



2013

Endomorphisms of Abelian Groups with Small Algebraic Entropy

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Recommended Citation

Dikranjan, D., Gong, K. and Zanardo, P. (2013). Endomorphisms of abelian groups with small algebraic entropy. *Linear Algebra and its Applications*, vol. 439, no. 7, pg. 1894–1904 doi.org/10.1016/j.laa.2013.05.021

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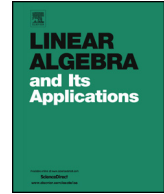


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Endomorphisms of abelian groups with small algebraic entropy [☆]



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ARTICLE INFO

Article history:

Received 11 January 2011

Accepted 6 May 2013

Available online 14 June 2013

Submitted by T.J. Laffey

Dedicated to Professor Sergey Yuzvinskii on his seventieth birthday

Keywords:

Yuzvinskii formula

Algebraic entropy

Mahler measure

Characteristic polynomial

Eigenvalue

Abelian group

ABSTRACT

We study the endomorphisms ϕ of abelian groups G having a “small” algebraic entropy h (where “small” usually means $h(\phi) < \log 2$). Using essentially elementary tools from linear algebra, we show that this study can be carried out in the group \mathbb{Q}^d , where an automorphism ϕ with $h(\phi) < \log 2$ must have all eigenvalues in the open circle of radius 2, centered at 0 and ϕ must leave invariant a lattice in \mathbb{Q}^d , i.e., be essentially an automorphism of \mathbb{Z}^d . In particular, all eigenvalues of an automorphism ϕ with $h(\phi) = 0$ must be roots of unity. This is a particular case of a more general fact known as Algebraic Yuzvinskii Theorem. We discuss other particular cases of this fact and we give some applications of our main results.

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1. Introduction

The algebraic entropy of an endomorphism ϕ of an abelian group G can be defined as follows [3] (see also Remark 2.2(c)). For a finite subset F of G define the n -trajectory of ϕ with respect to F by

$$T_n(\phi, F) = F + \phi(F) + \cdots + \phi^{n-1}(F),$$

[☆] Some results of this paper, under appropriate modification, are included in the second author's PhD thesis, discussed at the School of Mathematical Sciences, Dublin Institute of Technology, in June 2012.

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where, as usual, $A + B$ denotes the sum of the subsets A, B of G and let

$$H(\phi, F) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, F)|}{n},$$

where $|X|$ denotes the cardinality of a set X . For the existence of this limit, the reader is referred to [3]. The algebraic entropy of ϕ is defined as

$$h(\phi) = \sup\{H(\phi, F) : F \text{ is a finite subset of } G\}. \tag{1}$$

This note is dedicated mainly to the study of the endomorphisms of \mathbb{Q}^d with small algebraic entropy, where “small” usually means $< \log 2$. In fact, a general investigation of small algebraic entropy may be reduced to this case, see Remark 2.5 for an extended comment on this non-trivial issue. An endomorphism of \mathbb{Q}^d can be represented by a $d \times d$ matrix over \mathbb{Q} . So the endomorphism ring of \mathbb{Q}^d is isomorphic to the matrix ring $M_d(\mathbb{Q})$. Let ϕ be an endomorphism of \mathbb{Q}^d with eigenvalues λ_i , $i = 1, 2, \dots, d$, and let s be the positive leading coefficient of the primitive characteristic polynomial of ϕ over \mathbb{Z} . We call the number

$$m(\phi) = \log s + \sum_{|\lambda_i| > 1} \log |\lambda_i|$$

the (logarithmic) Mahler measure of ϕ [9]. This number is an important dynamical invariant (see, for example [20]) of ϕ as the following theorem shows:

Algebraic Yuzvinskii Theorem. $h(\phi) = m(\phi)$ for every endomorphism ϕ of \mathbb{Q}^d .

No direct proof of this fact is available so far.¹ It can be deduced from two important deep facts: Yuzvinskii’s formula for the topological entropy of the solenoid automorphisms [15] and a “duality theorem” of Peters [16,17] (see also [5]) connecting the topological entropy of continuous automorphisms of compact metrizable abelian groups to the algebraic entropy of their discrete Pontryagin dual (see [4, Theorem 6.8] for such a deduction). A proof in the case of endomorphisms of \mathbb{Z}^d , based on properties of an appropriate extension of the algebraic entropy to continuous endomorphisms of arbitrary locally compact abelian groups, can be found in [18]; the proof makes extensive use of the Haar measure on locally compact abelian groups.

The proof of the Algebraic Yuzvinskii Theorem is beyond the aim of this note, although a direct proof of this theorem based on purely algebraic tools is certainly desirable. Indeed, the proof of the Yuzvinskii’s formula for the topological entropy is highly non-trivial (see [15] for a comprehensive proof) and the proof of the duality theorem [16,17] apparently contains some flaws (see [8] for more details). We propose instead a self-contained straightforward proof of two immediate consequences of this deep fact. The first one implies as a by-product a particular case of the Algebraic Yuzvinskii Theorem that if either $h(\phi) = 0$ or $m(\phi) = 0$, then they coincide (this is the equivalence of (a) and (b) in Corollary 1.3).

Theorem 1.1. *If $\phi \in \text{End}(\mathbb{Q}^d)$ with $h(\phi) < \log 2$, then $s = 1$ and $|\lambda_i| < 2$ for all eigenvalues λ_i of ϕ .*

The conclusion of this theorem obviously follows from $m(\phi) < \log 2$; yet, without the equality $h(\phi) = m(\phi)$ the theorem seems less obvious.

We deduce the second part of Theorem 1.1 from the following theorem that presents another immediate consequence of the Algebraic Yuzvinskii Theorem.

Theorem 1.2. *Let $\phi \in \text{End}(\mathbb{Q}^d)$. Then $h(\phi) \geq \log |\lambda_i|$ for all non-zero eigenvalues of ϕ .*

¹ A proof was recently given by A. Giordano Bruno and S. Virili [11].

From these theorems one obtains

Corollary 1.3. *The following are equivalent for every $\phi \in \text{End}(\mathbb{Q}^d)$.*

- (a) $h(\phi) = 0$;
- (b) $m(\phi) = 0$, i.e., $s = 1$ and $|\lambda_i| = 1$ for all non-zero eigenvalues of ϕ ;
- (c) $s = 1$ and all non-zero eigenvalues of ϕ are roots of unity.

The equivalence of items (b) and (c) of the above corollary is known (see [9], where one can find relevant information on the logarithmic Mahler measure). Here it is obtained as a by-product of the proof of Theorem 1.1. Our proof of the equivalence of items (a) and (c), making no recourse to the Algebraic Yuzvinskii Theorem, provides a solution to an open problem in [4].

For an abelian group G and $\phi \in \text{End}(G)$, an element $x \in G$ is said to be *quasi-periodic*, if $\phi^n(x) = \phi^m(x)$ for some $n > m$ in \mathbb{N} . Interest in quasi-periodic elements comes from the following characterization of zero algebraic entropy endomorphisms of torsion abelian groups G : $h(\phi) = 0$ for some $\phi \in \text{End}(G)$ if, and only if, every element of G is quasi-periodic [7]. The next corollary follows from the implication (a) \rightarrow (c) of Corollary 1.3.

Corollary 1.4. *Every $\phi \in \text{End}(\mathbb{Q}^d)$ with $h(\phi) = 0$ has non-trivial quasi-periodic points.*

This is a particular case of [4, Theorem 6.8] (case (e)), asserting that an endomorphism $\phi \in \text{End}(G)$ with zero algebraic entropy has non-trivial quasi-periodic points for every abelian group G . This is the hardest case in that proof, making recourse to the topological Yuzvinskii formula and Peters' duality theorem from [16]. Therefore, Corollary 1.4 allows for an alternative approach to the study of zero algebraic entropy (Pinsker subgroup, quasi-periodic points, etc.), free from the use of Yuzvinskii formula in any form.

We now propose another application of Corollary 1.3. It was proved in [7, Lemma 2.5] that $h(\phi + \psi) = 0$ whenever $h(\phi) = h(\psi) = 0$ for two commuting endomorphisms ϕ, ψ of torsion abelian groups. Hence it seems natural to ask whether this property remains true in the general case. Easy counterexamples suggest the need to impose a stronger condition than just $h(\phi) = h(\psi) = 0$ on the commuting endomorphisms ϕ, ψ (see Example 2.6). We will require that ψ be nilpotent, that is, there exists an integer k such that $\psi^k = 0$. This gives the following theorem (its proof will be given in Section 3):

Theorem 1.5. *Let ϕ, ψ be two commuting endomorphisms of an abelian group G . If $h(\phi) = 0$ and ψ is nilpotent, then $h(\phi + \psi) = 0$.*

A particular case of this theorem, when $\phi = id_G$, was proved in [3, Claim 5.9].

The small values of algebraic entropy are related to the celebrated Lehmer problem formulated about the small values of Mahler measure. Indeed, provided the equality $h(\phi) = m(\phi)$ is available for every endomorphism ϕ of \mathbb{Q}^d , Lehmer asked whether a positive lower bound of all positive entropies $h(\phi)$ exists. We will not discuss this topic here, more information can be found in [9,12,6].

A similar, but different kind of algebraic entropy in vector spaces G is considered in the survey [10] (the finite set F is replaced by a finite-dimensional subspace F of G and in the limit defining $H(\phi, F)$, the finite number $\log |T_n(\phi, F)|$ is replaced by $\dim T_n(\phi, F)$). In this setting all entropies in a finite-dimensional space are zero.

This note is organized as follows: Section 2 contains some general properties of the algebraic entropy necessary for the proofs of our main results, but having also independent interest. The proofs of Theorems 1.1, 1.2 and 1.5, and Corollary 1.3 are given in Section 3. Here the key point is our Main Lemma, establishing that for an endomorphism ϕ of \mathbb{Q}^d with $h(\phi) < \log t$, where $t > 1$ is an integer, all eigenvalues are roots of some polynomials with coefficients from $\{-t+1, \dots, -1, 0, 1, 2, \dots, t-1\}$. The careful reader will notice that a "perfect" weaker counterpart of the Algebraic Yuzvinskii formula, in the spirit of Theorem 1.2, must necessarily contain also the inequality

$$h(\phi) \geq \log s \tag{2}$$

which is missing in the current form of the theorem. We discuss (2) in the final Remark 3.6.

2. General properties of the algebraic entropy

Here we recall some known general properties of the algebraic entropy and give some new ones. The set of all positive integers will be denoted by \mathbb{N} .

In the next theorem we collect the major results concerning the algebraic entropy h needed for our proofs.

Theorem 2.1. *Let G be an abelian group and $\phi \in \text{End}(G)$.*

- (a) [Addition Theorem] [3, Theorem 1.3] *If H is a ϕ -invariant subgroup of G (i.e., $\phi(H) \subseteq H$) and $\bar{\phi} : G/H \rightarrow G/H$ is the endomorphism induced by ϕ , then $h(\phi) = h(\phi \upharpoonright_H) + h(\bar{\phi})$.*
- (b) [Logarithmic Law] [3, Proposition 2.8(b)] *For every $k \in \mathbb{N}$, $h(\phi^k) = kh(\phi)$. If ϕ is an automorphism, then $h(\phi^{-1}) = h(\phi)$.*
- (c) [Extension Theorem] [4, Proposition 2.12] *If G is torsion-free, then ϕ (uniquely) extends to an endomorphism $\tilde{\phi}$ of the divisible hull $D(G)$ of G with $h(\tilde{\phi}) = h(\phi)$.*

Remark 2.2.

- (a) [Special Addition Theorem] The proof of the Addition Theorem makes use of the Algebraic Yuzvinskii Theorem. This is why we prefer to emphasize this dependence and avoid the use of the Addition Theorem. In the special case when $H = t(G)$ (where $t(G)$ denotes the torsion subgroup of G) the proof can be obtained from [3, Proposition 5.2] combined with [3, Proposition 3.3]. So this particular case does not make recourse to the Algebraic Yuzvinskii Theorem and we shall use it in the sequel.

The Addition Theorem was proved by Yuzvinskii [19] in the framework of measure-theoretic entropy on separable compact groups (where it coincides also with the topological entropy in the sense of [1]). For this reason some authors call it also *Yuzvinskii's Addition Formula*. For another manifestation of Yuzvinskii's Addition Formula see [2], where the reader may also find entropy in the context of Markov processes.

- (b) [Addition Theorem for zero algebraic entropy] The inequalities $h(\phi) \geq h(\phi \upharpoonright_H)$ and $h(\phi) \geq h(\bar{\phi})$ trivially follow from item (a) of Theorem 2.1 (they are also easily obtained from the definitions). Hence $h(\phi) = 0$ trivially implies $h(\phi \upharpoonright_H) = h(\bar{\phi}) = 0$ in item (a). The reverse implication

$$h(\phi \upharpoonright_H) = h(\bar{\phi}) = 0 \implies h(\phi) = 0$$

is non-trivial. It was obtained in [4, Corollary 6.9] as a consequence of [4, Theorem 6.8] (ensuring the existence of non-trivial quasi-periodic points of endomorphisms of zero algebraic entropy). Due to our Corollary 1.4, this weaker form of the Addition Theorem (for zero algebraic entropy) becomes independent of the Algebraic Yuzvinskii Theorem.

- (c) [Peters's entropy] Peters [16] introduced an entropy function h_P for an automorphism ϕ of an abelian group G , that in terms of h is given by $h_P(\phi) := h(\phi^{-1})$. Due to the equality $h(\phi^{-1}) = h(\phi)$ from item (b) of Theorem 2.1, one has $h_P(\phi) = h(\phi)$.

In the sequel we use $\lfloor x \rfloor$ (resp., $\lceil x \rceil$) to denote the greatest (resp., least) integer “below” (resp., “above”) a real number x , i.e.,

$$\lfloor x \rfloor, \lceil x \rceil \in \mathbb{Z} \quad \text{with} \quad \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1 \quad \text{and} \quad \lceil x \rceil - 1 < x \leq \lceil x \rceil.$$

Proposition 2.3. *Let G be an abelian group, $\phi \in \text{End}(G)$ and $t > 1$ be a real number. Then*

- (a) *if $h(\phi) < \log t$, then for every finite subset F in G there exists $N \in \mathbb{N}$, such that $|T_n(\phi, F)| < t^n$ for all $n > N$;*
- (b) *if $|T_n(\phi, F)| < t^n$ holds for some $n \in \mathbb{N}$, $v \in G$ and $F = \{0, v, 2v, \dots, (\lceil t \rceil - 1)v\}$, then there exists a non-zero n -tuple $(\alpha_0, \dots, \alpha_{n-1}) \in \{-\lceil t \rceil + 1, \dots, -1, 0, 1, \dots, \lceil t \rceil - 1\}^n$, such that $\sum_{i=0}^{n-1} \alpha_i \phi^i(v) = 0$.*

Proof. (a) Fix a finite subset F of G . By definition (1) we deduce $H(\phi, F) < \log t$. Hence there exists a natural number N , such that for every $n > N$, $(\log |T_n(\phi, F)|)/n < \log t$, hence $\log |T_n(\phi, F)| < n \log t = \log t^n$, i.e., $|T_n(\phi, F)| < t^n$ for all $n > N$.

(b) Consider the surjection $\eta : \{0, 1, \dots, \lceil t \rceil - 1\}^n \rightarrow T_n(\phi, F) = F + \phi(F) + \dots + \phi^{n-1}F$, defined by

$$\eta : (m_0, \dots, m_{n-1}) \mapsto \sum_{i=0}^{n-1} m_i \phi^i(v), \quad (m_0, \dots, m_{n-1}) \in \{0, 1, \dots, \lceil t \rceil - 1\}^n.$$

By our hypothesis $|T_n(\phi, F)| < t^n \leq \lceil t \rceil^n$, hence η cannot be injective. So there are two distinct n -tuples

$$(m_0, \dots, m_{n-1}) \in \{0, 1, \dots, \lceil t \rceil - 1\}^n, \quad (k_0, \dots, k_{n-1}) \in \{0, 1, \dots, \lceil t \rceil - 1\}^n$$

such that

$$\sum_{i=0}^{n-1} m_i \phi^i(v) = \sum_{i=0}^{n-1} k_i \phi^i(v).$$

Therefore, $\sum_{i=0}^{n-1} \alpha_i \phi^i(v) = 0$ holds with $\alpha_i = m_i - k_i \in \{-\lceil t \rceil + 1, \dots, -1, 0, 1, \dots, \lceil t \rceil - 1\}$, $i = 0, 1, 2, \dots, n - 1$ and not all $\alpha_i = 0$. \square

For a finite subset F of an abelian group G and $\phi \in \text{End}(G)$ let

$$T(\phi, F) = \bigcup_n T_n(\phi, F) \quad \text{and} \quad V(\phi, F) = \langle T(\phi, F) \rangle,$$

i.e., $V(\phi, F)$ is the smallest ϕ -invariant subgroup of G containing F . Clearly, $V(\phi, F)$ is finitely generated if and only if $V(\phi, F) = \langle T_n(\phi, F) \rangle$ for some $n \in \mathbb{N}$.

The next corollary describes important properties of the endomorphisms with small algebraic entropy.

Corollary 2.4. *If ϕ is an endomorphism of an abelian group G with $h(\phi) < \log 2$, then*

- (a) *for every $v \in G$ there exist $n \in \mathbb{N}$ and a non-zero n -tuple $(\alpha_0, \dots, \alpha_{n-1}) \in \{-1, 0, 1\}^n$, such that $\sum_{i=0}^{n-1} \alpha_i \phi^i(v) = 0$;*
- (b) *$V(\phi, F)$ is finitely generated for every finite subset F in G ;*
- (c) *if $\psi \in \text{End}(G)$ with $\psi\phi = \phi\psi$ and $h(\psi) < \log 2$, then for every finite set F in G there exists a finitely generated subgroup L of G containing F that is both ϕ -invariant and ψ -invariant;*
- (d) *$h(\phi|_{t(G)}) = 0$;*
- (e) *$h(\phi) = h(\bar{\phi})$, where $\bar{\phi}$ is the endomorphism of $G/t(G)$ induced by ϕ .*

Proof. Item (a) follows immediately from Proposition 2.3 with $t = 2$.

(b) Fix a finite subset F of G . It is not restrictive to assume that $0 \in F$. If $F = \{0\}$, there is nothing left to prove, so assume $F \neq \{0\}$. Then $F = \{0, v_1, \dots, v_t\}$, with $t \geq 1$. Assume the case $t = 1$ holds true. Then for every $i = 1, 2, \dots, t$ and $F_i = \{0, v_i\}$ we can find a finitely generated ϕ -invariant subgroup L_i of G such that $F_i \subseteq L_i$. Then $L = L_1 + L_2 + \dots + L_t$ is a finitely generated ϕ -invariant subgroup of G with $F \subseteq L$.

So it remains to consider the case $t = 1$. By (a) there exist $n \in \mathbb{N}$ and a non-zero n -tuple $(\alpha_0, \dots, \alpha_{n-1}) \in \{-1, 0, 1\}^n$, such that $\sum_{i=0}^{n-1} \alpha_i f^i(v_1) = 0$. Let k be the largest index with $\alpha_k \neq 0$. Then

$$\phi^k(v_1) = \pm \sum_{i=0}^{k-1} \alpha_i \phi^i(v_1). \tag{3}$$

The subgroup $L_1 = \langle v_1, \phi(v_1), \dots, \phi^{k-1}(v_1) \rangle$ is obviously finitely generated and contains F . Moreover, L_1 is ϕ -invariant as $\phi(L_1) \leq L_1$ by the relation (3).

(c) According to (b), the smallest ϕ -invariant subgroup $V(\phi, F)$ of G generated by F is finitely generated. Let F_1 be a finite set of generators of $V(\phi, F)$ containing F . So

$$\phi(F_1) \subseteq V(\phi, F). \tag{4}$$

Applying (b) to F_1 , we conclude that the ψ -invariant subgroup $V(\psi, F_1)$ containing F_1 (hence F , as well), is finitely generated. From (4) and the fact that ϕ commutes with all powers $\psi^i, i \in \mathbb{N}$, we deduce that $V(\psi, F_1)$ is ϕ -invariant as well.

Item (d) follows from Remark 2.2(b) and the fact that, for every torsion abelian group H and every $\psi \in \text{End}(H)$, one has $h(\psi) = \log n$ for some natural number $n > 0$, [7]. Hence, $h(\psi) < \log 2$ yields $h(\psi) = 0$.

(e) This is an easy consequence of (c) and the *Special Addition Theorem* of Remark 2.2. \square

Remark 2.5. Let G be an abelian group and let $\phi \in \text{End}(G)$ with $h(\phi) < \log 2$. From item (e) of the above corollary we conclude that $h(\tilde{\phi}) = h(\phi) < \log 2$. Since $G/t(G)$ is torsion-free, this shows that endomorphisms with small algebraic entropy can be studied on torsion-free groups. Furthermore, for a torsion-free group G , we can apply Theorem 2.1(c) extending ϕ to an endomorphism $\tilde{\phi}$ of the divisible hull $D(G)$ of G with $h(\tilde{\phi}) = h(\phi)$. Therefore, this reduction step leads to torsion-free divisible groups, namely linear spaces over \mathbb{Q} . Finally, by item (b) of Corollary 2.4, the hypothesis $h(\phi) < \log 2$ implies that every finite subset F of G is contained in some ϕ -invariant subspace of finite dimension. This justifies the restriction of the study of the endomorphisms with small algebraic entropy to the groups \mathbb{Q}^d only.

Example 2.6. Here we give some easy examples to show that the hypothesis $\phi\psi = \psi\phi$ and the nilpotency are both crucial in Theorem 1.5.

- (a) If the endomorphisms are not commuting, the theorem fails already for torsion abelian groups, as shown in [7]. Here is a counterexample in \mathbb{Q}^2 . Consider $\phi, \psi \in \text{End}(\mathbb{Q}^2)$ defined by $\phi(e_1) = e_1, \phi(e_2) = e_1 + e_2$, and $\psi(e_1) = 2e_2, \psi(e_2) = 0$. Then $h(\phi) = 0$ and $\psi^2 = 0$, yet $h(\phi + \psi) > 0$.
- (b) If the nilpotency of ψ is omitted, then a counterexample can be found already in \mathbb{Z} . Take $\phi = \psi = \text{id}_{\mathbb{Z}}$. Now $h(\phi) = h(\psi) = 0$, but $h(\phi + \psi) = \log 2$ (that $h(\text{id}_{\mathbb{Z}}) = 0$ follows from the more general fact that $h(\text{id}_G) = 0$ for all abelian groups G).

3. Proofs

In the sequel ϕ will be an endomorphism of \mathbb{Q}^d associated with a matrix $A \in M_d(\mathbb{Q})$. We shall keep the same notation also for the \mathbb{C} -linear endomorphism of \mathbb{C}^d induced by the matrix A . In this sense we can consider now the Jordan normal form B of A and $B = PAP^{-1}$ with $P \in GL(d, \mathbb{C})$. We keep the notation B also for the \mathbb{C} -linear endomorphism of \mathbb{C}^d induced by the matrix B .

For a fixed endomorphism ϕ with associated matrix A , we fix the matrices P and B as above and we write

$$B = \begin{pmatrix} J_{d_1}(\lambda_1) & 0 & 0 & 0 & \dots & 0 \\ 0 & J_{d_2}(\lambda_2) & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & J_{d_r}(\lambda_r) \end{pmatrix}$$

where $\lambda_i, i = 1, 2, \dots, r$, are the eigenvalues of A , $J_{d_i}(\lambda_i)$ are $d_i \times d_i$ Jordan blocks relative to λ_i , and $d_1 + d_2 + \dots + d_r = d$.

Lemma 3.1. *If $h(\phi) < \log t$ for some integer $t > 1$, then for every $v \in \mathbb{Q}^d$, there exists $N \in \mathbb{N}$, such that for every $n > N$ there exists a non-zero n -tuple $(\alpha_0, \dots, \alpha_{n-1}) \in \{-t + 1, \dots, -1, 0, 1, \dots, t - 1\}^n$ with*

$$\sum_{i=0}^{n-1} \alpha_i B^i \cdot P v = 0. \tag{5}$$

Proof. Let $F = \{0, v, 2v, \dots, (t - 1)v\}$. By item (a) of Proposition 2.3 there exists a natural number N , such that $|T_n(A, F)| < t^n$ for all $n > N$.

Let $E = PF$. Since the isomorphism $P : \mathbb{C}^d \rightarrow \mathbb{C}^d$ takes $T_n(A, F)$ to $T_n(B, PF)$, we have $|T_n(A, F)| = |T_n(B, PF)|$. So, $|T_n(B, E)| < t^n$. Now by item (b) of Proposition 2.3, applied to the endomorphism $B : \mathbb{C}^d \rightarrow \mathbb{C}^d$ and $P v \in \mathbb{C}^d$, we conclude that (5) holds for some non-zero n -tuple $(\alpha_0, \dots, \alpha_{n-1}) \in \{-t + 1, \dots, -1, 0, 1, \dots, t - 1\}^n$. \square

Here comes the central result of this section. Its power becomes clear (see Corollary 3.3) when combined with an easy lemma on the distribution of roots (see Lemma 3.2).

Main Lemma 1. *If $t > 1$ is an integer such that $h(\phi) < \log t$, then for any eigenvalue λ of ϕ , there exists a non-zero polynomial $f(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_{n-1} x^{n-1}$ with coefficients $\alpha_i \in \{-t + 1, \dots, -1, 0, 1, \dots, t - 1\}$, such that $f(\lambda) = 0$.*

Proof. It is not restrictive to assume that the eigenvalue in question is exactly λ_r .

There exists a vector $v \in \mathbb{Q}^d$ such that the (last) d -th entry of $P v$ is non-zero. Indeed, let H be the hyperplane in \mathbb{C}^d formed by all vectors with last entry zero. Arguing for a contradiction, assume that P takes the whole \mathbb{Q}^d into H . Since \mathbb{Q}^d spans \mathbb{C}^d , we deduce that P takes also \mathbb{C}^d into H . Since $\dim H \leq d - 1$, this contradicts our assumption that P is non-singular.

Let $b_d \neq 0$ be the d -th entry of $P v$. According to Lemma 3.1, for this vector v , there exist $n \in \mathbb{N}$ and a non-zero n -tuple $(\alpha_0, \dots, \alpha_{n-1}) \in \{-t + 1, \dots, -1, 0, 1, \dots, t - 1\}^n$ satisfying (5).

From

$$B^i = \begin{pmatrix} J_{d_1}(\lambda_1)^i & 0 & 0 & 0 & \dots & 0 \\ 0 & J_{d_2}(\lambda_2)^i & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & J_{d_r}(\lambda_r)^i \end{pmatrix},$$

we conclude that

$$\sum_{i=0}^{n-1} \alpha_i B^i = \begin{pmatrix} B_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & B_2 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & B_r \end{pmatrix},$$

where $B_j = \sum_{i=0}^{n-1} \alpha_i J_{d_j}(\lambda_j)^i$ for all $j = 1, 2, \dots, r$. Hence the last (d -th) term of $\sum_{i=0}^{n-1} \alpha_i B^i \cdot P v$ is $\sum_{i=0}^{n-1} \alpha_i \lambda_r^i b_d$. By (5) it equals zero. So $b_d \neq 0$ yields $\sum_{i=0}^{n-1} \alpha_i \lambda_r^i = 0$. This gives the desired polynomial $f(x)$ with $f(\lambda_r) = 0$. \square

The next lemma on the root distribution of polynomials can be deduced from a much stronger classical result (known as Rouché’s theorem). We prefer to give a simple self-contained proof.

Lemma 3.2. *Assume that a non-zero polynomial $f(x) = a_0 + a_1 x + \dots + a_n x^n \in \mathbb{C}[x]$ satisfies $|a_i| \leq r$ for some real number r , for all $i = 0, 1, \dots, n$ and non-zero coefficients have modulus at least 1. Then $1/(r + 1) < |\lambda| < r + 1$ holds for every non-zero root of $f(x)$.*

Proof. Let λ be a root of $f(x)$ with $|\lambda| > 1$. Then $f(\lambda) = a_0 + a_1\lambda + \dots + a_n\lambda^n = 0$. We can assume without loss in generality that $a_n \neq 0$. Hence $|a_n| \geq 1$, our assumption on the coefficients, entails $|a_n| \geq 1$. So we have

$$\begin{aligned} |\lambda|^n &\leq |a_n\lambda^n| = |a_0 + a_1\lambda + \dots + a_{n-1}\lambda^{n-1}| \\ &\leq |a_0| + |a_1||\lambda| + \dots + |a_{n-1}||\lambda|^{n-1} \\ &\leq r(1 + |\lambda| + \dots + |\lambda|^{n-1}) \\ &= r \cdot \frac{|\lambda|^n - 1}{|\lambda| - 1}. \end{aligned}$$

Then $|\lambda|^n \cdot (|\lambda| - 1) \leq r|\lambda|^n - r$, so dividing by $|\lambda|^n$ we get $|\lambda| \leq r + 1 - r/|\lambda|^n < r + 1$.

If $0 < |\lambda| < 1$, then $\mu = \lambda^{-1}$ is a root of the reciprocal polynomial $g(x)$ of $f(x)$, i.e. of the polynomial $g(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$, which still has the same properties since the set of its coefficients is the same as that of $f(x)$. Therefore $|\mu| < r + 1$, so that $|\lambda| > 1/(r + 1)$. \square

Corollary 3.3. *If $\phi \in \text{End}(\mathbb{Q}^d)$ and $|\lambda_i| \geq t$ for some eigenvalue of ϕ and some integer $t > 0$, then $h(\phi) \geq \log t$. In particular, $h(\phi) \geq \log \lfloor |\lambda_i| \rfloor$ for all eigenvalues λ_i of ϕ .*

Proof. Assume on the contrary that $h(\phi) < \log t$. According to Main Lemma and Lemma 3.2, this entails $|\lambda_i| < t$ for all eigenvalues of ϕ , a contradiction. \square

Now we can easily deduce Theorem 1.2 from Corollary 3.3:

Proof of Theorem 1.2. Fix an eigenvalue λ_j of ϕ and let $a := |\lambda_j|$. We have to prove that $h(\phi) \geq \log a$. If $a \leq 1$, this is obvious, so from now on assume $a > 1$. According to Corollary 3.3, $h(\phi) \geq \log \lfloor a \rfloor$.

For any natural number k , the eigenvalues of ϕ^k are exactly λ_i^k for all $1 \leq i \leq d$. Hence, arguing in the same way, we have $h(\phi^k) \geq \log(\lfloor a^k \rfloor)$. By the logarithmic law Theorem 2.1(b), $h(\phi^k) = kh(\phi)$, so we get $kh(\phi) \geq \log(\lfloor a^k \rfloor)$, and consequently

$$h(\phi) \geq \frac{1}{k} \log(\lfloor a^k \rfloor) = \log(\lfloor a^k \rfloor^{1/k}).$$

Hence, to prove the required inequality $h(\phi) \geq \log a$ it suffices to check that

$$\lim_{k \rightarrow \infty} (\lfloor a^k \rfloor)^{1/k} = a. \tag{6}$$

By our assumption $a > 1$, the sequence $\{\lfloor a^k \rfloor : k = 1, 2, \dots, \infty\}$ converges to infinity. Hence

$$(\lfloor a^k \rfloor + 1)^{1/k} - \lfloor a^k \rfloor^{1/k} \rightarrow 0 \text{ when } k \rightarrow \infty. \tag{7}$$

From $\lfloor a^k \rfloor \leq a^k < \lfloor a^k \rfloor + 1$, we deduce $\lfloor a^k \rfloor^{1/k} \leq a < (\lfloor a^k \rfloor + 1)^{1/k}$. So relation (7) implies the (6). \square

Proof of Theorem 1.1. If $\phi \in \text{End}(\mathbb{Q}^d)$ and $h(\phi) < \log 2$, then $|\lambda_i| < 2$ for all eigenvalues of ϕ by Theorem 1.2.

To prove that $s = 1$ we use the fact that the characteristic polynomial $p(x) \in \mathbb{Z}[x]$ of ϕ is primitive. Factorize $p(x) = p_1(x) \cdot p_2(x) \cdot \dots \cdot p_k(x)$ as a product of irreducible polynomials in $\mathbb{Z}[x]$ and denote by s_i the leading coefficient of $p_i(x)$. Then $s = s_1 \cdot s_2 \cdot \dots \cdot s_k$. Fix arbitrarily $j = 1, 2, \dots, k$ and pick a root λ_j of $p_j(x)$. Then it is an eigenvalue of A , thus $f(\lambda_j) = 0$ for some polynomial $f(x)$ with coefficients from $\{-1, 0, 1\}$ according to the Main Lemma and our hypothesis $h(\phi) < \log 2$. Since $p_j(x)$ is primitive and $f(x) \in \mathbb{Z}[x]$, we conclude that $p_j(x)|f(x)$ in $\mathbb{Z}[x]$ as well. Since $f(x)$ is a monic polynomial, we deduce that $p_j(x)$ is monic as well, i.e., $s_j = 1$. Therefore, $s = s_1 \cdot s_2 \cdot \dots \cdot s_k = 1$.

An alternative proof of the equality $s = 1$ can be obtained as follows. Take any finite set F of generators of the subgroup \mathbb{Z}^d of \mathbb{Q}^d . By means of item (b) of Corollary 2.4 we can produce a finitely

generated ϕ -invariant subgroup L of \mathbb{Q}^d containing \mathbb{Z}^d . Since $L \cong \mathbb{Z}^d$, this gives again $s = 1$ with a standard argument. \square

Remark 3.4. If ϕ is an automorphism of \mathbb{Q}^d with $h(\phi) < \log 2$, then Theorem 1.1 applied to ϕ^{-1} , along with the fact that $h(\phi^{-1}) = h(\phi)$ (Theorem 2.1(b)), gives $1/2 < |\lambda_i| < 2$ for all eigenvalues of ϕ (the recourse to the equality $h(\phi^{-1}) = h(\phi)$ can be avoided by a direct application of the Main Lemma and Lemma 3.2). Moreover, the first argument used to prove $s = 1$ gives also $\prod_j \lambda_j = \pm 1$.

Proof of Corollary 1.3. To prove the implication (a) \rightarrow (b) note that $s = 1$ follows from Theorem 1.1. By Theorem 1.2, $|\lambda_i| \leq 1$ for all eigenvalues of ϕ . Therefore, since the coefficients of the characteristic polynomial are integers and $s = 1$, the product of the non-zero eigenvalues of ϕ has modulus ≥ 1 . This yields $|\lambda_i| = 1$ for all non-zero eigenvalues of ϕ . Now the implication (b) \rightarrow (c) follows from Kronecker’s theorem [9,14].

To prove the implication (c) \rightarrow (a) assume that $s = 1$ and all λ_i are roots of unity. Then there exists $k \in \mathbb{N}$ such that $\lambda_i^k = 1$ for all i . Let $\xi = \phi^k$. Then all eigenvalues of ξ are 1. So $(\xi - id_{\mathbb{Q}^d})^k = 0$. So $\psi = \xi - id_{\mathbb{Q}^d}$, satisfies $\psi^k = 0$ and $\xi = \psi + id_{\mathbb{Q}^d}$. Now $h(\xi) = 0$ by [3, Claim 5.9]. For readers’ convenience we give here a different, direct and self-contained proof of this fact.

According to (1), to prove $h(\xi) = 0$ we have to check that $H(\xi, F) = 0$ for every finite subset F of \mathbb{Q}^d . Fix such an $F \subseteq \mathbb{Q}^d$.

From $\psi^k = 0$ one can deduce that $\psi^d = 0$. Hence, for $i \in \mathbb{N}$ and $x \in F$

$$\xi^i(x) = \begin{cases} x + C_1^i \psi(x) + \dots + C_i^i \psi^i(x) & \text{if } i < d, \\ x + C_1^i \psi(x) + \dots + C_{d-1}^i \psi^{d-1}(x) & \text{if } i \geq d - 1, \end{cases}$$

where C_k^i are the binomial coefficients (number of k -element subsets of an i -element set). Therefore,

$$\xi^i(F) \subseteq \begin{cases} F + C_1^i \psi(F) + C_2^i \psi^2(F) + \dots + C_i^i \psi^i(F) & \text{if } i < d, \\ F + C_1^i \psi(F) + C_2^i \psi^2(F) + \dots + C_{d-1}^i \psi^{d-1}(F) & \text{if } i \geq d - 1. \end{cases}$$

Let $M_j = \sum_{i=j}^{n-1} C_i^i \psi^j(F)$ for $j = 0, 1, \dots, d - 1$ we get

$$T_n(\xi, F) = \sum_{i=0}^{n-1} \xi^i(F) \subseteq \sum_{j=0}^{d-1} \sum_{i=j}^{n-1} C_j^i \psi^j(F) = M_0 + M_1 + \dots + M_{d-1}.$$

Therefore,

$$|T_n(\xi, F)| \leq \left| \sum_{j=0}^{d-1} M_j \right|. \tag{8}$$

To evaluate the latter cardinality fix an arbitrary finite subset $S = \{f_1, \dots, f_t\}$ of \mathbb{Q}^d and let c_1, c_2, \dots, c_n be an increasing sequence of naturals. Then the set $S_n = c_1 S + c_2 S + \dots + c_n S$ satisfies

$$|S_n| \leq (nc_n + 1)^{|S|}. \tag{9}$$

Indeed, for $S_n \ni x = \sum_{r=1}^n c_r f_{i_r}$ with $f_{i_r} \in S$ one has $x = \sum_{j=1}^t a_j f_j$, with $a_j = \sum_{i_r=j} c_r$. So $0 \leq a_j \leq nc_n$ for every $j = 1, 2, \dots, t$ by our hypothesis on c_r . Then the inequality (9) follows from the surjectivity of the map $\eta : \{0, 1, \dots, nc_n\}^t \rightarrow S_n$.

For $n > 2d$ and $j < d$ we apply (9) to $S = \psi^j(F)$ and the sequence $\{C_j^k : j \leq k < n\}$ to conclude that $|M_j| \leq (nC_j^{n-1} + 1)^{|F|}$. Along with the inequality (8), this gives

$$|T_n(\xi, F)| \leq |M_0 + M_1 + \dots + M_{d-1}| \leq \prod_{j=0}^{d-1} (nC_j^{n-1} + 1)^{|F|}. \tag{10}$$

Since the right-hand side of the inequality (10) is a polynomial of n , this proves that $H(\xi, F) = 0$, since $|T_n(\xi, F)|$ has polynomial growth. Of course, $h(\xi) = kh(\phi)$ yields $h(\phi) = 0$, as desired. \square

The following fact from matrix analysis [13, pp. 92–95] will be needed in the proof of Theorem 1.5:

Theorem 3.5. *Let A, B be two $d \times d$ commuting matrices over the field of complex numbers with eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_d$ and $\beta_1, \beta_2, \dots, \beta_d$ respectively. Then there exists a permutation σ of $\{1, 2, \dots, d\}$ such that the eigenvalues of $A + B$ are $\alpha_1 + \beta_{\sigma(1)}, \alpha_2 + \beta_{\sigma(2)}, \dots, \alpha_d + \beta_{\sigma(d)}$.*

Proof of Theorem 1.5. First we consider the case $G = \mathbb{Q}^d$. By Corollary 1.3, all non-zero eigenvalues of ϕ are roots of unity as $h(\phi) = 0$. On the other hand, ψ is nilpotent, so its eigenvalues are all 0. Hence by Theorem 3.5, all the non-zero eigenvalues of $\phi + \psi$ are roots of unity (being the same as those of ϕ). Again by Corollary 1.3, we have $h(\phi + \psi) = 0$.

Our next step is to extend the theorem to an arbitrary torsion-free abelian divisible group G , i.e., a linear space over \mathbb{Q} . To establish $h(\phi + \psi) = 0$ fix a finite subset F of G . Then by item (c) of Corollary 2.4, F is contained in a finite-dimensional subspace L of G that is both ϕ -invariant and ψ -invariant. Now, the above argument applies to the subspace $L \cong \mathbb{Q}^d$ and the restrictions $\phi \upharpoonright_L$ and $\psi \upharpoonright_L$ and this gives $h(\phi \upharpoonright_L + \psi \upharpoonright_L) = 0$. Since L is $(\phi + \psi)$ -invariant, this yields that $H(\phi + \psi, F) = 0$. Therefore, $h(\phi + \psi) = 0$.

Now assume that G is torsion-free and consider the extensions $\tilde{\phi}$ and $\tilde{\psi}$ of ϕ and ψ , respectively, to the divisible hull $D(G)$ of G . According to item (c) of Theorem 2.1, $h(\tilde{\phi}) = 0$. Moreover, from $\phi\psi = \psi\phi$, and the uniqueness of the extensions one can easily deduce that

$$\tilde{\phi}\tilde{\psi} = \tilde{\phi}\tilde{\psi} = \tilde{\psi}\tilde{\phi} = \tilde{\psi}\tilde{\phi}, \quad (\tilde{\psi})^n = \tilde{\psi}^n = 0, \quad \text{and} \quad \widetilde{\phi + \psi} = \tilde{\phi} + \tilde{\psi}. \tag{11}$$

Applying the above argument to the torsion-free divisible group $D(G)$ and its commuting endomorphisms $\tilde{\phi}$ and $\tilde{\psi}$, we conclude from (11) that $h(\tilde{\phi} + \tilde{\psi}) = 0$. So the last equality in (11) gives $h(\widetilde{\phi + \psi}) = 0$. By Remark 2.2(b), we deduce that $h(\phi + \psi) \leq h(\widetilde{\phi + \psi}) = 0$ is zero as well.

Now assume that G is an arbitrary abelian group and let $L = t(G)$. Then L is both ϕ -invariant and ψ -invariant. The restrictions $\phi \upharpoonright_L$ and $\psi \upharpoonright_L$ commute and have algebraic entropy zero, so $h(\phi \upharpoonright_L + \psi \upharpoonright_L) = 0$ by [7, Lemma 2.5]. Therefore, for $\xi = \phi + \psi$ one has $h(\xi \upharpoonright_L) = 0$. Using the *Special Addition Theorem* (Remark 2.2(a)), it is possible to deduce that $h(\xi) = h(\tilde{\xi})$, where $\tilde{\xi} : G/L \rightarrow G/L$ is the induced endomorphism. The induced endomorphisms $\tilde{\phi}, \tilde{\psi} \in \text{End}(G/L)$ commute, $h(\tilde{\phi}) = 0$ and $\tilde{\psi}^n = 0$. Since G/L is torsion-free, we get $h(\phi + \psi) = h(\tilde{\xi}) = 0$. \square

Remark 3.6. Both versions of the argument of the implication “ $h(\phi) < \log 2 \implies s = 1$ ” in Theorem 1.1 can easily be extended to prove the inequality (2) in the case when ϕ has no proper invariant \mathbb{Q} -subspaces (so that its characteristic polynomial $p(x)$ is irreducible). We believe that this remains true without this additional restraint on ϕ , but we have no proof at hand.

Acknowledgements

It is a pleasure to thank A. Giordano Bruno, B. Goldsmith, L. Salce, D. Toller and S. Virili for many helpful discussions related to the topic of this note. Thanks are due also to the referee for her/his careful reading and helpful comments. The second author would like to thank the Department of Mathematics and Computer Science at University of Udine for its kind hospitality and support during the visit when part of the work for this paper was being done.

References

[1] R.L. Adler, A.G. Konheim, M.H. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.* 114 (1965) 309–319.
 [2] L. Bowen, Non-abelian free group actions: Markov processes, the Abramov–Rohlin formula and Yuzvinskii’s formula, *Ergodic Theory Dynam. Systems* 30 (2010) 1629–1663.
 [3] D. Dikranjan, A. Giordano Bruno, Entropy for abelian groups, preprint, arXiv:1007.0533.

- [4] D. Dikranjan, A. Giordano Bruno, The Pinsker subgroup of an algebraic flow, *J. Pure Appl. Algebra* 216 (2) (2012) 364–376.
- [5] D. Dikranjan, A. Giordano Bruno, The connection between topological and algebraic entropy, *Topology Appl.* 159 (13) (2012) 2980–2989.
- [6] D. Dikranjan, A. Giordano Bruno, Topological entropy and algebraic entropy for group endomorphisms, in: *Proceedings ICTA2011, Islamabad, Pakistan, July 4–10 2011*, Cambridge Scientific Publishers, 2012, pp. 133–214.
- [7] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, Algebraic entropy of endomorphisms of abelian groups, *Trans. Amer. Math. Soc.* 361 (2009) 3401–3434.
- [8] D. Dikranjan, M. Sanchis, S. Virili, New and old facts about entropy in uniform spaces and topological group, *Topology Appl.* 159 (7) (2012) 1916–1942.
- [9] G. Everest, T. Ward, *Heights of Polynomials and Entropy in Algebraic Dynamics*, Universitext, Springer-Verlag London, Ltd., London, 1999.
- [10] A. Giordano Bruno, L. Salce, A soft introduction to algebraic entropy, *Arabian J. Math.* 1 (1) (2012) 69–87.
- [11] A. Giordano Bruno, S. Virili, Algebraic Yuzvinski formula, *J. Algebra* (2013), in press.
- [12] E. Hironaka, What is ... Lehmer's number?, *Notices Amer. Math. Soc.* 56 (3) (2009) 374–375.
- [13] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985, reprinted with corrections 1990.
- [14] L. Kronecker, Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten, *J. Reine Angew. Math.* 53 (1857) 173–175.
- [15] D. Lind, T. Ward, Automorphisms of solenoids and p -adic entropy, *Ergodic Theory Dynam. Systems* 8 (3) (1988) 411–419.
- [16] J. Peters, Entropy on discrete Abelian groups, *Adv. Math.* 33 (1) (1979) 1–13.
- [17] J. Peters, Entropy of automorphisms on L.C.A. groups, *Pacific J. Math.* 96 (2) (1981) 475–488.
- [18] S. Virili, Entropy for endomorphisms of LCA groups, *Topology Appl.* 159 (9) (2012) 2546–2556.
- [19] S.A. Yuzvinskii, Metric properties of the endomorphisms of compact groups, *Izv. Akad. Nauk SSSR Ser. Mat.* 29 (1965) 1295–1328 (in Russian).
- [20] S. Yuzvinskii, Calculation of the entropy of a group endomorphism, *Sibirsk. Mat. Zh.* 8 (1967) 230–239.