\,\omega K \leq |1 + L_o(j\omega)|$

i.e.: $$\boxed{|T(j\omega)| \leq 1/\omega K}$$

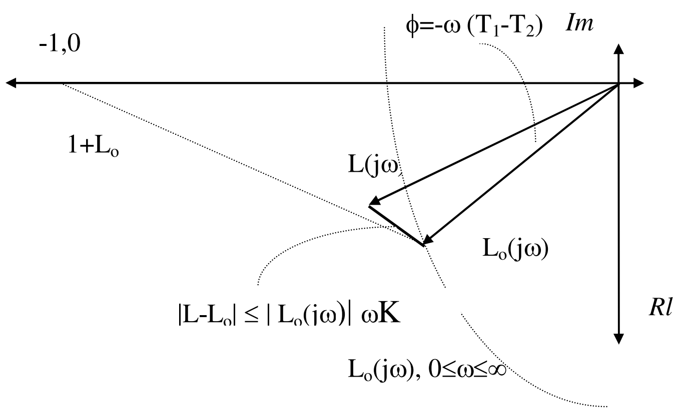

Figure 4: Robust Stability for Smith Predictor

4. H_∞ Control Law

H_∞ control is employed since it enables the designer to shape the closed-loop sensitivity functions via the appropriate selection of cost function weighting polynomials. The emphasis is on the selection of the weighting functions for stability in the presence of unmodelled time delay, or modelling uncertainty introduced by Pade approximation of delay. The H_∞ algorithm employed is described in (5) where the cost function to be minimised is:

$$J = J_\infty = \left\| \frac{B_q}{A_q} S \frac{C}{A} \right\|_\infty^2 + \left\| \frac{B_r}{A_r} M \frac{C}{A} \right\|_\infty^2$$

and $\lambda^2_{opt} = \min\{J_\infty\}$ with C_o stabilising, where λ_{opt} is a scalar. A_q , B_q , A_r and B_r are design weighting polynomials in the Laplace operator, s, which are used to encapsulate the inevitable trade-off between Sensitivity, S, and Control Sensitivity, M (which for design purposes is related to the Complementary Sensitivity, T). To simplify the notation define the following polynomials:

$$A_2 = AB_rA_q$$
$$B_2 = BA_rB_q$$
$$C_2 = CB_qB_r$$

Also define the spectral factor, D_c, where:

$$D_c^*D_c = B_2^*B_2 + A_2^*A_2$$

The following assumptions must be made:

- $A_2^{-1}[B_2C_2]$ must be proper in s.
- D_c has no imaginary roots.
- C is a strictly stable polynomial.
- The polynomials A_q and A_r are stable.

H_∞ Algorithm:

Step1: Solve the following set of coupled Diophantine equations for the polynomials G, H and F:

$$D_c^*G + FAA_q = B_2^*B_qC$$
$$D_c^*H - FBA_r = A_2^*B_rC$$

Step2: Evaluate the upper and lower bounds on λ, λ_{hi} and λ_{lo} as follows:

$$\lambda^2_{hi} = \left\| \frac{F^*F + C_2^*C_2}{D_C^*D_C} \right\|_\infty \qquad \lambda^2_{lo} = \left\| \frac{C_2^*C_2}{D_C^*D_C} \right\|_\infty$$

Step3: Try $\lambda = \lambda_{lo}$ and solve the H_∞ equations for the unknowns u, N and A_σ:

$$A_\sigma^*A_\sigma = D_c^*D_c - \lambda^{-1}C_2^*C_2 \qquad (A_\sigma \text{ stable})$$
$$D_c^*N - AA_\sigma u = Fu^*$$

If u^* is stable this is the non-generic case and two solutions exist. obtain the solution for $\lambda = -\lambda_{lo}$ and go to step 5.

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Step4: For the generic case iterate for λ between λ_{hi} and λ_{lo}. An eigenvalue problem is solved given a $A_\sigma(\lambda)$ for N, u and λ. Then the new λ is used to evaluate the spectral factor A_σ. If the new λ is greater than the old one then the old λ is less than the optimal value, λ_{opt} and vice versa. If the new λ is not between λ_{hi} and λ_{lo} then a bisection rule is used (i.e. $\lambda = (\lambda_{hi} + \lambda_{lo})/2$). Each time the values of λ_{hi} and λ_{lo} are updated as appropriate.

Step5: Evaluate the controller $C_o = C_{on}/C_{od}$ where:

$$C_{on} = A_r(Gu^* + AA_qN)/D_c$$

$$C_{od} = A_q(Hu^* - BA_rN)/D_c$$

(Note in the generic case there are two controllers)

Selection of Weighting Functions:

1. In the results which follow it is assumed that the disturbance colouring filter C/A is 1 (i.e. the process disturbance is assumed to be a White Gaussian Noise signal. Hence, in the H_∞ cost function the weight on S reduces to B_q/A_q and the weight on M is B_r/A_r.
2. The weighting function B_q/A_q is specified by the user to shape the Sensitivity function, S. In general, for good low frequency disturbance rejection, S should be low at low frequency. Hence the weighting on S to achieve this should be high at low frequency. In the results that follow the weight on S is selected to be (1/s) i.e. $B_q = 1$ and $A_q = s$. This particular choice of weight ensures that the controller will contain an integrator for steady state offset removal (refer to the definition of the controller denominator polynomial, C_{od}, which contains the polynomial A_q as one of its roots).
3. Although the requirements for robust stability in section 3 were presented in terms of the Complementary Sensitivity, T , they could equally be presented in terms of the related control Sensitivity, $M = T/G_o$. The weighting on M is thus selected to shape this sensitivity appropriately. For each of the three design scenarios considered, the key requirement for stability involves Complementary Sensitivity roll off at a rate greater than $1/\omega$. For the case of a pure time delay only and using Smith predictor compensation this applies at all frequencies. Using a Pade approximation of the delay in the design, the Complementary Sensitivity should roll off at a rate of $1/\omega$ at least at frequencies greater than the break frequency of π/T. Hence the weighting function B_r/A_r for pure time delay or Smith Predictor is selected to either increase linearly with frequency i.e. $B_r = K.s$ and $A_r = 1$. Where a Pade approximation is included in the design $B_r = K(1+\pi/T.s)$ and $A_r = 1$. The magnitude of the constant K is used to "tune" the controller according to the acceptable performance limits.

5. Results

The process model used in the simulations is $G = Be^{-sT}/A$, where B=2, A=(s+1)(s+2) and T=3. Figure 5 shows the response of the system with the H_∞ controller designed for the plant $G_o = B/A$. The weighting functions are selected to provide robustness to the modelling error introduced by the time delay. The result is a fairly conservative controller design exhibiting a damped response. Including some 'knowledge' of the time delay via a Pade approximation in the controller design results in a less conservative design (Figure 6) - the controller is designed for robust stability to the residual time delay modelling error. Finally, Figures 7 and 8 compare the responses achieved for the Smith Predictor for time delay mismatch in the compensation (T_2 = 3, 3.5, 4, 4.5) showing the improved stability resulting from the robust design outlined in section 3. In all cases the resulting H_∞ controller is of relatively low order (Fourth order for a second order process).

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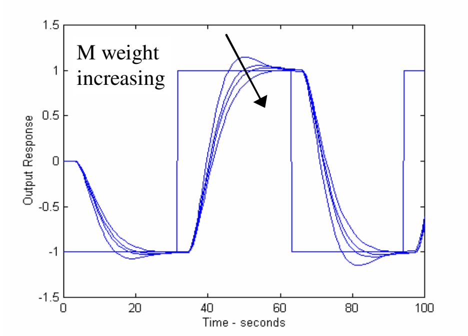

Figure 5: G_o=B/A, M weight=k*s (k=50,70,80,100)

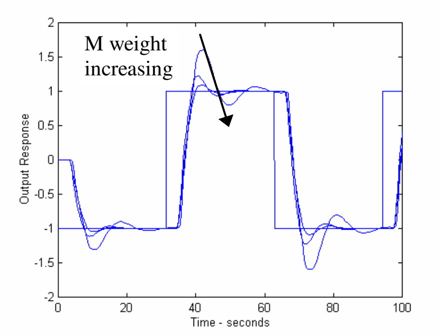

Figure 6: G_o=B(-T/2s+1)/A(T/2s+1), M weight =k*(1+π/3s) (k=5,10,20)

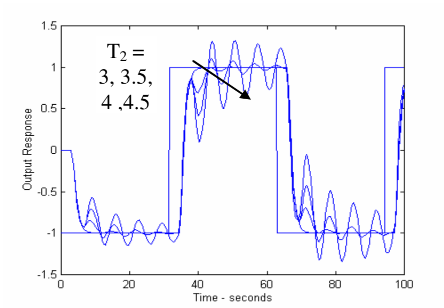

Figure 7: Smith Predictor - (no robustness)

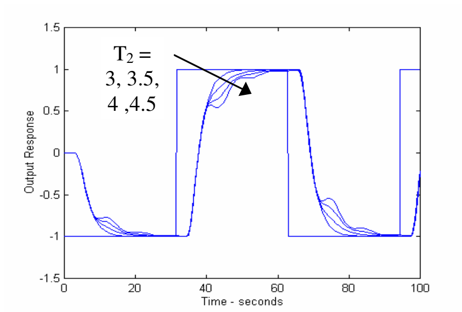

Figure 8: Smith predictor - Robust Design

6. Conclusions

The results presented demonstrate the capability of the standard H_∞ approach in robust controller design for time delay systems. In particular, the performance of the Smith predictor compensator, which provides effective time delay compensation but is notoriously unstable in the presence modelling error in the delay, is considerably improved.

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