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# A topological approach to a problem of Nunke

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# A topological approach to a problem of Nunke

By

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**1. Introduction.** If $P$ denotes the Specker group i.e. the direct product of countably many infinite cyclic groups, then the following question was raised by Nunke [6]: "If $P$ has a subgroup $A$ such that $P/A \cong \mathbb{Q}$, the additive group of rationals, is $A \cong P$?"

The question has been answered negatively by Meijer [4]. However the problem arose from essentially homological work of Nunke and Meijer's solution was also homological. We present here a solution to the problem which uses a topological idea introduced by Lady [3] and avoids all homological machinery.

The word group will be understood to refer to an additively written abelian group and we refer to the standard works [1] and [2] of Fuchs for notation and terminology.

**2. The Main Result.** A Specker group $P$ may be topologized by using the product topology obtained by considering each component as being discretely topologized. We refer to this topology on $P$ simply as the product topology. $P$ may also be topologized by choosing as a basis of neighbourhoods of 0 the subgroups $nP$ ($n \in \mathbb{Z}$, $n \neq 0$). This is the familiar $\mathbb{Z}$-adic topology (see [1] page 30). We will make use of a topology, which, following Lady [3], we call the strong topology. This is the topology on $P$ which is the supremum of the $\mathbb{Z}$-adic and product topologies on $P$.

The result we require can be stated as

**Theorem 1** (Meijer). *Let $G$ be a torsion-free group of at most countable rank and let $X$ be an extension of a Specker group $P$ by $G$. Then $X \cong P$ if and only if $G$ is free of finite rank.*

We use the following results:

**Lemma 2.** *If $Y \leqq P$ and $Y \cong P$ then there are integers $k_n$ and elements $a_n$ of $P$ ($n = 1, 2, 3, \ldots$) such that*

$$P = \prod^{\infty} \langle a_n \rangle \quad \text{and} \quad Y = \prod^{\infty} \langle k_n a_n \rangle$$

Proof. Suppose $Y = \prod_{n=1}^{\infty} \langle y_n \rangle$. Define elements $w_n$ of $P$ by

$$\begin{aligned} w_1 &= (y_1, 0, 0, \ldots), \\ w_2 &= (0, y_2, y_3, y_4, \ldots), \\ w_3 &= (0, 0, y_3, y_4, \ldots), \\ &\vdots \\ w_n &= (0, 0, \ldots 0, y_n, y_{n+1}, \ldots) \quad \text{etc.} \end{aligned}$$

Since, for every positive integer $j$, the $j$'th components of almost all the $w_n$ vanish, the infinite sum $\sum_{n=1}^{\infty} s_n w_n$ ($s_n$ integers) makes sense and so the subgroup $W$ consisting of all such infinite sums is a product in the sense defined by Fuchs [2] § 95.

Claim $W = Y$. For if $w \in W$ then $w = \sum_{n=1}^{\infty} s_n w_n$ $(s_n \in \mathbb{Z})$ and so

$$w = (s_1 y_1, s_2 y_2, (s_2 + s_3) y_3, \ldots (s_2 + s_3 + \cdots + s_n) y_n, \ldots) \in Y.$$

Thus $W \leqq Y$. While conversely if $y \in Y$ then $y = (t_1 y_1, t_2 y_2, \ldots, t_n y_n, \ldots)$ for integers $t_n$, and so $y = \sum_{n=1}^{\infty} s_n w_n$ where the $s_n$ are given by

$$s_1 = t_1, \quad s_2 = t_2, \quad s_3 = t_3 - s_2, \quad s_4 = t_4 - s_3, \ldots \quad \text{etc.}$$

Thus $y \in W$ and so $W = Y$ as claimed.

Thus $Y$ is a product in the sense of Fuchs [2] § 95 and the result follows from [2] Lemma 95.1.

**Proposition 3** (Nunke). *Every epimorphic image of a Specker group $P$ is the direct sum of a cotorsion group and a direct product of at most countably many infinite cyclic groups.*

Proof. See Nunke [5] or Fuchs [2] 95.2.

Proof of Theorem 1. Suppose $X$ is isomorphic to $P$, then applying Proposition 3 to $G$ where $X/P \cong G$, we see that $G$ is a direct sum of a cotorsion group and a product of infinite cyclic groups. Since $G$ has at most countable rank it is clear that the product of infinite cyclic groups must be a finite product. Moreover since a reduced torsion-free cotorsion group must contain a summand isomorphic to the $p$-adic integers, for some $p$, it is clear that the cotorsion summand of $G$ must be divisible. Thus $G = D \oplus F$ where $D$ is divisible and $F$ is a finite product of infinite cyclic groups i.e. $F$ is free of finite rank. It remains to show that $D = 0$.

Choose a subgroup $Y$ of $X$ such that $Y/P \cong F$. Clearly $Y \cong P$ and $X/Y \cong D$. Now applying Lemma 2 we may write

$$X = \prod_{n=1}^{\infty} \langle a_n \rangle \quad \text{and} \quad Y = \prod_{n=1}^{\infty} \langle k_n a_n \rangle.$$

We topologize $X$ with the strong topology. (We remark that we are only concerned with this presentation of $X$ and we don't need to show that the product topology on $X$ is independent of the presentation although this is indeed true.) The proof is completed by the following result:

**Lemma 4.** *$Y$ is closed and dense in $X$ in the strong topology on $X$.*

Proof. (i) *$Y$ is dense.* If $x$ is in $X$ then a basis for the neighbourhoods of $x$ in the strong topology is $x + m \prod_{n \notin J} \langle a_n \rangle$, where $J$ is a finite subset of $\{1, 2, \ldots\}$ and $m \in \mathbb{Z} \setminus \{0\}$. To show density it suffices to show that for any finite subset $J$ and any integer $m \neq 0$,

$$Y \cap (x + m \prod_{n \notin J} \langle a_n \rangle) \quad \text{is non-null.}$$

Since $Y$ is dense in $X$ in the $\mathbb{Z}$-adic topology, we may write for any non-zero integer $m$, $x = mx' + y$, some $x' \in X$, $y \in Y$. Say $x' = (\ldots, r_i a_i, \ldots)$. Then set $x' = u + v$ where $u = (\ldots, r_i a_i, \ldots)\, i \notin J$, and $v = (\ldots, r_i a_i, \ldots)\, i \in J$. Then $x - mu = mv + y$. But $Y$ is pure in $X$ and $k_n a_n \in Y \cap k_n X$, so $a_n \in Y$ each $n$. Since $v$ is just a finite sum of multiples of elements of $Y$, $v$ is in $Y$. Thus $x - mu \in Y$ i.e.

$$Y \cap (x + m \prod_{n \notin J} \langle a_n \rangle) \quad \text{is non-null.}$$

(ii) *$Y$ is closed in $X$.* Since the strong topology is the supremum of the product and $\mathbb{Z}$-adic topologies, it is clearly sufficient to show $Y$ is closed in the product topology on $X$. If $\bar{Y}$ denotes the closure of $Y$ in the product topology on $X$ then we see that $x \in \bar{Y}$ if, and only if, for each $n = (1, 2, \ldots)$ there is a $u^n$ in $Y$ such that $x_i = u_i^n$ for all $i < n$. (Here the subscript $i$ denotes the $i$-th component.)

So if $x \in \bar{Y}$ then there is a $u^2$ in $Y$ such that

$$x_1 = u_1^2 = r_1 k_1 a_1 \quad \text{some } r_1 \in \mathbb{Z}.$$

Similarly $\quad x_2 = u_2^3 = r_2 k_2 a_2 \quad$ some $r_2 \in \mathbb{Z}$ etc.

But then $x = (r_1 k_1 a_1, r_2 k_2 a_2, \ldots, r_i k_i a_i, \ldots)$ is clearly in $Y$. Thus $Y$ is closed as required.

We can easily deduce

**Corollary 5.** *If $X$ is a pure subgroup of the Specker group $P$ such that $P/X$ is of at most countable rank, then $X \cong P$ if and only if $P/X \cong \mathbb{Z}^n$ for some non-negative integer $n$.*

Remark. It is hoped that this paper, along with Lady's paper [3], will illustrate how useful simple topological techniques may be in areas which have hitherto used homological machinery.

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