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The trace of an abelian group:an application to digraphs

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THE TRACE OF AN ABELIAN GROUP—AN APPLICATION TO DIGRAPHS

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ABSTRACT

The trace of a finite Abelian group is used to give a short proof of a result of Arlinghaus and Harary on the digraph number of such a group.

Introduction

The *trace* of a finite Abelian group was introduced as an invariant of the group by Hoffman in [5], where he gave conditions, in terms of this invariant, for a finite Abelian group to be embedded in the full symmetric group Σ_n. This invariant was subsequently used in [3] to give a simple combinatorial proof of the characterisation of maximal order Abelian subgroups of Σ_n. Our objective here is to give a further application of this invariant to automorphism groups of digraphs. The result we obtain is not new (see [2]) but, similar to the situation in [3], we again obtain a very simple short proof. To make the paper as far as possible self-contained, we have included some elementary material on digraphs. All necessary references to digraphs and graph theory in general can be found in [4].

Let us begin by recalling some basic facts: if A is a finite Abelian group, then A can be written as $A = \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_k}$, where $\mathbb{Z}_{m_i}$ is a cyclic group of order m_i and each m_i is a prime power. The trace of A, $\mathrm{tr}(A)$, is defined to be $\mathrm{tr}(A) = \sum_{i=1}^{k} m_i$. In [5] Hoffman establishes the following theorem.

Theorem 1. *If a finite Abelian group A can be embedded in the full symmetric group Σ_n, then $\mathrm{tr}(A) \leq n$.*

Notice that the converse is trivially true so we have a complete characterisation of when a finite Abelian group can be embedded in Σ_n.

The *digraph number*, $d(G)$, of a finite group G is defined as the minimum number of vertices of a digraph having automorphism group G. Since the automorphism group of a digraph is necessarily a subgroup of the full symmetric group on the set of vertices of the graph, we have immediately the following corollary.

Corollary 2. *If A is a finite Abelian group then* $\mathrm{tr}(A) \leq d(A)$.

It is precisely this corollary which simplifies our approach. The result we seek to derive is the following theorem of Arlinghaus and Harary [2, Teorema].

Main theorem. *If A is a finite Abelian group, then the digraph number of G is equal to the trace of A, $d(A) = \mathrm{tr}(A)$.*

2. Digraphs and automorphism groups

In the sequel we shall have need of the following fundamental results, which may be found in [5, chapter 14].

Proposition 3. (*i*) *The automorphism group of the complement $\bar{H}$ of the digraph H is isomorphic to the automorphism group of H,* $\mathrm{Aut}\,\bar{H} \cong \mathrm{Aut}\,H$.

(*ii*) *If H_1 and H_2 are disjoint, connected, non-isomorphic graphs, then the automorphism group of the disjoint union $H_1 \cup H_2$ is the direct product of* $\mathrm{Aut}\,H_1$ *and* $\mathrm{Aut}\,H_2$.

Recall that the directed coupling of two digraphs H_1, H_2 is obtained from the disjoint union $H_1 \cup H_2$ by joining every vertex in H_1 to every vertex in H_2—see Diagram 2.

Lemma 4. *Let H_1, H_2 be digraphs and H_0 the directed coupling of H_1 and H_2. Then* $\mathrm{Aut}(H_0) = \mathrm{Aut}(H_1) \times \mathrm{Aut}(H_2)$.

PROOF. The result is easily derived via a matrix calculation. Let A_1, A_2 be adjacency matrices for H_1, H_2. Then

$$A_0 = \begin{pmatrix} A_1 & J \\ 0 & A_2 \end{pmatrix},$$

where J is the matrix of all 'ones ', is an adjacency matrix of H_0. The automorphism group of H_0 is isomorphic to the group of all permutation matrices P commuting with A_0. Writing

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

this condition is equivalent to

$$\begin{pmatrix} A_1P_{11} + JP_{21} & A_1P_{12} + JP_{22} \\ A_2P_{21} & A_2P_{22} \end{pmatrix} = \begin{pmatrix} P_{11}A_1 & P_{11}J + P_{12}A_2 \\ P_{21}A_1 & P_{21}J + P_{22}A_2 \end{pmatrix}.$$

The equation yields trace $JP_{21} = 0$, so $P_{21} = 0$ (since P_{21} is a $(0,1)$-matrix), which implies that P_{11}, P_{22} are invertible. They are therefore permutation matrices and thus $P_{12} = 0$.

In order to obtain the inequality $d(A) \leq \mathrm{tr}(A)$ in our main theorem, we need to exhibit certain groups as automorphism groups of digraphs. We collect these fundamental results in the following proposition.

Proposition 5. (*i*) *If p is a prime and a is a positive integer, then the cyclic group $\mathbb{Z}_{p^a}$, of order p^a, has digraph number $d(\mathbb{Z}_{p^a}) = p^a$.*

(*ii*) *If G is a cyclic group of order p^a, p prime, then for each positive integer n there is a digraph H_n having np^a vertices and automorphism group $G \times G \times \ldots \times G$ (n factors), i.e.* $\mathrm{Aut}(H_n) = G^{(n)}$.

PROOF. (i) Since $\mathrm{tr}(\mathbb{Z}_{p^a}) = p^a$, it follows from Corollary 2 that $p^a \leq d(\mathbb{Z}_{p^a})$. However, it is easy to check that a directed cycle of length p^a has automorphism group $\mathbb{Z}_{p^a}$—see Diagram 1 for the case $p^a = 4$. Hence $d(\mathbb{Z}_{p^a}) \leq p^a$ and the result follows.

(ii) Let H_1 be the directed cycle of length p^a so that $\mathrm{Aut}(H_1) = \mathbb{Z}_{p^a} = G$. If H_2 is the directed coupling of H_1 with a disjoint copy of H_1, then it follows from Lemma 4 that $\mathrm{Aut}(H_2) = G \times G$. If we now take H_3 to be the directed coupling of H_1 and H_2, it follows from Lemma 4 that $\mathrm{Aut}(H_3) = G \times G \times G$. Continuing in this way we derive the desired result; note that H_n has np^a vertices.

PROOF OF MAIN THEOREM. Suppose that A is the given finite Abelian group of the form $A = G_1 \times G_2 \times \ldots \times G_r$ where each G_i is a direct product of k_i copies of the cyclic group of prime power order $p_i^{t_i}$. It follows from Proposition 5(ii) that there is a digraph H_i with automorphism group G_i for each $i, 1 \leq i \leq r$. Moreover, by taking complements if necessary, we may assume that each H_i is connected. But then a simple induction using Proposition 3(ii) yields a digraph H with $\mathrm{Aut}\, H = A$. So $d(A) \leq$ the number of vertices in H. However, as the number of vertices in a digraph equals the number of vertices in its complement, this number can be calculated from Proposition 5(ii):

$$\text{number of vertices in } H = \sum_{i=1}^{r} k_i p_i^{t_i} = \mathrm{tr}(A).$$

So $d(A) \leq \mathrm{tr}(A)$. Since the reverse inequality follows from Corollary 2, the proof is complete.

Concluding remarks

If one drops the requirement that one works with digraphs and considers the analogous situation for graphs, then it is possible to define a *graph number* $g(G)$

for a finite group G. However, even when G is Abelian, the calculation of $g(G)$ is vastly more complicated—see [1] for a complete solution of this difficult problem in the Abelian case.

References

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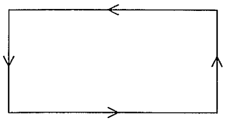

DIAGRAM 1

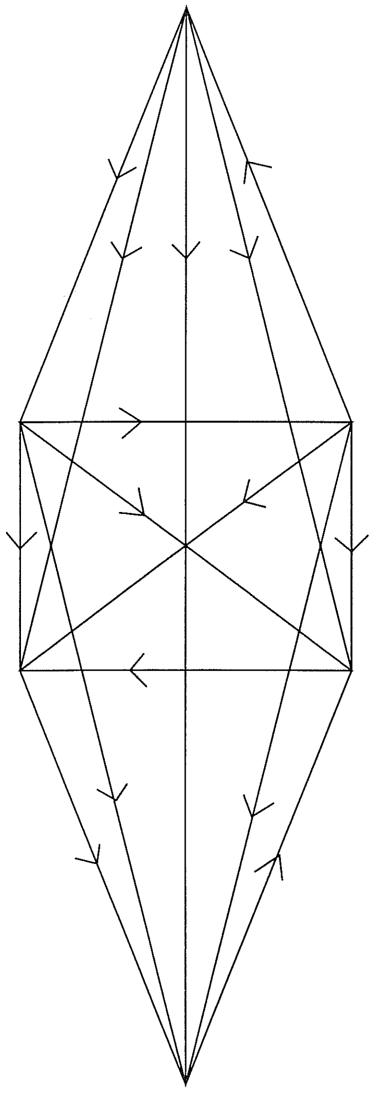

DIAGRAM 2