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Essentially Indecomposable Modules Over a Complete Discrete Valuation Ring.

B. GOLDSMITH (*)

1. Introduction.

Torsion-free modules over a complete discrete valuation ring R are markedly different from abelian groups or modules over an incomplete discrete valuation ring in that the only indecomposable modules which exist have rank 1 and so are isomorphic to R itself or the field, Q , of fractions of R ([7], p. 45). In this paper we investigate how close a reduced torsion-free R -module of infinite rank can come to being indecomposable. In particular we establish in § 4 the existence of essentially indecomposable modules with basic submodules of countable rank. The results here bear a strong resemblance to results on p -groups ([2], [5], [9] and [10]). Notation follows the standard works of Fuchs [3], [4] while set-theoretic concepts, which are kept to a minimum, may be found in Jech [6].

2. Maximal pure submodules.

Let R denote a complete discrete valuation ring of cardinality ν with unique prime p . For an infinite cardinal λ we let S_λ (or just S if no ambiguity is possible) denote a free R -module of rank λ . Clearly S_λ is not complete in the p -adic topology and we denote its completion by \hat{S}_λ (or just \hat{S}).

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DEFINITION. A R -module X is said to be a maximal pure submodule of the complete R -module \hat{S} if X is a pure submodule of \hat{S} containing S and $\hat{S}/X \cong Q$, the field of fractions of R . We remark that if X is a maximal pure submodule of \hat{S} then for any $x \in \hat{S} \setminus X$, we have $\hat{S} = \langle X, x \rangle_*$.

LEMMA 2.1. If G and K are pure submodules of \hat{S} containing S then $G \cong K$ if and only if there is an automorphism θ of \hat{S} with $G\theta = K$.

PROOF. The sufficiency is clear, we establish necessity. Let φ be an isomorphism from G onto K . Then φ extends uniquely to an endomorphism $\hat{\varphi}$ of \hat{S} . Similarly if ψ is the inverse of φ , it extends to an endomorphism $\hat{\psi}$ of \hat{S} . However since G and K are dense subsets of the Hausdorff space \hat{S} , it follows easily that $\hat{\varphi}\hat{\psi}$ and $\hat{\psi}\hat{\varphi}$ act as the identity on \hat{S} . Thus $\theta = \hat{\varphi}$ is the required automorphism.

Before examining the endomorphism rings of maximal pure submodules of \hat{S} , we introduce the concept of an inessential endomorphism (cf. [5]). Let X be a pure submodule of \hat{S} containing S , then, as we noted in the proof of Lemma 2.1, any endomorphism φ of X has a unique extension $\hat{\varphi}$ to an endomorphism of \hat{S} . We define an endomorphism of X to be inessential if its unique extension to \hat{S} maps \hat{S} into X . It is easily seen that the difference of two inessential endomorphisms is inessential while the composition of two endomorphisms is inessential when either factor is inessential. Thus the inessential endomorphisms of X form a two-sided ideal $I(X)$ in the endomorphism ring $E(X)$ of X .

THEOREM 2.2. For any infinite cardinal λ , there exists an R -module G , with basic submodule of rank λ , such that $E(G)$ is the ring split extension of R by $I(G)$,

$$E(G) = R \oplus I(G).$$

PROOF. Let S be a free R -module of rank λ and let G be any maximal pure submodule of \hat{S} . Clearly S is basic in G . We may identify $E(G)$ as a subring of $E(\hat{S})$ by identifying each endomorphism φ in $E(G)$ with its unique extension $\hat{\varphi}$ in $E(\hat{S})$. With this identification $I(G)$ is a left ideal of $E(\hat{S})$.

Pick $x \in \hat{S} \setminus G$. Then for arbitrary φ in $E(G)$ we must have $q(x\hat{\varphi}) = tx + g$, some $q, t \in R, g \in G$. Since every element of R is a product of a power of p and a unit, there is no loss in generality in supposing

$q = p^r, t = p^s$. We consider two cases:

(i) $r \leq s$.

In this case $p^r(x\phi - p^{s-r}x) = g$. The purity of G in \hat{S} implies that $x(\phi - p^{s-r}1)$ belongs to G . Since G is invariant under $\phi - p^{s-r}1$ and $\langle G, x \rangle_* = \hat{S}$, it is clear that $\hat{S}(\phi - p^{s-r}1)$ is contained in G . Thus $\phi - p^{s-r}1 \in I(G)$ and so $E(G) = R + I(G)$.

(ii) $r > s$.

We show that this case cannot arise. As before we can show that $x(p^{r-s}\phi - 1) \in G$ and deduce that $p^{r-s}\phi - 1 \in I(G)$. Suppose $p^{r-s}\phi - 1 = \theta$, where $\theta \in I(G)$. Since $r > s$, $p^{r-s}\phi$ belongs to the Jacobson radical of $E(\hat{S})$ (see [8]) and this forces θ to be a unit in $E(\hat{S})$. However since G is certainly not a homomorphic image of \hat{S} , $I(G)$ is a proper left ideal of $E(\hat{S})$ which contains a unit-contradiction. So case (ii) does not arise.

Since G is pure in \hat{S} , $R \cap I(G) = 0$ and *quâ* modules, $E(G) = R \oplus \oplus I(G)$. However this is clearly a ring split extension also and we have established the result.

3. Essentially-rigid systems of R -modules.

As we noted in the introduction indecomposable R -modules have rank 1 whereas indecomposable abelian groups of arbitrary large rank exist [11]. One useful tool in the investigation of indecomposable abelian groups was the concept of a rigid system of groups (see [4], § 88). In this section we define and explore an analogous concept for R -modules.

We extend the concept of inessential to homomorphisms between different reduced torsion-free R -modules X_i, X_j by defining $I_i(X_j) = \{\varphi \in \text{Hom}(X_i, X_j) \mid \hat{X}_i \phi \leq X_j\}$ where ϕ denotes the unique extension of φ to a map $\hat{X}_i \rightarrow \hat{X}_j$.

DEFINITION. A family $\{X_j\}$ ($j \in J$) of reduced torsion-free R -modules is said to be essentially rigid if

$$\text{Hom}(X_i, X_j) = \begin{cases} R \oplus I(X_i) & \text{if } i = j \\ I_i(X_j) & \text{if } i \neq j, \end{cases}$$

for all $i, j \in J$.

Suppose throughout this section that R is a complete discrete valuation ring of cardinality ν and λ is an infinite cardinal satisfying $\mu = \lambda^{\aleph_0} = 2^\lambda$ and $\nu < \mu$. For an infinite cardinal σ , let σ^+ denote the successor of σ . We can now state the main result of this section.

THEOREM 3.1. If λ is an infinite cardinal with the property that $\mu = \lambda^{\aleph_0} = 2^\lambda$ and R is a complete discrete valuation ring of cardinality $\nu < \mu$, then there exists an essentially-rigid family of R -modules having μ^+ members.

REMARK. (i) By assuming G.C.H. we may, of course replace μ^+ by 2^μ .

(ii) Cardinals of the form λ do exist for values of λ

other than $\lambda = \aleph_0$ e.g. assuming G.C.H., any cardinal of cofinality \aleph_0 will do.

LEMMA 3.2. Let V be a vector space of dimension α , an infinite cardinal, over a field. Let $\{W_i\}$ ($i < \beta$) be a family of subspaces of V indexed by the cardinal $\beta < \alpha$, such that $\dim W_i = \alpha$ for all $i < \beta$. Then there exist α^+ subspaces $\{U\}$ of V such that each subspace U is of codimension 1 in V and no subspace W_i is contained in any subspace U .

PROOF. By a result of Beaumont and Pierce ([1], Lemma 5.2) there exists at least one such subspace, U_0 say. Suppose that the subspaces $\{U_i\}$ ($i < \zeta$) have been constructed and $\zeta < \alpha^+$. Then the set of subspaces consisting of the given W_i together with the constructed subspaces U_i constitutes a family of at most α subspaces each of dimension α . Applying Beaumont and Pierce's result to this family yields another subspace of codimension 1. Call this subspace U_ζ . The result follows easily by transfinite induction.

LEMMA 3.3. If S is free of rank λ then there exist μ^+ maximal pure submodules of \hat{S} with the property that none of them contains a submodule isomorphic to \hat{S} .

PROOF. Since S is free of rank λ , $|\hat{S}| = \max(\lambda^{\aleph_0}, \nu^{\aleph_0})$. But $\nu < \mu$ implies that $\nu^{\aleph_0} < \mu^{\aleph_0} = (\lambda^{\aleph_0})^{\aleph_0} = \lambda^{\aleph_0} = \mu$. So \hat{S}/S is a Q -vector space of dimension $\mu = \lambda^{\aleph_0}$. Now let $\{W_k\}$ ($k \in K$) be the collection of submodules of \hat{S} which are isomorphic to \hat{S} . Each of these submodules is determined by an endomorphism of \hat{S} . However each endomorphism

of \hat{S} is completely determined by its action on S which has rank λ , so $|E(\hat{S})| = \mu^\lambda = (2^\lambda)^\lambda = 2^\lambda = \mu$. Hence $\{W_k\}$ ($k \in K$) is a family of at most μ submodules of \hat{S} .

Let $\bar{W}_k = \langle W_k + S \rangle_*/S$. Then $\{\bar{W}_k\}$ ($k \in K$) is a family of at most μ subspaces of the Q -vector space \hat{S}/S which has dimension μ . Since $\bar{W}_k \cong (Q \otimes_R (W_k + S))/Q \otimes_R S$, it follows that each \bar{W}_k has dimension μ . By Lemma 3.2 we can find μ^+ subspaces U such that no \bar{W}_k is contained in any U and, moreover, each U has codimension 1 in \hat{S}/S . If G is a submodule of \hat{S} with $G/S = U$, then G is a maximal pure submodule of \hat{S} and clearly no W_k is contained in any G . Thus we have constructed the required family of μ^+ maximal pure submodules.

Let \mathfrak{G}_λ denote the collection of all maximal pure submodules of \hat{S}_λ which do not contain an isomorphic copy of \hat{S}_λ .

LEMMA 3.4. If $\{G_\alpha\}$ ($\alpha < \beta$) is a subset of \mathfrak{G}_λ and $|\beta| \leq \mu$, then there exist μ^+ submodules G in \mathfrak{G}_λ such that $\text{Hom}(G_\alpha, G) = I_\alpha(G)$ for all $\alpha < \beta$.

PROOF. The proof is similar to that of Lemma 3.3. Suppose $\{W_{\alpha_i}\}$ denotes the set of endomorphic images of G_α which have rank μ . Then for each α , the set $\{W_{\alpha_i}\}$ is of cardinality at most μ . Since $|\beta| \leq \mu$, the union of all such collections is a set of at most μ submodules of \hat{S} . Call this set \mathcal{W} . Now let \mathcal{U} denote the set of endomorphic images of \hat{S} which are isomorphic to \hat{S} . Then $\mathcal{W} \cup \mathcal{U}$ is a collection of at most μ submodules of \hat{S} , say $\mathcal{W} \cup \mathcal{U} = \{X_i\}$ ($i < \mu$). Note that each X_i has rank μ . Set $\bar{X}_i = \langle X_i + S \rangle_*/S$; then $\{\bar{X}_i\}$ ($i < \mu$) is a collection of μ subspaces of the Q -vector space \hat{S}/S and each \bar{X}_i has dimension μ . By Lemma 3.2 there exist μ^+ maximal subspaces U such that no \bar{X}_i is contained in a U . Choose a maximal pure submodule G such that $G/S = U$. Clearly $G \in \mathfrak{G}_\lambda$.

Now consider $\text{Hom}(G_\alpha, G)$ for any α . If $\varphi: G_\alpha \rightarrow G$ is not inessential then $\text{Ker } \varphi$ is contained in G_α which forces $\text{Ker } \varphi$ to have rank less than μ . But then $\text{Im } \varphi \cong G_\alpha/\text{Ker } \varphi$ is an endomorphic image of G_α of rank μ and is contained in G -contradiction. So we conclude that $\text{Hom}(G_\alpha, G) = I_\alpha(G)$ for each α .

LEMMA 3.5. Given any maximal pure submodule G of \hat{S}_λ , there are at most μ maximal pure submodules G_i of \hat{S}_λ for which $\text{Hom}(G_i, G) \neq I_i(G)$.

PROOF. Suppose there exists a family $\{G_i\}$ ($i \in J$) of more than μ submodules. For each $i \in J$, pick a homomorphism $\varphi_i: G_i \rightarrow G$. Then $\{\hat{\varphi}_i\}$ ($i \in J$) is a family of more than μ endomorphisms of \hat{S}_λ . Since $|E(\hat{S}_\lambda)| = \mu$, we must have $\hat{\varphi}_i = \hat{\varphi}_j$ for some $i \neq j \in J$. But then

$$\hat{S}\hat{\varphi}_i = (G_i + G_j)\hat{\varphi}_i \leq G_i\hat{\varphi}_i + G_j\hat{\varphi}_j = G.$$

Thus φ_i is inessential-contradiction. This establishes the lemma.

PROOF OF THEOREM 3.1. Choose G_0 to be any member of \mathfrak{G}_λ . Suppose the essentially-rigid family $\{G_\alpha\}$ ($\alpha < \beta$) has been constructed for $\beta < \mu^+$. By Lemma 3.4 there exist μ^+ maximal pure submodules G such that $\text{Hom}(G_\alpha, G) = I_\alpha(G)$. However for each $\alpha < \beta$, there are, by Lemma 3.5, at most μ of these submodules G for which $\text{Hom}(G, G_\alpha) \neq I(G_\alpha)$. Then deleting all such submodules G deletes at most μ submodules from the original collection since $\beta \leq \mu$. So there exists $G \in \mathfrak{G}_\lambda$ with $\text{Hom}(G, G_\alpha) = I(G_\alpha)$ and $\text{Hom}(G_\alpha, G) = I_\alpha(G)$ for all $\alpha < \beta$. Set $G_\beta = G$. Then the family $\{G_\alpha\}$ ($\alpha \leq \beta$) is essentially-rigid. The proof is completed by transfinite induction.

4. Essentially-indecomposable modules.

In this section we show that a slightly stronger result than Theorem 3.1 can be deduced and apply this new result to the construction of essentially indecomposable modules.

We shall use the term basic rank of a homomorphism to denote the rank of a basic submodule of the image of the homomorphism.

DEFINITION. If S is a free R -module of infinite rank λ and X is a pure submodule of \hat{S} containing S , then we define

$$I_\lambda(X) = \{\varphi \in E(X) \mid \hat{S}\varphi \leq X \text{ and } \varphi \text{ has basic rank } < \lambda\}$$

Clearly $I_\lambda(X)$ is an ideal in $E(X)$.

THEOREM 4.1. If R is a complete discrete valuation ring of cardinality ν and λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$ and $\nu \leq \mu$, then there exists a family of μ^+ R -modules $\{G_j\}$ ($j \in J$) such that

- (i) for each $j \in J$, $E(G_j) = R \oplus I_\lambda(G_j)$;
- (ii) for distinct $j, k \in J$, every homomorphism $G_j \rightarrow G_k$ is inessential and has basic rank less than λ

PROOF. This stronger result comes by observing in the proof of Theorem 3.1 that all of the modules constructed actually belong to \mathfrak{G}_λ . Since the image of an inessential homomorphism is complete, it must be the completion of a free module of rank less than λ . This gives the desired result.

Recall that $E_0(G)$ denotes the ideal of $E(G)$ consisting of all endomorphisms of finite rank.

COROLLARY 4.2 (G.C.H.). If R is a complete discrete valuation ring of cardinality 2^{\aleph_0} , then there exists a family of $2^{2^{\aleph_0}}$ R modules $\{G_j\}$ ($j \in J$) each with basic submodules of rank \aleph_0 such that

- (i) for each $j \in J$, $E(G_j) = R \oplus E_0(G_j)$;
- (ii) for distinct $j, k \in J$, every homomorphism $G_j \rightarrow G_k$ has finite rank.

PROOF. Since $2^{\aleph_0} \leq (\aleph_0)^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{2^{\aleph_0}}$, we see that $\lambda = \aleph_0$ satisfies the cardinality requirements of Theorem 4.1. However if a homomorphism from G_j has finite basic rank then it clearly also has finite rank. The result now follows from Theorem 4.1 and G.C.H.

DEFINITION. If λ is an infinite cardinal we say that a reduced torsion-free R -module G is λ -essentially indecomposable if in any decomposition $G = A \oplus B$, one of A, B is the completion of a free module of rank less than λ .

In the case $\lambda = \aleph_0$ we are requiring that in any direct decomposition one of the summands is complete of finite rank. A module with this property is said to be essentially indecomposable (cf. essentially indecomposable p -groups, [9], § 15).

The existence of λ -essentially indecomposable modules follows rather easily from Theorem 4.1 in the case $\lambda^{\aleph_0} = 2^\lambda$. For if G is one of the modules constructed in Theorem 4.1 and we have a decomposition $G = A \oplus B$ with associated projections π_1 and π_2 , then one of π_1, π_2 belongs to $I_\lambda(G)$ since the quotient $E(G)/I_\lambda(G)$ is a domain. If $\pi_1 \in I_\lambda(G)$ then clearly A is the completion of a free R -module of rank less than λ . In particular if $\lambda = \aleph_0$ we have established the existence of $2^{2^{\aleph_0}}$ essentially indecomposable R -modules for any complete discrete valuation ring R of cardinality 2^{\aleph_0} .

We conclude this section by constructing an essentially indecomposable module which is not a maximal pure submodule of a complete module.

PROPOSITION 4.3. If R is a complete discrete valuation ring of cardinality 2^{\aleph_0} and S is a free R -module of countably infinite rank, then there exists a pure submodule H of \hat{S} containing S with $\hat{S}/H \cong \cong Q \oplus Q$ and such that $E(H) = R \oplus E_0(H)$.

PROOF. Choose distinct maximal pure submodules G and G_1 belonging to the family constructed in Corollary 4.2. Set $H = G \cap G_1$. Clearly $S \leq H \leq \hat{S}$ and both inclusions are pure. Also $\hat{S}/H \cong Q \oplus Q$. Let $\hat{S} = \langle H, x, y \rangle_*$ where $G = \langle H, x \rangle_*$ and $G_1 = \langle H, y \rangle_*$. Let φ be any endomorphism of H . Then as in the proof of Theorem 2.2 we may write

$$q(x\hat{\varphi}) = h_0 + \alpha x + \beta y$$

$$q(y\hat{\varphi}) = h_1 + \gamma x + \delta y$$

where $q, \alpha, \beta, \gamma, \delta \in R$ and $h_0, h_1 \in H$.

Now $x(q\hat{\varphi} - \alpha 1) \in G_1$ and so $q\hat{\varphi} - \alpha 1$ maps G into G_1 . From the properties of G and G_1 we conclude that $q\hat{\varphi} - \alpha 1$ is an inessential endomorphism. It follows as in the proof of Theorem 2.2, that $\hat{\varphi}|_G$ belongs to $R \oplus I(G_1)$. Hence we can write $\varphi = r + \theta$ where $r \in R$, $\theta \in I(G_1)$. But then $\theta \in E(H) \cap I(G_1) = E(H) \cap E_0(G) = E_0(H)$. So $\varphi \in R \oplus E_0(H)$ and $E(H) \leq R \oplus E_0(H)$. The reverse inequality is clearly true so $E(H) = R \oplus E_0(H)$.

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