



1985-01-01

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## Recommended Citation

Goldsmith, Brendan and B. Franzen: On endomorphism algebras of mixed modules. *Journal London Mathematical Society*, (2), 31, (1985), pp.468-472.

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ON ENDOMORPHISM ALGEBRAS OF MIXED MODULES

B. FRANZEN AND B. GOLDSMITH

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## 1. Introduction

In a remarkable paper [1] some twenty years ago, Corner showed that every countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free abelian group. This has been the starting point for many investigations of the so-called realization problem, which may be stated as follows.

Given an algebra  $A$  over a commutative ring  $R$ , when will  $A$  be the endomorphism algebra of an  $R$ -module  $G$  which belongs to some suitably restricted class  $\mathcal{C}$ . Complete characterizations of such algebras  $A$  have been obtained in the case where  $R$  is a complete discrete valuation ring and  $\mathcal{C}$  is the class of torsion-free reduced  $R$ -modules [11] and also in the case where  $R = \mathbb{Z}$  and  $\mathcal{C}$  is the class of separable  $p$ -groups [10; 9, Section 109]. Such characterizations are, inevitably, much too complicated to lend themselves readily to applications. Consequently Corner [2] tackled the realization problem for primary abelian groups from a different angle. He showed that a suitably large class of rings  $A$  could not be realized as full endomorphism rings, but rather that the full endomorphism algebra would be the split extension of the given ring  $A$  by some ideal whose presence was unavoidable; in the case of primary groups this ideal is precisely the ideal of small endomorphisms [2]. This idea was subsequently extended to large primary groups in [6] and a similar result was produced in [8] for torsion-free modules over a complete discrete valuation ring.

The results in [2, 6, 8] are all capable of translation into results on endomorphism algebras in a suitable quotient category. Thus, for example, if  $\mathcal{S}$  is the category having primary abelian groups as objects, and morphisms

$$\text{Hom}_{\mathcal{S}}(G, H) = \text{Hom}(G, H)/\text{Hom}_s(G, H),$$

where  $\text{Hom}_s(G, H)$  consists of the small homomorphisms of  $G$  into  $H$ , then Corner's result is that if  $A$  is a ring whose additive group is the completion of a free  $p$ -adic module of at-most countable rank, then there exists a primary group  $G$  with  $E_{\mathcal{S}}(G) = A$ .

When dealing with mixed abelian groups (or more generally mixed  $R$ -modules), there is a natural category in which to work, viz. the category  $\text{Walk}$  ( ${}_R\text{Walk}$ ). The objects of  ${}_R\text{Walk}$  are  $R$ -modules and its morphisms are given by

$$\text{Hom}_W(G, H) = \text{Hom}(G, H)/\text{Hom}_t(G, H),$$

where  $\text{Hom}_t(G, H)$  consists of the  $R$ -homomorphisms of  $G$  into  $H$  with torsion image (see [12]). Recently Dugas [4] has shown that each torsion-free reduced ring  $A$  is the  $\text{Walk}$ -endomorphism ring of a mixed abelian group  $G$ . The groups  $G$  so realized are all of large infinite rank even when the ring  $A$  is of comparatively small cardinality.

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Received 14 December 1983; revised 28 May 1984.

1980 *Mathematics Subject Classification* 20K21.

*J. London Math. Soc.* (2) 31 (1985) 468-472

The cardinalities of these groups have been significantly reduced in [3], which unifies the torsion, torsion-free and mixed cases of the realization problem.

Our approach will be to construct a (non-trivial) full embedding of the category of torsion-free reduced  $R$ -modules into the category  ${}_R\text{Walk}$ , where  $R$  is a principal ideal domain. As a consequence of this full embedding we may immediately lift established results from the category of reduced torsion-free  $R$ -modules to the category  ${}_R\text{Walk}$ . A typical, but by no means exhaustive, list of such results is contained in Corollaries 2.4–2.6. We note, in particular, that many of the results in [7] can now be established immediately. It is, by now, standard to use such realization results to exhibit a wide range of pathologies and so we desist from such repetition.

We conclude this introduction by noting that all unexplained terms may be found in the standard works of Fuchs [9]; our notation is in accord with [9] except that maps are written on the right.

2. The embedding theorem

Throughout let  $R$  be a principal ideal domain. We begin with an arbitrary reduced, separable torsion  $R$ -module  $T$ , and  $T'$  any pure extension of  $T$  by  $Q/R$  such that  $T'$  is also separable and reduced. Thus we have a pure-exact sequence of  $R$ -modules

$$0 \longrightarrow T \longrightarrow T' \longrightarrow Q/R \longrightarrow 0; \tag{*}$$

these will be fixed for the rest of the section. Note that provided  $T$  has no torsion-complete  $p$ -component  $T_p$ , such a sequence exists (see [9, Corollary 68.5]).

Now, if  $X$  is an arbitrary  $R$ -module, then (\*) yields another pure-exact sequence (see [9, Theorem 60.4]):

$$0 \longrightarrow T \otimes X \longrightarrow T' \otimes X \longrightarrow Q/R \otimes X \longrightarrow 0. \tag{*_X}$$

Since  $Q/R \otimes X$  is canonically an epimorphic image of  $Q \otimes X$  we can form the pullback  $H(X)$  of  $*_X$  with respect to this canonical epimorphism  $\eta_X$ . This yields the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T \otimes X & \longrightarrow & H(X) & \xrightarrow{\pi_X} & Q \otimes X \longrightarrow 0 \\ & & \parallel & & \downarrow \sigma_X & & \downarrow \eta_X \\ 0 & \longrightarrow & T \otimes X & \longrightarrow & T' \otimes X & \longrightarrow & Q/R \otimes X \longrightarrow 0 \end{array}$$

in which  $\sigma_X$  is epic since  $\eta_X$  is epic. Note that, by the construction of a pullback,  $\text{Ker } \sigma_X$  is mapped isomorphically onto  $\text{Ker } \eta_X$  by  $\pi_X$ . Also  $\text{Ker } \eta_X$  is canonically isomorphic to  $X/t(X)$ . The  $R$ -module  $H(X)$  has the same torsion-free rank as  $X$  and its torsion submodule is isomorphic to  $T \otimes X$ . Note that, if  $X$  is torsion-free reduced, then  $H(X)$  is reduced and hence non-split. If  $U(M) = \bigcap_{0 \neq r \in R} rM$  denotes the first Ulm submodule of an  $R$ -module, then the purity of (\*) implies the following result.

LEMMA 2.1.  $\text{Ker } \sigma_X = U(H(X))$ .

*Proof.* Note first that it follows from [9, Theorem 61.1] that  $T'_p \otimes X \cong T'_p \otimes B_p$ , where  $B_p$  is a  $p$ -basic submodule of  $X$ . Thus  $T'_p \otimes X = \bigoplus_p T'_p \otimes B_p$  and since  $T'$  is separable it follows readily that  $U(T' \otimes X) = 0$ . But  $U(H(X))\sigma_X \subseteq U(T' \otimes X) = 0$  and hence  $U(H(X)) \subseteq \text{Ker } \sigma_X$ .

Conversely, let  $m$  be an arbitrary element of  $\text{Ker } \sigma_X$  and let  $r$  be an arbitrary non-zero element of  $R$ . Then there is an element  $y \in H(X)$  with  $m - ry = z \in t(H(X))$ . But then

$$z = z\sigma_X = m\sigma_X - ry\sigma_X = -ry\sigma_X \in r(T' \otimes X) \cap T \otimes X = r(T \otimes X)$$

by the purity of the sequence  $(*_X)$ . Hence  $m \in rH(X)$ . Since  $r$  is arbitrary non-zero, we have  $m \in U(H(X))$  and so  $\text{Ker } \sigma_X \subseteq U(H(X))$ .

We remark that the construction of  $H(X)$  is functorial: every  $f \in \text{Hom}(X, Y)$  yields homomorphisms  $Q \otimes X \rightarrow Q \otimes Y$  and  $T' \otimes X \rightarrow T' \otimes Y$  which in turn give rise to a unique homomorphism  $H(f): H(X) \rightarrow H(Y)$  by the universal property of the pullback. We denote this functor by  $H$ . In order to place our construction in a functorial setting let  $U$  be the subfunctor of the identity defined by  $U(X) = \bigcap_{0 \neq r \in R} rX$  and  $U(f) = f|U(X)$ , the restriction of  $f$  to  $U(X)$ ; let  $F$  be the functor defined by  $F(X) = X/t(X)$  and  $F(f) = \bar{f}$ , where  $\bar{f}$  is the mapping induced by  $f$  on the quotient.

**PROPOSITION 2.2.** *The functors  $UH$  and  $F$  are naturally equivalent.*

*Proof.* By Lemma 2.1  $UH(X) = \text{Ker } \sigma_X$  and since  $\pi_X$  maps  $\text{Ker } \sigma_X$  isomorphically onto the kernel of  $\eta_X$  the assertion follows from the observation that  $\text{Ker } \eta_X \cong X/t(X)$ .

In the following let  ${}_R\mathcal{C}$  denote the category of torsion-free reduced  $R$ -modules.

**THEOREM 2.3.** *Let  $R$  be a principal ideal domain, let  $T$  be a separable reduced torsion  $R$ -module and let  $T'$  be a pure extension of  $T$  by  $Q/R$  such that  $T'$  is separable and reduced. Then there is a full embedding  $\bar{H}: {}_R\mathcal{C} \rightarrow {}_R\text{Walk}$  such that*

- (i)  $\bar{H}(X)$  is reduced, non-split and of the same torsion-free rank as  $X$ ,
- (ii)  $t(\bar{H}(X)) \cong T \otimes X$ ,
- (iii)  $\bar{H}(X)/t(\bar{H}(X))$  is divisible,
- (iv)  $U\bar{H}(X) = X$  and  $\bar{H}(X)/U\bar{H}(X) = T' \otimes X$ .

*Proof.* Let  $\bar{H}(X) = H(X)$  for  $X \in {}_R\mathcal{C}$  and  $\bar{H}(f) = H(f) + \text{Hom}_t(H(X), H(Y))$  for  $f: X \rightarrow Y$ . The only assertion still to be verified is that  $\bar{H}$  is a full embedding. By Proposition 2.2  $UH$  is naturally equivalent to  $F$  which is the identity functor on  ${}_R\mathcal{C}$ . Therefore we may identify  $X$  and  $UH(X)$ . Consider the homomorphisms  $h: \text{Hom}(X, Y) \rightarrow \text{Hom}(H(X), H(Y))$  and  $u: \text{Hom}(H(X), H(Y)) \rightarrow \text{Hom}(X, Y)$  induced by  $H$  and  $U$  respectively. Then  $hu$  is the identity on  $\text{Hom}(X, Y)$ , thus  $h$  is monic and  $u$  is epic. Furthermore  $\text{Ker } u = \text{Hom}_t(H(X), H(Y))$  since  $g|UH(X) = 0$  implies that  $\text{Im } g$  is torsion as an epimorphic image of the torsion module  $H(X)/UH(X) \cong T' \otimes X$ . On the other hand, if  $\text{Im } g$  is torsion, then  $g(UH(X)) = 0$  because  $UH(Y) \cap t(H(Y)) = 0$ . Thus we conclude that the map  $f \mapsto \bar{H}(f)$  is an isomorphism and  $\bar{H}$  is a full embedding.

**REMARKS.** (a) An alternative way to construct the functor  $H$  is the following. Let  $M = H(R)$ , a mixed module of torsion-free rank one. Then it is readily seen that the functors  $H$  and  $M \otimes -$  are naturally equivalent.

(b) As indicated in the above proof,  $E(H(X))$  is the split extension of  $E(X)$  by  $\text{Hom}_t(H(X), H(X))$ , that is, there are ring homomorphisms

$$E(X) \xrightarrow{h} E(H(X)) \xrightarrow{u} E(X)$$

such that  $hu = \text{id}_{E(X)}$  and  $\ker u = \text{Hom}_t(H(X), H(X))$ .

**COROLLARY 2.4.** *Let  $R$  be a principal ideal domain. If  $A$  is a countable reduced torsion-free  $R$ -algebra then there are  $2^{\aleph_0}$  countable mixed  $R$ -modules  $M_i$  with  $M_i/t(M_i)$  divisible,  $E_W(M_i) \cong A$  and  $\text{Hom}_W(M_i, M_j) = 0$  for  $i \neq j$ .*

*Proof.* By an unpublished extension of a well-known theorem of Corner [1] there exist countable reduced torsion-free modules with  $E(X_i) \cong A$  and  $\text{Hom}(X_i, X_j) = 0$  for  $i \neq j$ . Now Theorem 2.3 yields the assertion by choosing an appropriate torsion module  $T$ , for example an unbounded countable direct sum of cyclics.

In the finite rank case Corner's result gives the following.

**COROLLARY 2.5.** *Let  $R$  be a principal ideal domain and let  $A$  be a countable reduced torsion-free algebra of finite rank  $n$ . Then there exists a reduced mixed module  $M$  of torsion-free rank  $2n$  such that  $M/t(M)$  is divisible and  $E_W(M) = A$ .*

**COROLLARY 2.6.** *If  $R$  is a principal ideal domain and not a complete discrete valuation ring and  $A$  is any cotorsion-free  $R$ -algebra, then there exists a reduced mixed  $R$ -module  $M$  with  $M/t(M)$  divisible and  $E_W(M) = A$ .*

*Proof.* This is a consequence of [5, Corollary 5.4], which ensures the existence of a cotorsion-free  $R$ -module  $X$  with  $E(X) = A$ .

**REMARK.** It was shown in [3] that  $X$  can be chosen to be of cardinality  $|A|^{\aleph_0}$ . Thus  $M$  can be made to be of cardinality  $|A|^{|R|}$ .

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