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Introduction

In a recent paper Shelah [9] has established the existence of a rigid-like family of 2^λ separable p-groups each of cardinality λ, where λ is a strong limit cardinal of cofinality $> \aleph_0$, that is a family such that (i) the endomorphism ring of each group is the split extension of the p-adic integers by the ideal of small endomorphisms and (ii) every homomorphism between different members of the family is small. (See Pierce [8] or Fuchs [3] for the concept of small homomorphisms.) Then assuming G.C.H. this leaves open the following problem:—

If $\mu = \lambda^{\aleph_0} = 2^\lambda$, is there a rigid-like family of 2^μ separable p-groups, each of cardinality μ?

This problem seems to be extremely difficult and in this paper we derive a weaker result and in so doing obtain a partial answer to Fuchs [4; Problem 53]. We remark that our technique is mainly group-theoretic and uses a minimal number of notions from set theory.

Finally all groups are additively written abelian groups and we refer to Fuchs [3] and [4] for standard results and notation; for set theoretic concepts we refer to Jech [5].

1. *Essentially-rigid families of p-groups*

Let λ be an infinite cardinal and suppose $\bar{B}$ is the torsion-completion of the group B which is a standard p-group of final rank λ. Recall that if G is a reduced p-group containing B as a basic subgroup then we can regard G as a pure subgroup of $\bar{B}$ and then any endomorphism ϕ of G will have a unique extension $\bar{\phi}$ to $\bar{B}$. Thus we may, and do, regard endomorphisms of G as endomorphisms of $\bar{B}$. Let $E(X)$ denote the endomorphism ring of any group X.

Define the ideal of inessential endomorphisms of G by $I(G) = \{\phi \in E(G) | \bar{B}\bar{\phi} \leqslant G\}$. Clearly $I(G)$ is a 2-sided ideal of $E(G)$ and a left ideal of $E(\bar{B})$.

THEOREM 1.1. *For any infinite cardinal λ, there exists a group G, with basic subgroups of final rank λ, such that $E(G)$ is the split extension of the p-adic integers, $Q_p{}^*$, by the ideal $I(G)$, $E(G) = Q_p{}^* \oplus I(G)$.*

Proof. Let B be a standard basic group of final rank λ and choose G such that $B \leqslant G \leqslant \bar{B}$ and $\bar{B}/G \cong Z(p^\infty)$. Then G has rank $\lambda^{\aleph_0}$ but has a basic subgroup of final rank λ. We show $E(G) = Q_p{}^* \oplus I(G)$.

Now we have the exact sequence

$$0 \to G \to \bar{B} \to Z(p^\infty) \to 0$$

which yields

$$0 \to \operatorname{Hom}(\bar{B}, G) \to \operatorname{Hom}(G, G) \to P\operatorname{ext}(Z(p^\infty), G)$$

$$0 \to \operatorname{Hom}(Z(p^\infty), Z(p^\infty)) \to P\operatorname{ext}(Z(p^\infty)\, G), \to 0.$$

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So $P\text{ext}(Z(p^\infty), G)$ is a cyclic $Q_p{}^*$-module and since the image of $\text{Hom}(\bar{B}, G)$ in $\text{Hom}(G, G)$ is just $I(G)$, we have that $E(G) = Q_p{}^* + I(G)$. Since $Q_p{}^* \cap I(G) = 0$, the extension is a ring split extension.

LEMMA 1.2 (Leptin [6]). *If G and H are pure subgroups of $\bar{B}$ containing B then $G \cong H$ if and only if there is an automorphism θ of $\bar{B}$ with $G\theta = H$.*

Definition. A group G is said to be a maximal pure subgroup of $\bar{B}$ if G is a pure subgroup of $\bar{B}$ containing B and $\bar{B}/G \cong Z(p^\infty)$.

LEMMA 1.3. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there are 2^μ non-isomorphic maximal pure subgroups of $\bar{B}$, the torsion-completion of the standard basic group B of final rank λ.*

Proof. Since $\bar{B}$ has cardinality $\lambda^{\aleph_0} = \mu$ we have that $\bar{B}/B \cong \oplus_\mu Z(p^\infty)$. Now a maximal pure subgroup G of $\bar{B}$ corresponds to a divisible subgroup of corank 1 in this quotient and so there are clearly 2^μ such groups. But by Lemma 1.2 any two maximal pure subgroups will be isomorphic only if there is an automorphism θ of $\bar{B}$ mapping one to the other. However any automorphism of $\bar{B}$ is determined by its action on B, of cardinality λ, and so there are at most μ automorphisms of $\bar{B}$. Hence there are 2^μ non-isomorphic maximal pure subgroups of $\bar{B}$.

For the rest of this section suppose that λ is an infinite cardinal such that $\lambda^{\aleph_0} = 2^\lambda = \mu$ and let $\bar{B}$ denote the torsion-completion of the standard basic group B of final rank λ.

Definition. If G_j and G_k are maximal pure subgroups of $\bar{B}$, let

$$I_j(G_k) = \{\phi \in \text{Hom}(G_j, G_k) \mid \bar{B}\bar{\phi} \leqslant G_k\}.$$

LEMMA 1.4. *For an arbitrary maximal pure subgroup G of $\bar{B}$, there are at most μ maximal pure subgroups G_k of $\bar{B}$ such that* $\text{Hom}(G_k, G) \neq I_k(G)$.

Proof. Suppose there exists a family $\{G_k\}_{k \in K}$ of more than μ maximal pure subgroups of $\bar{B}$ with $\text{Hom}(G_k, G) \neq I_k(G)$. Then we can construct a family $\{\bar{\phi}_k\}_{k \in K}$ of endomorphisms of $\bar{B}$ such that ϕ_k does not belong to $I_k(G)$. Since $E(\bar{B})$ has cardinality μ and K has cardinality greater than μ, we must have $\bar{\phi}_k = \bar{\phi}_j$ for some different k, j in K. But then

$$\bar{B}\bar{\phi}_k = (G_j + G_k)\bar{\phi}_k \leqslant G_j\bar{\phi}_k + G_k\bar{\phi}_k = G$$

which is contrary to our choice of ϕ_k.

LEMMA 1.5. *If G_0 is an arbitrary maximal pure subgroup of $\bar{B}$ then there are at most μ maximal pure subgroups G_k such that* $\text{Hom}(G_0, G_k) \neq I_0(G_k)$.

Proof. Suppose there exists a family $\{G_k\}_{k \in K}$, where $|K| > \mu$, of maximal pure subgroups of $\bar{B}$ such that $\text{Hom}(G_0, G_k) \neq I_0(G_k)$. Then we have a family $\{\bar{\phi}_k\}_{k \in K}$ of endomorphisms of $\bar{B}$ such that $\phi_k \notin I_0(G_k)$. But then $\bar{\phi}_k = \bar{\phi}_j$ for $k, j \in K'$ with $|K'| > \mu$. So $G_0\phi_k \leqslant \bigcap_{j \in K'} G_j$ for $k \in K'$. Let $G_k = \langle H_k, x_k \rangle_*$ where $H_k \geqslant \bigcap_{j \in K'} G_j$

and let π_k denote the projection of $\bar{B}$ onto $\langle x_k \rangle_*$. Set $\bar{\phi}_k' = \bar{\phi}_k + \pi_k$. Then $\phi_k' \in \text{Hom}(G_0, G_k)$ but $\phi_k' \notin I_0(G_k)$. But since there are more than μ such x_k's we have exhibited more than μ distinct endomorphisms $\bar{\phi}_k'$ of $\bar{B}$—a contradiction.

Definition. A family $\{G_j\}_{j \in J}$ of separable p-groups is said to be essentially-rigid if

$$\text{Hom}(G_j, G_k) = \begin{cases} Q_p^* \oplus I(G_j) & j = k \\ I_j(G_k) & j \neq k. \end{cases}$$

THEOREM 1.6. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there exists an essentially-rigid family of 2^μ groups, each of cardinality μ.*

Proof. Let ε denote the least ordinal of cardinality 2^μ. We construct a family $\{G_\alpha\}_{\alpha < \varepsilon}$ inductively. For G_0 choose any maximal pure subgroup of $\bar{B}$, the torsion-completion of the standard basic group of final rank λ. Now suppose the family $\{G_\alpha\}_{\alpha < \beta}$ has been constructed where $0 < \beta < \varepsilon$.

For each $\alpha < \beta$ consider the set of maximal pure subgroups G_i of $\bar{B}$ such that $\text{Hom}(G_\alpha, G_i) \neq I_\alpha(G_i)$ or $\text{Hom}(G_i, G_\alpha) \neq I_i(G_\alpha)$. By Lemmas 1.4 and 1.5 we know there are at most μ indices i for which this is true. By the minimality of ε we know that the set of G_i having this property for any α less than β contains less than 2^μ members. So by Lemma 1.3 there exists a maximal pure subgroup G^1 of $\bar{B}$ not in this set. Set $G_\beta = G^1$. Then it follows easily that $\{G_\alpha\}_{\alpha \leqslant \beta}$ is essentially-rigid. The construction is completed by transfinite induction. The proof is then completed by the observation that each maximal pure subgroup of $\bar{B}$ has cardinality $= |\bar{B}| = \mu$.

2. *Prescribing the ideal of inessential endomorphisms*

The results in §1 imposed very little restriction on the ideal over which the endomorphism ring of a maximal pure subgroup of $\bar{B}$ splits. In this section we show that this ideal can be restricted somewhat. Ideally we would like to show that splitting can occur over the ideal of small endomorphisms; however, this seems to be very difficult. We offer instead a weaker splitting result.

Definition. If G_j and G_k are separable p-groups with a common basic subgroup B, then we define, for an infinite cardinal λ,

$$I_j^\lambda(G_k) = \{\phi \in \text{Hom}(G_j, G_k) | \bar{B}\bar{\phi} \leqslant G_k \quad \text{and} \quad |\bar{B}\bar{\phi}| \leqslant \lambda\}.$$

When $G_j = G_k$ we simply write $I^\lambda(G_k)$.

THEOREM 2.1. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there exists a family $\{G_j\}$ of 2^μ separable p-groups, each of cardinality μ, having a common basic subgroup B of cardinality λ and such that*

$$\text{Hom}(G_j, G_k) = \begin{cases} Q_p^* \oplus I^\lambda(G_k) & j = k \\ I_j^\lambda(G_k) & j \neq k. \end{cases}$$

Proof. Let

$$B = \bigoplus_{n < \omega} B_n \quad \text{where} \quad B_n = \bigoplus_{\lambda} Z(p^n).$$

Let $\bar{B}$ denote the torsion-completion of B. Then $\bar{B}/B$ is a divisible p-group of rank μ.

Let $\{W_k\}_{k \in K}$ denote the set of endomorphic images of $\bar{B}$ which have rank μ. Since $|E(\bar{B})| = \mu$, there are at most μ such images, i.e. $|K| \leqslant \mu$. Let $W_k{}^*$ denote a minimal pure subgroup of $(W_k+B)/B$. Then $\{W_k{}^*[p]\}_{k \in K}$ is a family of at most μ subspaces of the vector space $(\bar{B}/B)[p]$. Moreover, by choice of W_k, each of these subspaces has dimension μ. Then appealing to Lemma 2.2 below we see that there exist 2^μ maximal subspaces of $(\bar{B}/B)[p]$ which contain no $W_k{}^*[p]$. Hence we can find 2^μ maximal pure subgroups of $\bar{B}$, each containing B, such that no W_k is contained in any of them. Then using Lemmas 1.4 and 1.5 we can refine this family to a family $\{G_j\}$ of 2^μ maximal pure subgroups such that no W_k is contained in a G_j and the family is essentially rigid.

To obtain the desired result we now observe that the basic subgroup of an image of $\bar{B}$ in a G_j has rank χ less than λ or rank λ. Since the rank of an image of $\bar{B}$ in a G_j cannot be $\lambda^{\aleph_0}$, any image of $\bar{B}$ in a G_j has rank at most $\chi^{\aleph_0}$ or λ. But if χ is less than λ then $\chi^{\aleph_0}$ is less than μ, hence λ has cofinality $\aleph_0$ and then χ less than λ implies that $\chi^{\aleph_0}$ is also less than λ. Thus in either case any image of $\bar{B}$ in a G_j has rank at most λ. The proof is completed by

LEMMA 2.2. *Let V be a vector space of dimension α over a field F, where α is an infinite cardinal. Let $\{W_i\}_{i<\alpha}$ be a family of subspaces each of dimension α. Then there exist 2^α subspaces U_j each of co-dimension* 1 *such that no W_i is contained in a U_j.*

Proof. This is a standard set-theoretic extension of a well-known result (see [**1**; Lemma 5.2]).

COROLLARY 2.3. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there exist 2^μ separable p-groups G_i, each of rank μ, with basic subgroups of final rank λ, such that every homomorphism between different members of the family has image of cardinality at most λ.*

Clearly the homomorphisms between different members of the above family are "small" (but not in the technical sense of Pierce). This gives a partial answer to Fuchs [**4**; Problem 53]. A similar result has been obtained by Shelah [**9**] but by entirely different methods.

COROLLARY 2.4. *There exists a family of $2^\mathfrak{c}$, where $\mathfrak{c} = 2^{\aleph_0}$, separable p-groups $\{G_j\}$ such that $E(G_j) = Q_p{}^* \oplus E_s(G_j)$ for each j and, moreover, every homomorphism between distinct members of the family is small.*

Proof. Clearly $\lambda = \aleph_0$ satisfies the conditions of Theorem 2.1. But, by a result of Megibben [7], if $\theta : \bar{B} \to G_j$ is not small then G_j contains an unbounded torsion-complete group. Since such a group must have rank $2^{\aleph_0}$ we deduce from the construction of the G_j's in Theorem 2.1, that no such group can be contained in any of the G_j. Thus every homomorphism from G_j to G_k ($j \neq k$) is small and the result follows from Theorem 2.1.

We remark that the members of such a family are essentially indecomposable and that this answers a problem raised by Shelah at the end of his paper [**9**]. We note however that such a family had been previously constructed by Corner [**2**].

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