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# THE WALKER ENDOMORPHISM ALGEBRA OF A MIXED MODULE

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Abstract

Archimedean valuation domains $R$ are characterised in terms of the endomorphism algebras of non-splitting mixed modules of rank 1, in the Walker category.

## Introduction

When studying mixed Abelian groups (or more generally mixed modules over some ring $R$) it is natural to consider the quotient category Walk: this is the category having objects $R$-modules and morphisms $\mathrm{Hom}_W(G, H) = \mathrm{Hom}_R(G, H)/\mathrm{Hom}_t(G, H)$ where $\mathrm{Hom}_t(G, H)$ consists of the $R$-homomorphisms of $G$ into $H$ having torsion image. In [5, problems 8 and 9] Warfield asks which algebras occur as endomorphism algebras in Walk of finite rank $R$-modules when $R = \mathbf{Z}$ or $R = J_p$, the $p$-adic integers.

Partial answers have been given in [1] and [2], where a full embedding of the category of torsion-free $R$-modules into Walk is exhibited. In this paper we look at the situation for modules of torsion-free rank 1 over more general valuation domains. In the first section we find results on the Walker endomorphism algebra and construct mixed modules of torsion-free rank 1, finally obtaining a complete characterisation in the case where $R$ is an Archimedean valuation domain. In the final section we look at the global situation for Abelian groups. Our arguments are direct and do not depend on the deep techniques used in [1] or [2]. Our notation is standard, and unexplained terms may be found in [3] and [4].

## The local case

We suppose throughout this section that $R$ is a valuation domain, with maximal ideal $P$ and field of quotients $Q$, and $M$ is a non-splitting mixed $R$-module of torsion-free rank 1. Note that the condition on $M$ of being non-splitting implies that $M$ does not contain copies of $Q$, since $Q$ is injective, and that $M/tM$ is not isomorphic to $R$; in general, $M/tM$ can be isomorphic either to $Q$ or to a non-principal ideal of $R$; in particular, $M/tM$ is a *uniserial* $R$-module (see [4]). An element $x \in M$ will be called torsion-free if $x \not\in tM$. The Walker endomorphism algebra $\text{End}_R(M)/\text{Hom}_R(M, tM)$ will be denoted by $\text{End}_W(M)$. It is readily seen that $\text{End}_W(M)$ embeds into $\text{End}_R(M/tM) \subseteq \text{End}_R(Q) = Q$; therefore $\text{End}_W(M) = R_J$ for some prime ideal $J$ of $R$. If $\varphi \in \text{End}_R(M)$, we shall denote by $\varphi_W$ the class of $\text{End}_W(M)$ containing $\varphi$.

**Proposition 1.** *Let us suppose that $R$ contains a non-zero minimal prime ideal $L$; if $M$ is a non-splitting mixed $R$-module of torsion-free rank 1, then* $\text{End}_W(M) \subseteq R_L$.

Proof. Let us choose a torsion-free element $x \in M$. If $\varphi \in \text{End}_R(M)$, then, since $M/tM$ is a uniserial $R$-module, either

$$\varphi(x) = \alpha x + t \text{ for some } \alpha \in R,\ t \in tM \tag{1}$$

or

$$\beta\varphi(x) = x + t_1 \text{ for some } \beta \in R,\ t_1 \in tM\,. \tag{2}$$

Since $M$ is of torsion-free rank 1, it follows at once that $\varphi - \alpha \in \text{Hom}(M, tM)$ in case (1), and $\beta\varphi - 1 \in \text{Hom}(M, tM)$ in case (2). We claim that, in case (2), we have necessarily $\beta \not\in L$. By contradiction, let us suppose $\beta \in L$, and let $\psi = \beta\varphi - 1 \in \text{Hom}(M, tM)$. Consider now ker $\psi$; for all $z \in \ker \psi$ we have

$$z = \beta\varphi(z) = \beta^2\varphi^2(z) = \cdots\,.$$

Moreover, since ker $\psi$ is clearly invariant under $\varphi$, this implies that ker $\psi$ is divisible by any power of $\beta$. Since $L$ is minimal, $\beta \in L$ implies that $\cap_n \beta^n R = 0$ so that, for all $r \in R$, there exists $m \in \mathbf{N}$ such that $r$ divides $\beta^m$. We deduce that ker $\psi$ is a divisible submodule of $M$. By (2) we also get that there exists a $0 \neq s \in R$ such that $sx \in \ker \psi$. But if a divisible submodule contains torsion-free elements, then it must contain a copy of $Q$, so that $Q \subseteq M$, and this is the required contradiction.

From (1) and (2) with $\beta \not\in L$ it is easy to conclude that either $\varphi_W = \alpha 1_W$ or $\beta\varphi_W = 1_W$ with $\alpha \in R$, $\beta \in R \setminus L$, i.e. $\text{End}_W(M) \subseteq R_L$. ■

In our next theorem we shall use the following obvious remark on valuation domains:

for every valuation domain $R$ there exists a set of elements $\{r_\alpha : \alpha < \sigma\}$ indexed by an ordinal $\sigma$, such that $r_\alpha R > r_\beta R$ if $\alpha < \beta$ and $\bigcap_{\alpha<\sigma} r_\alpha R = 0$. (*)

**Theorem 2.** *For every valuation domain $R$ there exists a non-splitting mixed $R$-module $M$ with $M/tM \cong Q$.*

Proof. Choose elements $r_\alpha \in R$, $\alpha < \sigma$, as in (*). Let $A = \{\xi, \eta_\alpha : \alpha < \sigma\}$ be a set of symbols; let $F$ be the free $R$-module with basis $A$ and let $N$ be the submodule of $F$ generated by the elements of the form: $\xi - r_\alpha \eta_\alpha$, $\alpha < \sigma$. Then let us define $M = F/N$, $x = \xi + N$, $y_\alpha = \eta_\alpha + N$, $\alpha < \sigma$. Straightforward calculations show that $x$ is a torsion-free element of $M$ and that, for every torsion-free $z \in M$, there exists $0 \neq s \in R$ such that $sz \in Rx$, so that $M/tM$ has rank 1; moreover, from $x = r_\alpha y_\alpha$ for all $\alpha < \sigma$, it follows that $M/tM \cong Q$, since the elements $r_\alpha^{-1}$, $\alpha < \sigma$, generate $Q$ as an $R$-module. It remains to show that $M$ is non-splitting; it is enough to verify that $M$ does not contain copies of $Q$. By contradiction, suppose that a torsion-free $z \in M$ is such that $Qz \subseteq M$. We can assume that $z = sx$ for some $0 \neq s \in P$. There exists $z_1 = bx + \Sigma_i b_{\alpha i} y_{\alpha i} \in Qz$ such that

$$sx = s^2 z_1 .$$

Not all the $b_{\alpha i}$ can be zero, otherwise $s(1 - sb)x = 0$: impossible, since $x$ is torsion-free and $1 - sb$ is a unit of $R$. Moreover, we can assume that $b_{\alpha i} \not\in r_{\alpha i} R$ for all $i$. Choose now $0 \neq q \in P$ such that $b_{\alpha i} \not\in qR$ for all $i$. Then there exists $z_2 = cx + \Sigma_j c_{\beta j} y_{\beta j} \in Qz$ such that $z_1 = qz_2$. This equality, read inside the free $R$-module $F$, implies that there exists an element $\Sigma_k d_{\gamma k}(\xi - r_{\gamma k}\eta_{\gamma k}) \in N$ such that

$$b\xi + \sum_i b_{\alpha i}\eta_{\alpha i} = qc\xi + \sum_j qc_{\beta j}\eta_{\beta j} + \sum_k d_{\gamma k}(\xi - r_{\gamma k}\eta_{\gamma k}) .$$

The above equality gives the required contradiction, because the first member's summand $b_{\alpha 1}\eta_{\alpha 1}$ cannot appear in the second member, since $b_{\alpha 1} \not\in r_{\alpha 1}R + qR$. The desired conclusion follows. ■

The following corollary will be used in Theorem 4.

**Corollary 3.** *For every non-zero prime ideal $L$ of $R$ there exists a non-splitting mixed $R$-module $M$ with $M/tM \cong Q$ and* $\mathrm{End}_W(M) \supseteq R_L$.

PROOF. Let us construct an $R_L$-module $M$ as in Theorem 2. Consider $M$ as an $R$-module; it is straightforward to verify that the $R$-module $M$ satisfies the requirements of the assertion. ■

Before stating our characterisation theorem, we recall that a valuation domain $R$ is said to be *Archimedean* if the maximal ideal $P$ is the unique non-zero prime ideal; equivalently, the value group of $R$ is a subgroup of the real numbers. Of course, a discrete valuation ring is Archimedean.

**Theorem 4.** *Let $R$ be a valuation domain. The following are equivalent:* (a) *$R$ is Archimedean*; (b) *for every non-splitting mixed $R$-module $M$ of torsion-free rank 1 we have* $\mathrm{End}_W(M) = R$.

PROOF. (a)$\rightarrow$(b). If $R$ is Archimedean, $P$ is the minimal non-zero prime ideal of $R$; by Proposition 1 we obtain that $\mathrm{End}_W(M) \subseteq R_P = R$, for every non-splitting mixed $R$-module $M$ of torsion-free rank 1.

(b)$\rightarrow$(a). If $R$ is not Archimedean, there exists a non-zero prime ideal $L \subset P$. In view of Corollary 3, for a convenient choice of $M$ we have $\mathrm{End}_W(M) \supseteq R_L \supset R$. ∎

However, it is possible to construct a non-splitting mixed module $M$ with $M/tM \cong Q$ and $\mathrm{End}_W(M) = R$, even when $R$ does not contain a minimal non-zero prime ideal, as shown in the following example.

*Example 6.* Let $R$ be a valuation domain satisfying the following condition: there exists a countable strictly decreasing sequence of non-zero prime ideals of $R$

$$P_0 = P > P_1 > \cdots > P_n > \cdots$$

such that $\cap_{n=0}^{\infty} P_n = 0$. The existence of such an $R$ is ensured, for instance, by proposition 5 of [6]. For all $n \in \mathbf{N}$, choose $p_n \in P_{n-1} \backslash P_n$ (here we set $P_{-1} = R$, $p_0 = 1$); clearly $\cap_n p_n R = 0$. Let $T = \oplus_{n=0}^{\infty} R/P_n$. For all $k \in \mathbf{N}$, we set $T_k = \oplus_{n=0}^{k} R/P_n$; let us note that, given $\varphi \in \mathrm{End}(T)$, we have $\varphi(T_k) \subseteq T_k$ for all $k$: in fact $\mathrm{Hom}_R(R/P_m, R/P_n) = 0$ if $n < m$, because the $P_i$ are prime ideals. We embed $T$ into the $R$-module $T' = \Pi_{n=0}^{\infty} R/P_n$. We denote the elements of $T'$ by formal series $\Sigma_{n=0}^{\infty} (a_n + P_n)$. Let us consider the following mixed $R$-submodule of $T'$,

$$M = \langle T, z_j \colon j \in \mathbf{N} \rangle ,$$

where $z_j = \Sigma_{n=0}^{\infty} (b_n + P_n) \in T'$ is defined as follows: $b_n = 0$ if $n < j$, $b_n = p_n p_j^{-1}$ if $n \geqslant j$. Note that, since $p_0 = 1$, we have $z_0 = \Sigma_{n=0}^{\infty} (p_n + P_n)$. It is easy to check that: (a) $T = tM$; (b) for all $j$, we have $z_0 - p_j z_j \in T_{j-1}$, so that $M/tM \cong Q$; (c) if $x = \Sigma_{n=0}^{\infty} (c_n + P_n)$, $y = \Sigma_{n=0}^{\infty} (d_n + P_n) \in M$ are such that $x - y \in T$, there exists a $k$ such that $c_n + P_n = d_n + P_n$ for all $n \geqslant k$.

Let us now prove that $\mathrm{End}_W(M) = R$. By contradiction, let us assume that $s \in R$, not a unit, is such that $s^{-1} 1_W \in \mathrm{End}_W(M)$; then there exists $\varphi \in \mathrm{End}(M)$ such that $s\varphi - 1 \in \mathrm{Hom}(M, tM)$. Let us now consider the element $z_0$; let $\varphi(z_0) = \Sigma_{n=0}^{\infty} (c_n + P_n)$. Since $s\varphi(z_0) - z_0 \in T$, by property (c) there exists a $k \in \mathbf{N}$ such that $sc_n + P_n = p_n + P_n$ for all $n \geqslant k$. We also know, by property (b), that $z_0 = p_k z_k + t$, for a suitable $t \in T_{k-1}$, from which

$$s\varphi(z_0) = sp_k \varphi(z_k) + \varphi(st)$$

where $\varphi(st) \in T_{k-1}$, since $\varphi(T_{k-1}) \subseteq T_{k-1}$. Therefore, if $\varphi(z_k) = \Sigma_{n=0}^{\infty} (d_n + P_n)$, we have, for all $n \geqslant k$,

$$sc_n + P_n = sp_k d_n + P_n .$$

In particular,

$$p_k + P_k = sc_k + P_k = sp_k d_k + P_k$$

implies $p_k(1 - sd_k) \in P_k$, from which $p_k \in P_k$, because $1 - sd_k$ is a unit, against our assumption on the $p_n$. From the contradiction we conclude that $\mathrm{End}_W(M) = R$, as desired.

We close this section by remarking that Fuchs' divisible module $\partial$ (see [4, ch. VI]) is mixed and of torsion-free rank 1, and is non-splitting *if and only if the projective dimension of $Q$ is* $\geqslant 2$, by [4, ch. VI, lemma 3.6]; hence a result similar to our Theorem 2 was already available, but with restrictions on $R$. It is also worth noting that the modules constructed in Theorem 2 have a much simpler structure than that of $\partial$; of course, they are reminiscent of the classical construction of the Prüfer group $H_{\omega+1}$ (see [3, 35]). Some relations can also be found between Example 6 and the mixed group in [3, example 3.100].

## The global case

Our results in the first section have a natural global generalisation to certain classes of Prüfer domains. It is not possible, however, to give a generalisation without the imposition of some additional properties on the Prüfer domain that essentially make it very similar to $\mathbf{Z}$. Thus we restrict our attention here to Abelian groups.

Recall that if $G$ is a mixed Abelian group and $x \in G \backslash tG$, then we can associate with $x$ a height matrix $\mathbf{H}(x)$ defined as follows: let $\mathbf{P} = \{p_n : n \in \mathbf{N}\}$ be the set of all primes; then $\mathbf{H}(x) = (\sigma_{nk})$ where $\sigma_{nk}$ is the ordinal or symbol $\infty$ that occurs as the generalised $p_n$-height in $G$ of $p_n^k x$ for all $n$ and $k$. We say that two such height matrices $(\sigma_{nk})$ and $(\rho_{nk})$ are equivalent if, for almost all $n$, the $n$th rows of the two matrices are identical and for the remaining $n$ there exist non-negative integers $h$, $m$ (depending on $n$) such that $\sigma_{n,k+h} = \rho_{n,k+m}$ for all $k$. It is well known (see [3, 103]) that if $G$ is of torsion-free rank 1 and $x$, $y$ are torsion-free elements of $G$, then $\mathbf{H}(x)$ and $\mathbf{H}(y)$ are equivalent. Thus, in this case, associated with $G$ we have an invariant that is the equivalence class of height matrices of any torsion-free element. Let $\mathbf{U} = \mathbf{U}(G)$ denote this invariant. With this invariant of $G$ we associate a set $\mathbf{P(U)}$ of primes as follows: $\mathbf{P(U)} = \{p_n \in \mathbf{P} : \sigma_{nk} < \infty \forall k \in \mathbf{N}\}$ where $(\sigma_{nk}) \in \mathbf{U}$. With this notation we can state our global result.

**Theorem 7.** *Let $G$ be a mixed Abelian group of torsion-free rank 1; then* $\mathrm{End}_W(G) = \mathbf{Z}_{\mathbf{P(U)}}$, *the localisation of* $\mathbf{Z}$ *at* $\mathbf{P(U)}$.

PROOF. Since $\mathrm{End}_W(G)$ is a subring of the rational field $\mathbf{Q}$, it will suffice to compute the characteristic of the identity in $\mathrm{End}_W(G)$. If $1_G \in p\,\mathrm{End}_W(G)$ for some $p \in \mathbf{P}$, then $\psi = 1_G - p\varphi \in \mathrm{Hom}(G, tG)$ for some endomorphism $\varphi$. Since $\ker \psi$ must then contain a torsion-free element $y$, we have that

$$y = p\varphi(y) = p^2\varphi^2(y) = \cdots$$

and so $ht_p(y) = \infty$, implying that $p \not\in \mathbf{P(U)}$. Thus if $q \in \mathbf{P(U)}$ then $\chi(1_G)$, the characteristic of $1_G$, is zero at the place corresponding to $q$. Equivalently we have $\mathrm{End}_W(M) \subseteq \mathbf{Z}_{\mathbf{P(U)}}$.

To prove the reverse inequality we make use of an observation of Warfield [5, lemma 1], which he refers to as the Abelian group theorist's 'Hasse principle'; see Lemma 8 below. Suppose that $q \in \mathbf{P} \setminus \mathbf{P}(\mathbf{U})$ and $x$ is a torsion-free element of $G$ such that $x$ has infinite $q$-height in $G$. Let $x = qy$ for some $y \in G$ and take $S = \langle x \rangle$, $T = \langle y \rangle$. Then $f: S \to T$ with $f(x) = y$ is a homomorphism. Now if $p$ is a prime distinct from $q$, $G_p = G \otimes \mathbf{Z}_p$ is a $\mathbf{Z}_p$-module and division by $q$ is a homomorphism of $G_p$ that extends the induced map $f_p$. Moreover, since $q \not\subseteq \mathbf{P}(\mathbf{U})$, $G_q$ is a split extension of $tG_p$ by $\mathbf{Q}$, so that

$$G_q = \mathbf{Q}x \oplus tG_q = \mathbf{Q}y \oplus tG_q ,$$

and then $f_q$ lifts to an endomorphism of $G_q$. Hence, by Lemma 8, there is an endomorphism $g$ of $G$ with $g|_S = f$. However, we now have that $(qg - 1_G)(x) = (qf - 1_G)(x) = qy - x = 0$, so that $qg - 1_G \in \mathrm{Hom}(G, tG)$. Thus $1_G$ is divisible in $\mathrm{End}_W(G)$ by $q$ and so $q^{-1} \in \mathrm{End}_W(G)$; it follows that $\mathbf{Z}_{\mathbf{P}(\mathbf{U})} \subseteq \mathrm{End}_W(G)$. ■

**Lemma 8** [5]. *Let $G$, $H$ be groups and $S$ and $T$ subgroups such that $G/S$ and $H/T$ are torsion, and for each prime $p$ identify $S_p = S \otimes \mathbf{Z}_p$, $T_p = T \otimes \mathbf{Z}_p$ with submodules of $G_p = G \otimes \mathbf{Z}_p$, $H_p = H \otimes \mathbf{Z}_p$, respectively. Let $f: S \to T$ be a homomorphism and $f_p: S_p \to T_p$ the induced map. Then $f$ extends to a homomorphism from $G$ to $H$ if and only if, for each prime $p$, $f_p$ extends to a homomorphism from $G_p$ to $H_p$.*

Proof. The proof is identical to [5, lemma 1], where it is shown for localisations at the $p$-adic integers $J_p$. However, the proof by Warfield makes no use of the completion. ■

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