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On the N-wave Equations with $\mathcal{PT}$-symmetry

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Abstract

We study extensions of N-wave systems with $\mathcal{PT}$-symmetry. The types of (nonlocal) reductions leading to integrable equations invariant with respect to $\mathcal{P}$- (spatial reflection) and $\mathcal{T}$- (time reversal) symmetries is described. The corresponding constraints on the fundamental analytic solutions and the scattering data are derived. Based on examples of 3-wave (related to the algebra $sl(3,\mathbb{C})$) and 4-wave (related to the algebra $so(5,\mathbb{C})$) systems, the properties of different types of 1- and 2-soliton solutions are discussed. It is shown that the $\mathcal{PT}$ symmetric 3-wave equations may have regular multi-soliton solutions for some specific choices of their parameters.

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1 Introduction

The N-wave equations [43, 44, 45]

$$i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0, \tag{1}$$

appear to be one of the important classes of integrable nonlinear evolution equations (NLEE) with various applications in physics (nonlinear optics, plasma physics, fluid mechanics, quantum filed theory, to name but a few). Generically, $Q(x, t)$ is assumed to be a $n \times n$ traceless matrix with vanishing diagonal entries and J and I are constant diagonal matrices:

$$J = \operatorname{diag}(a_1, a_2, \ldots, a_n), \qquad I = \operatorname{diag}(b_1, b_2, \ldots, b_n), \qquad \operatorname{tr} J = \operatorname{tr} I = 0. \tag{2}$$

Here also the subscripts x and t stay for differentiation with respect to x and t. Typically $Q(x, t)$ can be chosen to be hermitian $Q(x, t) = Q^\dagger(x, t)$.

The integrability of the N-wave system is based on its Lax representation discovered by Zakharov and Manakov [43, 44]. It can be formulated as the condition $[L, M] = 0$ of two ordinary differential operators L and M

$$
\begin{aligned}
L\psi &\equiv i\frac{\partial \psi}{\partial x} + U(x, t, \lambda)\psi(x, t, \lambda) = 0, & U(x, t, \lambda) &= [J, Q](x, t) - \lambda J, \\
M\psi &\equiv i\frac{\partial \psi}{\partial t} + V(x, t, \lambda)\psi(x, t, \lambda) = 0, & V(x, t, \lambda) &= [I, Q](x, t) - \lambda I
\end{aligned}
\tag{3}
$$

commute identically with respect to the spectral parameter λ. Then the the N-wave equations can be solved using the inverse scattering method (ISM) [43, 44, 12, 26]: the associated spectral problem for the operator L is known as Zakharov-Shabat (ZS) type (matrix) spectral problem.

N-wave interaction models describe a special class of wave-wave interactions which are not sensitive on the physical nature of the waves and bear an universal character. This explains why they find numerous applications in physics and attract attention of the scientific community over the last few decades [5, 13, 17, 18, 19, 20, 22, 23, 31, 33, 34, 35, 42].

In the last decade artificial heterogenic media gained a special interest in Nonlinear Optics. This is related to the fact that such media exhibit a number of properties that cannot be observed in homogeneous media. Such new properties, due to the resonance type of interaction of the media and light are observed in photonic crystals, random lasers, etc (for an up-to-date review, see [46]). An important type of such models, allowing rigorous treatment are the so-called $\mathcal{PT}$-symmetric (Parity-Time) symmetric systems [1, 6, 7, 8, 9, 36, 37, 39].

The initial interest in such systems was motivated by quantum mechanics [8, 36]. In [8] it was shown that quantum systems with a non-hermitian Hamiltonian admit states with real eigenvalues, i.e. the hermiticity of the Hamiltonian is not a necessary condition to have real spectrum. Using such Hamiltonians one can build up new quantum mechanics [8, 9, 36, 37]. Starting point is the fact that in the case of a non-Hermitian Hamiltonian with real spectrum, the modulus of the wave function for the eigenstates is time-independent even in the case of complex potentials.

Historically the first pseudo-hermittian hamiltonian with real spectrum is the $\mathcal{PT}$-symmetric one in [8]. Pseudo-hermiticity here means that the Hamiltonian $\mathcal{H}$ commutes with the operators of spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$: $\mathcal{PTH} = \mathcal{HPT}$. The action of these operators is defined as follows: $\mathcal{P}: x \to -x$ and $\mathcal{T}: t \to -t$.

Due to Wigner's theorem, operators of symmetries can be either linear and unitary or anti-linear and anti-unitary. The type of the operators $\mathcal{P}$ and $\mathcal{T}$ is determined by their action on

the imaginary unit i: a direct calculation shows that $\mathcal{P}$ is a linear and unitary operator. Since applying the spatial reflection twice will reproduce the initial state, so the wave functions $\psi(x,t)$ and $\mathcal{P}^2\psi(x,t)$ can differ only by a phase factor ϕ: $\mathcal{P}^2\psi(x,t) = e^{i\phi}\psi(x,t)$. In order to have the parity of the wave function to be observable quantity, the operator $\mathcal{P}$ must be Hermitian. The only phase factor keeping the operator $\mathcal{P}$ Hermitian is the trivial one: $\mathcal{P}^2\psi(x,t) = \psi(x,t)$. Supposing that the wave fuction is a scalar, this leads to the following action of the operator of spatial reflection on the space of states:

$$\mathcal{P}\psi(x,t) = \psi(-x,t). \tag{4}$$

Applying similar arguments to the (anti-linear and anti-unitary) time reversal operator $\mathcal{T}$ show that it should act on the space of states as follows:

$$\mathcal{T}\psi(x,t) = \psi^*(x,-t). \tag{5}$$

Therefore, the Hamiltonian and the wave function are $\mathcal{P}\mathcal{T}$-symmetric, if

$$\mathcal{H}(x,t) = \mathcal{H}^*(-x,-t), \qquad \psi(x,t) = \psi^*(-x,-t). \tag{6}$$

Integrable systems with $\mathcal{P}\mathcal{T}$-symmetry were studied extensively over the last two decades [14, 15]. Recently, in [2, 25] was proposed a nonlocal integrable equation of nonlinear Schrödinger type with $\mathcal{P}\mathcal{T}$-symmetry, due to the invariance of the so-called self-induced potential $V(x,t) = \psi(x,t)\psi^*(-x,-t)$ under the combined action of parity and time reversal symmetry (6). In the same paper, the 1-soliton solution for this model is derived and it was shown that it develops singularities in finite time. Soon after this, nonlocal $\mathcal{P}\mathcal{T}$-symmetric generalisations are found for the Ablowitz-Ladik model in [3]. All these models are integrable by the Inverse Scattering Method (ISM) [4].

The purpose of the present paper is to study $\mathcal{P}\mathcal{T}$-symmetric extensions of the N-wave equations: in what follows we will outline the $\mathcal{P}\mathcal{T}$-reductions for 3- and 4-wave systems (with Lax operators related to the algebras $sl(3,\mathbb{C})$ and $so(5,\mathbb{C})$ respectively) and will derive the constraints these reductions impose on the scattering data of the Lax operators. We will derive their 1- and 2-soliton solutions.

The paper is organized as follows. The next Section 2 contains preliminaries on the Mikhailov reduction group. In the next Section 3 we give some basic examples of three- and four-wave interactions. Here we also formulate the reduction conditions compatible with the $\mathcal{P}\mathcal{T}$-symmetry. In Section 4 we construct the FAS of the Lax operators and derive the constraints that the $\mathcal{P}\mathcal{T}$-symmetries impose on them, on the scattering matrix $T(\lambda,t)$ and on the Gauss factors of $T(\lambda,t)$. We also discuss the consequences of the reductions on the locations of the discrete eigenvalues of L. In Section 5 we derive and analyze the soliton solutions of 3-wave systems related to the algebra $sl(3,\mathbb{C})$.

2 Preliminaries

Here we outline basic facts about the direct scattering problem for the Lax operator (3). We outline also the main types of local and nonlocal reductions of (3). The reduction problem [38] plays a fundamental role one in the theory of integrable systems. The class of nonlocal reductions includes $\mathcal{P}$- (spatial reflection) and $\mathcal{T}$-(time reversal) symmetries [2, 4, 41].

2.1 Direct scattering transform and Jost solutions

The starting point here are the so-called Jost solutions for the operator $L(\lambda)$, which are defined by their asymptotics (see, e.g. [26] and the references therein):

$$\lim_{x \to -\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x \to \infty} \psi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1} \qquad (7)$$

and the scattering matrix $T(\lambda)$ is defined by $T(\lambda,t) \equiv \psi^{-1}\phi(x,t,\lambda)$. Here we assume that the potential $q(x,t)$ is tending to zero fast enough, when $|x| \to \infty$. We note also, that since $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are taking values in simple Lie algebras ($sl(3)$ and $so(5)$ as regards our examples), then the Jost solutions and the scattering matrix $T(t,\lambda)$ must take values in the corresponding Lie group.

An essential difficulty that comes up here is due to the fact that the Jost solutions $\psi(x,t,\lambda)$ and $\phi(x,t,\lambda)$ exist only if the potential $Q(x,t)$ is on finite support [11]. In this case not only the Jost solutions, but also the scattering matrix $T(\lambda,t)$ (here and hereafter by $\hat{X}$ we will denote the inverse matrix, $\hat{X} \equiv X^{-1}$):

$$T(t,\lambda) = \hat{\psi}(x,t,\lambda)\phi(x,t,\lambda). \qquad (8)$$

become meromorphic functions of λ. More details and more precise statements will be given in the next Section 3.

2.2 Local and Non-local Reductions

An important and systematic tool to construct new integrable NLEE is the so-called reduction group [38]. It will be instructive to start with the local reductions:

$$
\begin{array}{lll}
1) & C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), & C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), \\
2) & C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), & C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \\
3) & C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), & C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), \\
4) & C_4(U(\kappa_2(\lambda))) = U(\lambda), & C_4(V(\kappa_2(\lambda))) = V(\lambda).
\end{array}
\qquad (9)
$$

The consequences of these reductions and the constraints they impose on the FAS and the Gauss factors of the scattering matrix are well known, see [38, 44, 17, 18, 19, 23].

It is important to note, that for the N-wave equations *there are no* reductions compatible with either $\mathcal{P}$- or $\mathcal{T}$-symmetry separately. However the $\mathbb{Z}_2$ reductions

$$
\begin{array}{lll}
1) & C_1(U^\dagger(-x,-t,\kappa_1\lambda^*)) = -U(x,t,\lambda), & C_1(V^\dagger(-x,-t,\kappa_1\lambda^*)) = -V(x,t,\lambda), \\
2) & C_2(U^T(-x,-t,\kappa_2\lambda)) = U(x,t,\lambda), & C_2(V^T(-x,-t,\kappa_2\lambda)) = V(x,t,\lambda), \\
3) & C_3(U^*(-x,-t,\kappa_1\lambda^*)) = U(x,t,\lambda), & C_3(V^*(-x,-t,\kappa_1\lambda^*)) = V(x,t,\lambda), \\
4) & C_4(U(-x,-t,\kappa_2\lambda)) = -U(x,t,\lambda), & C_4(V(-x,-t,\kappa_2\lambda)) = -V(x,t,\lambda),
\end{array}
\qquad (10)
$$

are obviously $\mathcal{PT}$-symmetric [41]. Here $\kappa_i^2 = 1$ and C_i, $i = 1, \ldots, 4$ are involutive automorphisms of the relevant Lie algebra. Below we will present examples of such reductions for 3-wave and 4-wave interaction systems; it is natural to call them *non-local*.

3 Basic examples of 3-wave and 4-wave interactions

3.1 Examples of 3-wave Interactions

We consider a Lax pair of the form (3) related to $sl(3, \mathbb{C})$ algebra: in the typical representation the potential matrix adopts the form

$$Q = \begin{pmatrix} 0 & q_1(x,t) & q_3(x,t) \\ p_1(x,t) & 0 & q_2(x,t) \\ p_3(x,t) & p_2(x,t) & 0 \end{pmatrix}, \qquad \begin{aligned} J &= \text{diag}\,(a_1, a_2, a_3), \\ I &= \text{diag}\,(b_1, b_2, b_3). \end{aligned} \tag{11}$$

We presume that the potential matrices $Q(x,t)$, I and J are traceless (i.e. the Lax operators take values in the algebra $sl(3, \mathbb{C})$) and the eigenvalues of I and J are ordered as follows: $a_1 > a_2 > a_3$, $b_1 > b_2 > b_3$ ($a_1 + a_2 + a_3 = 0$ and $b_1 + b_2 + b_3 = 0$).

The compatibility condition (12) produces the generic system of 3-wave resonant interaction equations

$$\begin{aligned}
i(a_1 - a_2)q_{1,t} - i(b_1 - b_2)q_{1,x} + \kappa q_3(x,t)p_2(x,t) &= 0, \\
i(a_2 - a_3)q_{2,t} - i(b_2 - b_3)q_{2,x} + \kappa q_3(x,t)p_1(x,t) &= 0, \\
i(a_1 - a_3)q_{3,t} - i(b_1 - b_3)q_{3,x} + \kappa q_1(x,t)q_2(x,t) &= 0,
\end{aligned} \tag{12}$$

where the interaction constant κ is

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2). \tag{13}$$

In addition there are 3 "conjugated" equations, formally obtained from (12) by replacing $p_k(x,t)$ by $q_k^*(x,t)$. The standard reduction $p_k(x,t) = q_k^*(x,t)$ ($k = 1 \div 3$) leads to the canonical form of the 3-wave resonant interaction equations [43, 44].

The system (12) is Hamiltonian [43, 44] and possess a hierarchy of pairwise compatible Hamiltonian structures [24, 26]. The (canonical) Hamiltonian of (1) is given by [43, 44, 19, 33, 35]:

$$H_{3-\text{w}} = \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\sum_{k=1}^{3} v_k \left(q_k \frac{\partial p_k}{\partial x} - p_k \frac{\partial q_k}{\partial x} \right) + \kappa(q_3 p_1 p_2 - p_3 q_1 q_2) \right). \tag{14}$$

Here v_k are the group velocities of (12):

$$v_1 = \frac{b_1 - b_2}{a_1 - a_2}, \qquad v_2 = \frac{b_2 - b_3}{a_2 - a_3}, \qquad v_3 = \frac{b_1 - b_3}{a_1 - a_3}.$$

Example 1. *If we impose a reduction of type 1) from (10) with $C_1 = \text{id}$, we will get $a_k^* = \epsilon a_k$, $b_k^* = \epsilon b_k$ and $p_k(x,t) = -\epsilon q_k^*(-x,-t)$ ($k = 1 \div 3$). Thus the generic 3-wave system (12) will reduce to following nonlocal one*

$$\begin{aligned}
i(a_1 - a_2)q_{1,t} - i(b_1 - b_2)q_{1,x} - \epsilon\kappa q_3(x,t)q_2^*(-x,-t) &= 0, \\
i(a_2 - a_3)q_{2,t} - i(b_2 - b_3)q_{2,x} - \epsilon\kappa q_3(x,t)q_1^*(-x,-t) &= 0, \\
i(a_1 - a_3)q_{3,t} - i(b_1 - b_3)q_{3,x} + \kappa q_1(x,t)q_2(x,t) &= 0.
\end{aligned} \tag{15}$$

Example 2. *If we impose a reduction of type 1) from (10) with*

$$C_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

will give

$$a_3^* = \epsilon a_1, \qquad a_2^* = \epsilon a_2, \qquad b_3^* = \epsilon b_1, \qquad b_2^* = \epsilon b_2, \tag{16}$$
$$p_1(x,t) = -\epsilon q_2^*(-x,-t), \qquad p_2(x,t) = -\epsilon q_1^*(-x,-t), \qquad p_3(x,t) = -\epsilon q_3^*(-x,-t).$$

As a result, one can get the following nonlocal 3-wave system:

$$i(a_1 - a_2)q_{1,t} - i(b_1 - b_2)q_{1,x} - \kappa q_3(x,t)q_1^*(-x,-t) = 0,$$
$$i(a_2 - a_3)q_{2,t} - i(b_2 - b_3)q_{2,x} - \kappa q_3(x,t)q_2(-x,-t) = 0, \tag{17}$$
$$i(a_1 - a_3)q_{3,t} - i(b_1 - b_3)q_{3,x} + \kappa q_1(x,t)q_2(x,t) = 0.$$

<div align="right">□</div>

3.2   Examples of 4-wave interactions

Another important example are the 4-wave equations. Their Lax operators are related to the algebra $so(5, \mathbb{C})$. In the typical representation of $so(5, \mathbb{C})$ the (generic) potential $Q(x,t)$ takes the form:

$$Q(x,t) = \begin{pmatrix} 0 & q_1(x,t) & q_3(x,t) & q_4(x,t) & 0 \\ p_1(x,t) & 0 & q_2(x,t) & 0 & q_4(x,t) \\ p_3(x,t) & p_2(x,t) & 0 & q_2(x,t) & -q_3(x,t) \\ p_1(x,t) & 0 & p_2(x,t) & 0 & q_1(x,t) \\ 0 & p_4(x,t) & -p_3(x,t) & p_1(x,t) & 0 \end{pmatrix}; \tag{18}$$

and the elements of the Cartan subalgebra J and I can be represented as diagonal matrices

$$J = \text{diag}\,(a_1, a_2, 0, -a_2, -a_1), \qquad I = \text{diag}\,(b_1, b_2, 0, -b_2, -b_1). \tag{19}$$

The zero-curvature representation (1) will lead to the following system of 4 waves [19]:

$$i(a_1 - a_2)q_{1,t} - i(b_1 - b_2)q_{1,x} + 2\kappa q_3(x,t)p_2(x,t) = 0,$$
$$ia_2 q_{2,t} - ib_2 q_{2,x} + \kappa[q_4(x,t)p_3(x,t) + q_3(x,t)p_1(x,t)] = 0,$$
$$ia_1 q_{3,t} - ib_1 q_{3,x} + \kappa[q_4(x,t)p_2(x,t) - q_1(x,t)q_2(x,t)] = 0, \tag{20}$$
$$i(a_1 + a_2)q_{4,t} - i(b_1 + b_2)q_{4,x} - 2\kappa q_3(x,t)q_2(x,t) = 0,$$

where the interaction constant reads $\kappa = a_1 b_2 - a_2 b_1$. In addition we have a set of 4 "conjugated" equations, formally obtained from (20) by replacing $p_k(x,t)$ by $q_k^*(x,t)$. The standard reduction $p_k(x,t) = q_k^*(x,t)$ ($k = 1 \div 4$) leads to the canonical form of the 4-wave resonant interaction equations [44, 18, 19]. Here we have a local reduction of type 1).

Example 3. *Let us impose a reduction of type 3) from (10) with*

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{21}$$

This gives $C_1 J^* C_1 = J$ and $C_1 Q^*(-x, -t) C_1^{-1} = Q$; or in components we have $a_2 = a_1^*$ and

$$p_1(x,t) = q_1^*(-x,-t), \qquad q_2(x,t) = q_3^*(-x,-t), \qquad p_2(x,t) = p_3^*(-x,-t),$$
$$q_4(x,t) = -q_4^*(-x,-t), \qquad p_4(x,t) = -p_4^*(-x,-t). \tag{22}$$

Thus we find the following nonlocal analogue of the classical 4-wave equations

$$i(a_1 - a_2)q_{1,t} - i(b_1 - b_2)q_{1,x} + \kappa q_3(x,t)p_3^*(-x,-t) = 0,$$
$$ia_2 q_{2,t} - ib_2 q_{2,x} + \kappa[q_4(x,t)p_2^*(-x,-t) + q_1^*(-x,-t)q_1^*(-x,-t)] = 0,$$
$$ia_2 p_{2,t} - ib_2 p_{2,x} - \kappa[p_4(x,t)q_2^*(-x,-t) + q_1(x,t)p_2^*(-x,-t)] = 0, \tag{23}$$
$$i(a_1 + a_2)q_{4,t} - i(b_1 + b_2)q_{4,x} - \kappa q_2(x,t)q_2^*(-x,-t) = 0.$$
$$i(a_1 + a_2)p_{4,t} - i(b_1 + b_2)p_{4,x} + \kappa p_2(x,t)p_2^*(-x,-t) = 0.$$

Example 4. *If we impose a reduction of type 3) from (10) with $C_3 = \mathrm{id}$ is equivalent to imposing a reduction of type 1) with $C_1 = w_0$, where w_0 is the (inner) automorphism which maps the highest-weight vectors into the lowest-weight vectors of the algebra $so(5, \mathbb{C})$ (see e.g. [18]). This will lead to: $a_{1,2}^* = -\epsilon a_{1,2}$, $b_{1,2}^* = -\epsilon b_{1,2}$, $q_k(x,t) = \epsilon q_k^*(-x,-t)$ and $p_k(x,t) = \epsilon p_k^*(-x,-t)$ ($k = 1 \div 4$). This will lead to a generic 4-wave system (for 8 real-valued functions $q_k(x,t)$ and $p_k(x,t)$, $k = 1 \div 4$) of the type (20).*

Example 5. *If we impose a reduction of type 1) from (10) with C_1 given by (21) above we will get: $a_2^* = a_1$, $b_2^* = b_1$ and*

$$q_1(x,t) = q_1^*(-x,-t), \qquad p_2(x,t) = q_3^*(-x,-t), \qquad p_3(x,t) = q_2^*(-x,-t),$$
$$p_1(x,t) = p_1^*(-x,-t), \qquad p_4(x,t) = -q_4^*(-x,-t). \tag{24}$$

As a result, one gets the 5-wave system (containing 2 "real" (p_1 and q_1) and 3 complex (q_2, q_3 and q_4) functions):

$$i(a_1 - a_2)q_{1,t} - i(b_1 - b_2)q_{1,x} + \kappa q_3(x,t)q_3^*(-x,-t) = 0,$$
$$i(a_1 - a_2)p_{1,t} - i(b_1 - b_2)p_{1,x} - \kappa q_2(x,t)q_2^*(-x,-t) = 0,$$
$$ia_2 q_{2,t} - ib_2 q_{2,x} + \kappa[q_4(x,t)q_2^*(-x,-t) + q_3(x,t)p_1(x,t)] = 0, \tag{25}$$
$$ia_1 q_{3,t} - ib_1 q_{3,x} + \kappa[q_3^*(-x,-t)q_4(x,t) - q_1(x,t)q_2(x,t)] = 0,$$
$$i(a_1 + a_2)q_{4,t} - i(b_1 + b_2)q_{4,x} - \kappa q_2(x,t)q_3(x,t) = 0,$$

This is an example of reduction that imposes a symmetry condition directly on the solution profiles of the "real" solutions. □

All these examples and their reductions allow a Hamiltonian formulation. The reduction conditions will impose restrictions on the corresponding Hamiltonians, symplectic forms and integrals of motion [18].

4   The FAS and their reductions

In constructing the FAS of the relevant Lax operator L we have to consider separately the cases of real and complex-valued J.

4.1 The case of real $J \in sl(3)$

The important notion of FAS for generic $n \times n$ Zakharov-Shabat system with real-valued J (1) was introduced by Shabat [40]. If J has real eigenvalues then the Jost solutions of L exist for all real λ. They can be viewed as the solutions of a system of Volterra-type integral equations. Indeed, if we introduce

$$Y_+(x, t, \lambda) = \psi(x, t, \lambda)e^{iJ\lambda x}, \qquad Y_-(x, t, \lambda) = \phi(x, t, \lambda)e^{iJ\lambda x}, \tag{26}$$

then $Y_\pm(x, t, \lambda)$ must satisfy:

$$Y_{\pm;jk}(x, t, \lambda) = \delta_{jk} + i \int_{\pm\infty}^{x} dy \; e^{-i\lambda(a_j - a_k)(x-y)} \left([J, Q(y, t)]Y_\pm(x, t, \lambda)\right)_{jk}. \tag{27}$$

We will assume that the eigenvalues of J are ordered, i.e. $a_1 > a_2 > \cdots > a_n$ and also $\operatorname{tr} J = 0$.

The Volterra equations (26) always have solution for real λ. Analytic extension for $\lambda \in \mathbb{C}_+$ (resp. for $\lambda \in \mathbb{C}_-$) is possible only for the first column of $Y_-(x, t, \lambda)$ and for the last column of $Y_+(x, t, \lambda)$ (resp. for the last column of $Y_-(x, t, \lambda)$ and for the first column of $Y_+(x, t, \lambda)$. Shabat [40] proposed to change the integral equations (26) and consider two sets of integral equations:

$$\xi_{jk}^+(x, t, \lambda) = \delta_{jk} + i \int_{\epsilon_{jk}\infty}^{x} dy \; e^{-i\lambda(a_j - a_k)(x-y)} \left([J, Q(y, t)]\xi^+(y, t, \lambda)\right)_{jk}. \tag{28}$$

and

$$\xi_{jk}^-(x, t, \lambda) = \delta_{jk} + i \int_{-\eta_{jk}\infty}^{x} dy \; e^{-i\lambda(a_j - a_k)(x-y)} \left([J, Q(y, t)]\xi^-(y, t, \lambda)\right)_{jk}. \tag{29}$$

where

$$\epsilon_{jk} = \begin{cases} 1 & \text{for } j < k, \\ -1 & \text{for } j \geq k, \end{cases} \qquad \eta_{jk} = \begin{cases} -1 & \text{for } j \leq k, \\ 1 & \text{for } j > k, \end{cases}$$

Then, in [40] he proved that the equations (28) (resp. (29)) possess solutions $\xi^+(x, t, \lambda)$ (resp. $\xi^-(x, t, \lambda)$) which allow analytic extension for $\lambda \in \mathbb{C}_+$ (resp. for $\lambda \in \mathbb{C}_-$). The solutions $\xi^\pm(x, t, \lambda)$ can be viewed also as solutions to a Riemann-Hilbert problem (RHP) in multiplicative form [17, 40, 10, 44]

$$\xi^+(x, t, \lambda) = \xi^-(x, t, \lambda)G(x, t, \lambda), \qquad G(x, t, \lambda) = e^{i\lambda Jx}G_0(t, \lambda)e^{-i\lambda Jx} \tag{30}$$

with canonical normalization, i.e. $\lim_{\lambda \to \infty} \xi^\pm(x, t, \lambda) = \mathbb{1}$.

If we denote by $\chi^\pm(x, t, \lambda) = \xi^\pm(x, t, \lambda)e^{-i\lambda Jx}$ then $\chi^\pm(x, t, \lambda)$ will be the FAS of L. Besides it was established [16, 44], that the FAS are related to the Jost solutions by

$$\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda)S^\pm(t, \lambda), \qquad \chi^\pm(x, t, \lambda) = \psi(x, t, \lambda)T^\mp(t, \lambda)D^\pm(\lambda), \tag{31}$$

where $S^\pm$, $T^\pm$ and $D^\pm$ are the Gauss factors of the scattering matrix:

$$T(t, \lambda) = T^-(t, \lambda)D^+(\lambda)\hat{S}^+(t, \lambda) = T^+(t, \lambda)D^-(\lambda)\hat{S}^-(t, \lambda). \tag{32}$$

In other words $T^-(t, \lambda)$, $S^-(t, \lambda)$ (resp. $T^+(t, \lambda)$, $S^+(t, \lambda)$) are lower- (resp. upper-) triangular matrices whose diagonal elements are equal to 1; the matrices $D^+(\lambda)$ and $D^-(\lambda)$ are diagonal and allow analytic extension for $\lambda \in \mathbb{C}_\pm$ respectively. Then the sewing matrix $G_0(t, \lambda) = \hat{S}^-(t, \lambda)S^+(t, \lambda)$. Along with $\chi^\pm(x, t, \lambda)$ one may use also $\tilde{\chi}^\pm(x, t, \lambda)$ defined by:

$$\tilde{\chi}^\pm(x, t, \lambda) = \phi(x, t, \lambda)S^\pm(t, \lambda)\hat{D}^\pm(\lambda), \qquad \tilde{\chi}^\pm(x, t, \lambda) = \psi(x, t, \lambda)T^\mp(t, \lambda). \tag{33}$$

i.e. $\tilde{\chi}^\pm(x, t, \lambda) = \chi^\pm(x, t, \lambda)\hat{D}^\pm(\lambda)$.

4.2 The case of complex-valued $J \in sl(3)$

We mentioned above the problem, that the Jost solutions $\psi(x, t, \lambda)$ and $\phi(x, t, \lambda)$ exist only if the potential $Q(x, t)$ is on finite support [11]. In this case not only the Jost solutions, but also the scattering matrix $T(\lambda, t)$ and its factors of Gauss decompositions (8) become meromorphic functions of λ. However, following [11] and [27] one can construct and prove the existence of fundamental analytic solutions (FAS) $\chi_\nu(x, t, \lambda)$ of L. The construction is based on the set of integral equations:

$$\xi_{\nu;jk}(x, t, \lambda) = \delta_{jk} + i \int_{\varepsilon_{\nu;jk}\infty}^{x} dy \, e^{-i\lambda(a_j - a_k)(x-y)} \left([J, Q(y, t)] \xi_\nu(y, t, \lambda) \right)_{jk}, \tag{34}$$

where the appropriately chosen $\varepsilon_{\nu;jk}$ take values ± 1. It is obvious that for any choice of $\varepsilon_{\nu;jk}$ the solution of the equations (34) will be a solution also of the Lax operator.

Skipping the details, we will briefly outline the idea of constructing the FAS. The first step is to determine the regions (sectors) of the analyticity of $\xi_\nu(x, t, \lambda)$. In each of these regions $\lambda \in \Omega_\nu$ the equations (34) will possess a solution provided all the exponentials $e^{-i\lambda(a_j - a_k)(x-y)}$ are decaying for $x, y \to \pm\infty$. The regions Ω_ν will be divided by lines l_ν, on which the exponentials $e^{-i\lambda(a_j - a_k)(x-y)}$ oscillate. In other words the lines l_ν consists of those points of λ, for which

$$\text{Im}\,\lambda(a_j - a_k) = 0. \tag{35}$$

Therefore we have to solve the set of algebraic equations (35) for all possible choices of $j \neq k$; the signs $\varepsilon_{\nu;kk}$ can be chosen arbitrarily. The solution of eq. (35) is therefore provided by:

$$\arg \lambda = -\arg(a_j - a_k) = -\arctan \frac{\text{Im}(a_j - a_k)}{\text{Re}\,(a_j - a_k)} = -\beta_{jk}; \tag{36}$$

in other words, each line l_ν is in fact a ray closing angle $-\beta_{jk}$ with the real λ-axis.

Depending on the size of the matrices $n \times n$ and on the choice for the eigenvalues a_k of J we may have various sets of rays describing the spectrum of L. Note, that the number of rays is always even $2h$. Indeed, if l_ν is one the rays closing angle $-\beta_{jk}$ with the real λ-axis, then the ray $l_{\nu+h}$ closing an angle $\pi - \beta_{jk}$ will also satisfy the equation (36) and will be a part of the continuous spectrum of L. Below in Figure 1 we describe the spectrum of 3×3 L-operators with special choices for J, compatible with the reduction.

Remark 1. *We will take special care that the operators L and M have the same (continuous) spectra. Obviously, we can approach the operator M in the same way as L and construct its FAS. Obviously the FAS of M will be analytic in sectors determined by the eigenvalues of I. In order that both J and I determine the same sectors of analyticity Ω_ν they should have (up to a common constant) the same sets of eigenvalues. However, if $I = \text{const}\, J$ then the interaction constant $\kappa = 0$. Therefore below we choose $I = J^*$ which ensures the isospectrality of L and M and ensures also that interaction constant $\kappa \neq 0$.*

The next step consists in establishing the relations between the Jost solutions and the FAS $\chi_\nu(x, t, \lambda)$ in each of the sectors Ω_ν.

In order to be more specific we now consider 3×3 Lax operator and impose on it the reduction (10), type 1). It can be done in two ways:

$$\begin{aligned}
&\text{A)} && s_0 U^\dagger(-x, -t, -\lambda^*) s_0^{-1} = -U(x, t, \lambda), \\
&\text{B)} && s_0 U^\dagger(-x, -t, \lambda^*) s_0^{-1} = -U(x, t, \lambda).
\end{aligned} \tag{37}$$

	Ray	l_0	l_1	l_2	l_3	l_4	l_5
case A)	Root	$-e_1 + e_2$	$e_2 - e_3$	$e_1 - e_3$	$e_1 - e_2$	$-e_2 + e_3$	$-e_1 + e_3$
	Angle	β_{12}	$\pi/2$	$\pi - \beta_{12}$	$\pi + \beta_{12}$	$-\pi/2$	$-\beta_{12}$

	Ray	l_0	l_1	l_2	l_3	l_4	l_5
case B)	Root	$e_1 - e_2$	$e_1 - e_3$	$e_2 - e_3$	$-e_1 + e_2$	$-e_1 + e_3$	$-e_2 + e_3$
	Angle	$-\gamma_{12}$	0	γ_{12}	$\pi - \gamma_{12}$	π	$\pi + \gamma_{12}$

Table 1: The correspondence between the rays l_ν, roots α_ν and the angles that l_ν closes with the real axis for case A) (where $a_1 = j_0 + ij_1$ and $\beta_{12} = \arctan \frac{j_1}{3j_0}$) and case B) (where $a_1 = j_0 + ij_1$ and $\gamma_{12} = \arctan \frac{3j_1}{j_0}$).

where $s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. An immediate consequence of the reductions (37) is

$$
\begin{aligned}
\text{A)} \quad & s_0 J^\dagger s_0^{-1} = J, & s_0 Q^\dagger(-x, -t) s_0^{-1} = Q(x, t), \\
\text{B)} \quad & s_0 J^\dagger s_0^{-1} = -J, & s_0 Q^\dagger(-x, -t) s_0^{-1} = -Q(x, t),
\end{aligned}
\tag{38}
$$

We will also need two sets of FAS for each sector, namely:

$$
\begin{aligned}
\chi_\nu(x, t, \lambda) &= \phi(x, t, \lambda) S_\nu^+(\lambda, t) = \psi(x, t, \lambda) T_\nu^-(\lambda, t) D_\nu^+(\lambda), & \lambda \in l_\nu^+, \\
\chi_\nu(x, t, \lambda) &= \phi(x, t, \lambda) S_{\nu+1}^-(\lambda, t) = \psi(x, t, \lambda) T_{\nu+1}^+(\lambda, t) D_{\nu+1}^-(\lambda), & \lambda \in l_{\nu+1}^-,
\end{aligned}
\tag{39}
$$

and

$$
\begin{aligned}
\tilde{\chi}_\nu(x, t, \lambda) &= \phi(x, t, \lambda) S_\nu^+(\lambda, t) \hat{D}_\nu^+(\lambda) = \psi(x, t, \lambda) T_\nu^-(\lambda, t), & \lambda \in l_\nu^+, \\
\tilde{\chi}_\nu(x, t, \lambda) &= \phi(x, t, \lambda) S_{\nu+1}^-(\lambda, t) \hat{D}_{\nu+1}^-(\lambda) = \psi(x, t, \lambda) T_{\nu+1}^+(\lambda, t), & \lambda \in l_{\nu+1}^-,
\end{aligned}
\tag{40}
$$

where by $\lambda \in l_\nu^\pm$ we mean that $\lambda = \lambda' e^{\pm i0}$, where $\lambda' \in l_\nu$. In order to explain the notations above we use the fact that [27] to each of the rays l_ν one can relate a root α_ν of the algebra $sl(3)$ and an $sl_\nu(2)$ subalgebra generated by E_α, $E_{-\alpha}$ and H_α, see Table 1 below. Then the corresponding scattering matrix $T_\nu(\lambda)$ evaluated on l_ν takes values in $sl_\nu(2)$ and the Gauss factors $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$ and $D_\nu^\pm(\lambda)$ are given by:

$$
S_\nu^\pm(\lambda) = \exp\left(s_{\nu, \alpha_\nu}^\pm(\lambda) E_{\pm\alpha_\nu}\right), \quad T_\nu^\pm(\lambda) = \exp\left(t_{\nu, \alpha_\nu}^\pm(\lambda) E_{\pm\alpha_\nu}\right), \quad D_\nu^\pm(\lambda) = \exp\left(d_{\nu, \alpha_\nu}^\pm(\lambda) H_{\pm\alpha_\nu}\right),
\tag{41}
$$

It is well known that the local reductions impose constraints on each of the Jost solutions $\psi(x, t, \lambda)$ and $\phi(x, t, \lambda)$ separately.

The nonlocal reductions we are studying relate the two Jost solutions as follows:

$$
\text{A)} \quad s_0 \psi^\dagger(-x, -t, -\lambda^*) s_0^{-1} = \hat{\phi}(x, t, \lambda), \qquad \text{B)} \quad s_0 \psi^\dagger(-x, -t, \lambda^*) s_0^{-1} = \hat{\phi}(x, t, \lambda). \tag{42}
$$

Our next task is to derive the constraints that the nonlocal reductions impose on the FAS. Before to go into this we find how the mappings $\lambda \to -\lambda^*$ and $\lambda \to \lambda^*$ affect the the lines l_ν and the sectors Ω_ν:

$$
\text{A)} \quad \lambda \leftrightarrow -\lambda^* \quad l_\nu^+ \leftrightarrow l_{4-\nu}^- \quad \Omega_\nu \leftrightarrow \Omega_{3-\nu}, \quad \text{B)} \quad \lambda \leftrightarrow \lambda^* \quad l_\nu^+ \leftrightarrow l_{8-\nu}^- \quad \Omega_\nu \leftrightarrow \Omega_{7-\nu},
\tag{43}
$$

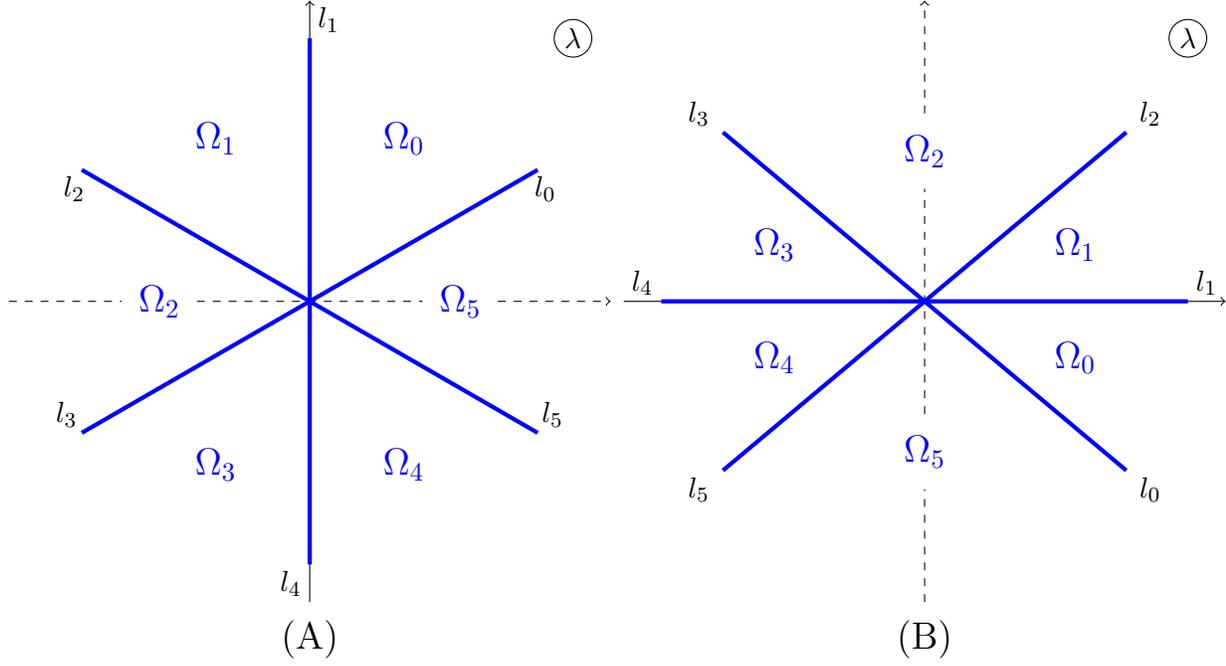where $4 - \nu$ and $3 - \nu$ in case A) and $8 - \nu$ and $7 - \nu$ in case B) are evaluated modulo 6.

Figure 1: The continuous spectra of a 3×3 Lax operator with complex-valued J: case A – left panel; case B – right panel subject to a reduction of type A)and B) in (37).

Next we compare the relations (7) with (39) and (40) fixing up λ to take values close to the rays l_ν. This allows us to derive the constraints on the scattering data.

In case A we find that:

$$s_0(\chi_\nu(-x, -t, -\lambda^*))^\dagger s_0^{-1} = \widehat{\widetilde{\chi}}_{3-\nu}(x, t, \lambda), \tag{44}$$

where $3 - \nu$ is evaluated modulo 6. From these relations there follows the constraints on the Gauss factors of the scattering matrix. Skipping the tedious details we obtain;

$$s_0(S_\nu^+(-\lambda^*))^\dagger s_0^{-1} = \hat{T}_{4-\nu}^+(\lambda), \quad s_0(S_\nu^-(-\lambda^*))^\dagger s_0^{-1} = \hat{T}_{4-\nu}^-(\lambda),$$
$$s_0(D_\nu^+(-\lambda^*))^\dagger s_0^{-1} = D_{4-\nu}^+(\lambda). \tag{45}$$

In case B we have

$$s_0(\chi_\nu(-x, -t, -\lambda^*))^\dagger s_0^{-1} = \widehat{\widetilde{\chi}}_{7-\nu}(x, t, \lambda), \tag{46}$$

where $7 - \nu$ is evaluated modulo 6. The constraints on the Gauss factors of the scattering matrix now read;

$$s_0(S_\nu^+(\lambda^*))^\dagger s_0^{-1} = \hat{T}_{8-\nu}^+(\lambda), \quad s_0(S_\nu^-(\lambda^*))^\dagger s_0^{-1} = \hat{T}_{8-\nu}^-(\lambda), \quad s_0(D_\nu^+(\lambda^*))^\dagger s_0^{-1} = D_{8-\nu}^+(\lambda), \tag{47}$$

and again $7 - \nu$ and $8 - \nu$ are evaluated modulo 6.

Let us also note, that the zeroes and the poles of the functions $D_\nu^\pm(\lambda)$ determine the discrete eigenvalues of L. From eq. (45) it follows that if λ_k is a discrete eigenvalue, then $-\lambda_k^*$ will also be an eigenvalue of L in case A. In case B eq. (47) ensures that if λ_k is a discrete eigenvalue, then λ_k^* will also be an eigenvalue of L.

Remark 2. *The same pairing of the discrete eigenvalues $\lambda_k, -\lambda_k^*$ like in our case A was discovered in [2] for the spectrum of the $\mathcal{P}$-symmetric 2×2 Zakharov-Shabat system. The substantial difference with our case is that in [2] both λ_k and $-\lambda_k^*$ lie in the same analyticity region. From Figure 1A it is obvious the two eigenvalues always must lie in different analyticity regions.*
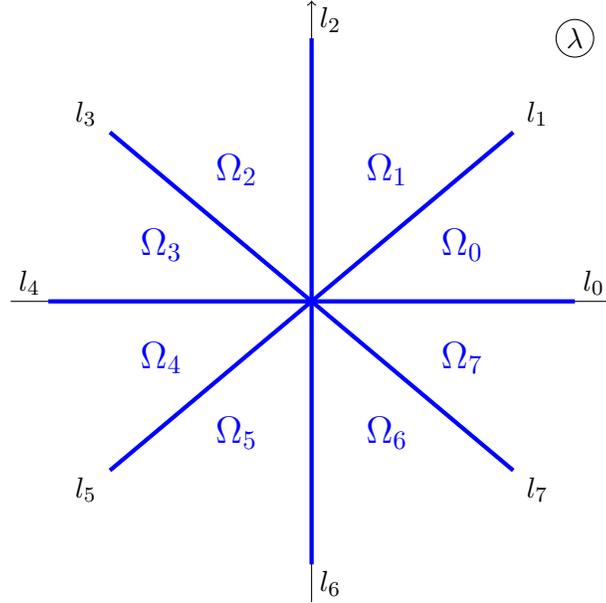
11

Figure 2: The continuous spectra of 5×5 Lax operator with complex-valued J subject to a reduction of type 1) or 3) in (10)

Ray	l_0	l_1	l_2	l_3	l_4	l_5	l_6	l_7
Root	$e_1 + e_2$	e_2	$e_1 - e_2$	e_1	$-(e_1 + e_2)$	$-e_2$	$-(e_1 - e_2)$	$-e_1$
Angle	0	φ_1	$\pi/2$	$\pi - \varphi_1$	π	$\pi + \varphi_1$	$-\pi/2$	$-\varphi_1$

Table 2: To each of the rays l_ν of the continuous spectrum of the $so(5)$ Lax determined by (48) one can relate a root of the $so(5)$ algebra and the angle that it closes with the real axis, as shown above (here $\varphi_1 = \arg a_1$).

4.3   The case of complex-valued $J \in so(5)$

The root system Δ of $so(5)$ is a special set of 8 vectors in two-dimensional Euclidean space $\mathbb{E}^2$. If we introduce in $\mathbb{E}^2$ the orthonormal basis e_1, e_2 then $\Delta \equiv \{e_1 \pm e_2, -e_1 \pm e_2, \pm e_1, \pm e_2\}$. Any element J from the Cartan subalgebra is a linear combination of two basic elements $J = a_1 H_{e_1} + a_2 H_{e_2}$. We can associate with J a vector $\vec{J} = a_1 e_1 + a_2 e_2$. For more details on the root system of $so(5)$ and other simple Lie algebras [30]. The 4-wave system with real-valued J and their soliton solutions have been analyzed in [44, 18, 19, 32, 29].

Therefore here we will consider mostly the case of complex-valued J; more specifically we will take $a_1 = a_2^*$. As in the previous Subsection we will take care that the M operators has the same continuous spectrum as L, which means that $I = a_2 H_{e_1} + a_1 H_{e_2} = J^*$.

The construction of the FAS is quite analogous to the one for $sl(3)$ already describe above, see [27, 28]. Skipping the details we note that the continuous spectrum of L is located on the lines

$$\operatorname{Im} \lambda(\alpha, \vec{J}) = 0, \qquad \alpha \in \Delta. \tag{48}$$

This is a set of four lines intersecting at the origin, see Figure 4.3. With each of the lines l_ν one can relate a root α (see Table 2) for which $\operatorname{Im} \lambda(\alpha, \vec{J}) = 0$ for $\lambda \in l_\nu$.

Quite analogously to the previous case, one can prove as particular case of the results in

[27, 28] that:

$$\chi_\nu(x,t,\lambda) = \phi(x,t,\lambda)S_\nu^+(\lambda,t) = \psi(x,t,\lambda)T_\nu^-(\lambda,t)D_\nu^+(\lambda), \qquad \lambda \in l_\nu^+,$$
$$\chi_\nu(x,t,\lambda) = \phi(x,t,\lambda)S_{\nu+1}^-(\lambda,t) = \psi(x,t,\lambda)T_{\nu+1}^+(\lambda,t)D_{\nu+1}^-(\lambda), \qquad \lambda \in l_{\nu+1}^-, \tag{49}$$

where $\nu = 0, \ldots 7$ and

$$\tilde{\chi}_\nu(x,t,\lambda) = \phi(x,t,\lambda)S_\nu^+(\lambda,t)\hat{D}_\nu^+(\lambda) = \psi(x,t,\lambda)T_\nu^-(\lambda,t), \qquad \lambda \in l_\nu^+,$$
$$\tilde{\chi}_\nu(x,t,\lambda) = \phi(x,t,\lambda)S_{\nu+1}^-(\lambda,t)\hat{D}_{\nu+1}^-(\lambda) = \psi(x,t,\lambda)T_{\nu+1}^+(\lambda,t), \qquad \lambda \in l_{\nu+1}^-, \tag{50}$$

The scattering matrix $T_\nu(\lambda,t)$ for $\lambda \in l_\nu$ takes values in the $SL_\nu(2)$ subgroup of $SO(5)$ generated by E_{α_ν}, $E_{-\alpha_\nu}$ and H_{α_ν}. By $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$ and $D_\nu^\pm(\lambda)$ we have denoted the Gauss factors of $T_\nu(\lambda,t)$ (see eq. (32) above) and

$$S_\nu^\pm(\lambda) = \exp\left(s_{\nu,\alpha_\nu}^\pm(\lambda)E_{\pm\alpha_\nu}\right), \quad T_\nu^\pm(\lambda) = \exp\left(t_{\nu,\alpha_\nu}^\pm(\lambda)E_{\pm\alpha_\nu}\right), \quad D_\nu^\pm(\lambda) = \exp\left(d_{\nu,\alpha_\nu}^\pm(\lambda)H_{\pm\alpha_\nu}\right), \tag{51}$$

By $\lambda \in l_\nu^\pm$ above we mean that $\lambda = \lambda' e^{\pm i0}$, where $\lambda' \in l_\nu$.

The solutions $\xi^\pm(x,t,\lambda)$ can be viewed also as solutions to a RHP

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad G(x,t,\lambda) = e^{i\lambda Jx}G_0(t,\lambda)e^{-i\lambda Jx} \tag{52}$$

where $\nu = 0, \ldots 7$ with canonical normalization, i,e, $\lim_{\lambda \to \infty} \xi^\pm(x,t,\lambda) = \mathbb{1}$.

Reducing the inverse scattering problem to a RHP first of all allows one to use the dressing method for constructing the soliton solutions of the relevant 4-wave equations.

5 Dressing method and soliton solutions

The dressing factor relates the FAS of two Lax operators:

$$\chi_\nu^{(1)}(x,t,\lambda) = g(x,t,\lambda)\chi_\nu^{(0)}(x,t,\lambda), \tag{53}$$

where $\chi_\nu^{(1)}(x,t,\lambda)$ and $\chi_\nu^{(0)}(x,t,\lambda)$ are Lax operators with potentials Q_1 and Q_0. We assume that $Q_0 = 0$, so the corresponding 'naked' solution is $\chi_\nu^{(0)}(x,t,\lambda) = e^{-i\lambda Jx}$. Obviously the dressing factor must satisfy:

$$ig_x + [J,Q]g - g[J,Q_0] - \lambda[J,g] = 0, \tag{54}$$

Besides, $q(x,t,\lambda)$ must satisfy all the reduction conditions imposed on the Lax operator:

$$\begin{aligned} \text{A)} \qquad & s_0 g^\dagger(-x,-t,-\lambda^*)s_0^{-1} = \hat{g}(x,t,\lambda), \\ \text{B)} \qquad & s_0 g^\dagger(-x,-t,\lambda^*)s_0^{-1} = \hat{g}(x,t,\lambda). \end{aligned} \tag{55}$$

We will present below the 1- and 2- soliton solutions for nonlocal reductions of class A). The solutions of class B) are derived analogously.

More about the dressing method the reader can find in [44, 21, 32, 40].

5.1 One soliton solution

Assume that for $\mathfrak{g} \simeq sl(3, \mathbb{C})$ the dressing factor $g(x, t, \lambda)$ and its inverse $\hat{g}(x, t, \lambda)$ have simple poles at $\lambda = \lambda_1^-$ and $\lambda = \lambda_1^+$ respectively, i.e.

$$g(x, t, \lambda) = 1 + \frac{\lambda_1^- - \lambda_1^+}{\lambda - \lambda_1^-} P_1(x, t), \qquad \hat{g}(x, t, \lambda) = 1 + \frac{\lambda_1^+ - \lambda_1^-}{\lambda - \lambda_1^+} P_1(x, t), \tag{56}$$

where $P_1(x, t)$ is the rank 1 projector:

$$P_1(x, t) = \frac{|n_0(x, t)\rangle \langle m_0(x, t)|}{\langle m_0(x, t)|n_0(x, t)\rangle}, \qquad P_1^2(x, t) = P_1(x, t). \tag{57}$$

From (55) it follows that

$$g(x, t, \lambda) s_0 g^\dagger(-x, -t, -\lambda^*) s_0^{-1} = \mathbb{1}, \tag{58}$$

which means that:

$$P_1(x, t) = s_0 P_1^\dagger(-x, -t) s_0^{-1}, \qquad \lambda_1^+ = -(\lambda_1^-)^* = \alpha_1 + i\beta_1. \tag{59}$$

From the general theory of the dressing method [45] it is known that $q(x, t, \lambda)$ must satisfy the equation

$$i\frac{dg}{dx} + ([J, Q_1] - \lambda J)g + g\lambda J = 0, \qquad i\frac{dg}{dt} + ([J^*, Q_1] - \lambda J^*)g + g\lambda J^* = 0, \tag{60}$$

where $Q_1(x, t)$ is the 'dressed' potential determined by the one-soliton solution. In particular, analyzing the residue for $\lambda \to \lambda_1^-$ as well as the limit for $\lambda \to \infty$ of this equation there follows that

$$i\frac{d\langle m_0|}{dx} + \lambda_1^- \langle m_0(x, t)|J = 0, \qquad i\frac{d\langle m_0|}{dt} + \lambda_1^- \langle m_0(x, t)|J^* = 0,$$
$$[J, Q_1(x, t)] = -2\alpha_1[J, P_1(x, t)]. \tag{61}$$

Thus we get that $m_{0k}(x, t) = m_{00k}e^{R_k + i\Omega_k}$, $n_{0k}(x, t) = (-1)^{k+1}m_{00\bar{k}}^* e^{-R_{\bar{k}} + i\Omega_{\bar{k}}}$, $\bar{k} = 4 - k$ where

$$
\begin{aligned}
R_1 &= (\alpha_1 j_1 - \beta_1 j_0)x - (\alpha_1 j_1 + \beta_1 j_0)t, & \Omega_1 &= -(\alpha_1 j_0 + \beta_1 j_1)x - (\alpha_1 j_0 - \beta_1 j_1)t, \\
R_2 &= 2\beta_1 j_0(x + t), & \Omega_2 &= 2\alpha_1 j_0(x + t), \\
R_3 &= -(\alpha_1 j_1 + \beta_1 j_0)x + (\alpha_1 j_1 - \beta_1 j_0)t, & \Omega_3 &= -(\alpha_1 j_0 - \beta_1 j_1)x - (\alpha_1 j_0 + \beta_1 j_1)t,
\end{aligned}
\tag{62}
$$

For simplicity we also assume that $m_{003} = m_{001}^*$. Then for the denominator of $P_1(x, t)$ we get:

$$\langle n_0(x, t)|m_0(x, t)\rangle = e^{-i\Omega_2}\Delta_1(x, t), \quad \Delta_1(x, t) = 2|m_{001}|^2 \cosh(R_1 - R_3) - |m_{002}|^2 e^{3i\Omega_2}. \tag{63}$$

Finally, for the one-soliton solution we get:

$$Q_{1;12}(x, t) = -\frac{2\alpha_1}{\Delta_1} m_{001} m_{002} e^{R_2 - R_3} e^{i(\Omega_2 - \Omega_1)}, \quad Q_{1;21}(x, t) = \frac{2\alpha_1}{\Delta_1} m_{001} m_{002}^* e^{R_1 - R_2} e^{i(\Omega_2 - \Omega_3)},$$

$$Q_{1;13}(x, t) = -\frac{2\alpha_1}{\Delta_1} |m_{001}|^2 e^{i(\Omega_3 - \Omega_1)}, \qquad Q_{1;31}(x, t) = -\frac{2\alpha_1}{\Delta_1} |m_{001}|^2 e^{i(\Omega_1 - \Omega_3)},$$

$$Q_{1;23}(x, t) = \frac{2\alpha_1}{\Delta_1} m_{001}^* m_{002}^* e^{R_3 - R_2} e^{i(\Omega_2 - \Omega_1)}, \quad Q_{1;32}(x, t) = -\frac{2\alpha_1}{\Delta_1} m_{001}^* m_{002} e^{R_2 - R_1} e^{i(\Omega_2 - \Omega_3)}.$$

$$\tag{64}$$

If we assume in addition $2|m_{01}|^2 > |m_{02}|^2$ then the denominator *never vanishes* and *the soliton is regular.*

It is easy to check that for $x \to \pm\infty$ the solution decays to zero, i.e. it falls into the class of potentials for which we constructed the FAS.

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5.2 Two soliton solution

Let us now consider the two-soliton solution corresponds to two discrete eigenvalues, λ_1 and λ_2 so that the anzatz for g, becomes

$$g(x, t, \lambda) = \mathbb{1} + \frac{A_1(x, t)}{\lambda - \lambda_1^-} + \frac{A_2(x, t)}{\lambda - \lambda_2^-} \tag{65}$$

for some residues $A_1(x, t)$ and $A_2(x, t)$. From the condition (58) there follows that $\hat{g}(x, t, \lambda)$ must have simple poles at $\lambda_k^+ = -(\lambda_k^-)^*$. In addition it leads to the following system of equations for A_1, A_2

$$\left(\mathbb{1} - \frac{A_1(x, t)}{\lambda_k^+ - \lambda_1^-} - \frac{A_2(x, t)}{\lambda_k^+ - \lambda_2^-} \right) s_0 A_k^\dagger(-x, -t) s_0^{-1} = 0, \qquad k = 1, 2 \tag{66}$$

which are the conditions of the vanishing of the residue of (58) at $-\lambda_k^+$ and $-\lambda_2^*$.

Now the dressing factor (65) relates the FAS of the 'naked' Lax operator with the one for L_2 whose potential Q_2 is the 2-soliton solution of the 3-wave equation. As before we consider $A_k(x, t)$ as rank-1 degenerate matrices in the form

$$A_k(x, t) = |n_k(x, t)\rangle\langle m_k(x, t)|, \tag{67}$$

where $|n_k(x, t)\rangle$ are 3-component vector-columns, and $\langle m_k(x, t)|$ are 3-component vector-rows. The system (66) produces the following linear system of two equations

$$\begin{pmatrix} s_0|m_1^*(-x, -t)\rangle \\ s_0|m_2^*(-x, -t)\rangle \end{pmatrix} + \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} |n_1(x, t)\rangle \\ |n_2(x, t)\rangle \end{pmatrix} = 0, \tag{68}$$

where

$$F_{kp} = \frac{\langle m_p(x, t)|s_0|m_k^*(-x, -t)\rangle}{\lambda_k^+ + \lambda_p^{+,*}}. \tag{69}$$

which allows the expression of the n- vectors via m- vectors. In addition, the condition (54) requires the residues to satisfy the equations

$$i\frac{dA_k}{dx} + [J, Q]A_k - \lambda_k^-[J, A_k] = 0, \qquad i\frac{dA_k}{dt} + [J^*, Q]A_k - \lambda_k^-[J^*, A_k] = 0, \tag{70}$$

$k = 1, 2$. These differential equations for A_i lead to the following equations for $|n\rangle$ and $\langle m|$:

$$\begin{aligned}
i\frac{d\langle m_k|}{dx} + \lambda_k^-\langle m_k|J = 0, \qquad & i\frac{d|n_k\rangle}{dx} + ([J, Q_2] - \lambda_k^- J)|n_k\rangle = 0, \\
i\frac{d\langle m_k|}{dt} + \lambda_k^-\langle m_k|J^* = 0, \qquad & i\frac{d|n_k\rangle}{dt} + ([J, Q_2] - \lambda_k^- J^*)|n_k\rangle = 0.
\end{aligned} \tag{71}$$

This result shows that while $\langle m_k|$ satisfy the 'naked' equation at $\lambda = \lambda_k^-$ then $|n_k\rangle$ are eigenfunctions of the spectral problem for $\lambda = \lambda_k^-$ for the (yet unknown) 2-soliton solution $Q_2(x, t)$. One can easily solve for $|m_k\rangle$ and then can recover $|n_k\rangle$ from (68). These solutions are

$$\langle m_k| = \langle m_{k0}|e^{i\lambda_k^-(Jx+J^*t)}, \tag{72}$$

Taking $\lambda_k^\pm = \pm\alpha_k + i\beta_k$ we find

$$\begin{pmatrix} |n_1(x, t)\rangle \\ |n_2(x, t)\rangle \end{pmatrix} = \frac{1}{\Delta_2(x, t)} \begin{pmatrix} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{pmatrix} \begin{pmatrix} s_0|m_1^*(-x, -t)\rangle \\ s_0|m_2^*(-x, -t)\rangle \end{pmatrix}, \tag{73}$$

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where $\Delta_2(x,t) = F_{12}F_{21} - F_{11}F_{22}$. Clearly $\Delta_2(x,t)$ is $\mathcal{PT}$ invariant. The 2-soliton solution is determined from

$$[J, Q_2(x,t) = [J, A_1(x,t) + A_2(x,t)] = [J, |n_1(x,t)\rangle\langle m_1(x,t)| + |n_2(x,t)\rangle\langle m_2(x,t)|] \qquad (74)$$

Thus the two-soliton solution will be complicated rational functions of the exponentials like $e^{-R_k \pm i\Omega_k}$. In the generic case it is rather difficult to find out if this solution is singular or not.

However for some special cases of the parameters the 2-soliton solutions simplify and one can see that they are *regular*. First we assume that $\lambda_1^+ = -\lambda_2^+ = \alpha_1 + i\beta_1$. That means that

$$\begin{aligned}
\langle m_1(x,t)| &= \langle \mu_0 | e^{\mathbf{R}+i\mathbf{\Omega}}, & |m_1^*(-x,-t)\rangle &= e^{-\mathbf{R}+i\mathbf{\Omega}}|\mu_0^*\rangle, & \mathbf{R} &= \operatorname{diag}(R_1, R_2, R_3) \\
\langle m_2(x,t)| &= \langle \tilde{\mu}_0 | e^{-\mathbf{R}-i\mathbf{\Omega}}, & |m_2^*(-x,-t)\rangle &= e^{\mathbf{R}-i\mathbf{\Omega}}|\tilde{\mu}_0^*\rangle, & \mathbf{\Omega} &= \operatorname{diag}(\Omega_1, \Omega_2, \Omega_3)
\end{aligned} \qquad (75)$$

Next, like for the one-soliton case above, we assume $\mu_{03} = \mu_{01}^*$ and $\tilde{\mu}_{03} = \tilde{\mu}_{01}^*$ and $\tilde{\mu}_{02} = 0$. Then the expressions for F_{kp} simplify into:

$$F_{11} = \frac{e^{-i\Omega_2}}{\alpha_1}\left(|\mu_{01}|^2 \cosh(R_1 - R_3) - \frac{1}{2}|\nu_{02}|^2 e^{-3i\Omega_2}\right), \quad F_{22} = -\frac{e^{i\Omega_2}}{\alpha_1}|\mu_{01}|^2 \cosh(R_1 - R_3),$$

$$F_{12} = \frac{e^{R_2}}{i\beta_1}|\mu_{01}\tilde{\mu}_{01}|\cos(\Omega_1 - \Omega_3 - \sigma_{01}), \qquad F_{21} = -\frac{e^{-R_2}}{i\beta_1}|\mu_{01}\tilde{\mu}_{01}|\cos(\Omega_1 - \Omega_3 + \sigma_{01}).$$

$$(76)$$

where $\sigma_{01} = \arg\mu_{01} + \arg\tilde{\mu}_{01}$. Inserting these expressions into Δ_2 we obtain:

$$\Delta_2(x,t) = \frac{1}{\alpha_1^2}|\tilde{\mu}_{01}|^2 \cosh(R_1 - R_3)\left(|\mu_{01}^2\cosh(R_1 - R_3) - \frac{1}{2}|\mu_{02}|^2 e^{-3i\Omega_2}\right)$$

$$+ \frac{1}{\beta_1^2}|\mu_{01}\tilde{\mu}_{01}|^2 \cos(\Omega_1 - \Omega_3 + \sigma_{01})\cos(\Omega_1 - \Omega_3 - \sigma_{01}). \quad (77)$$

The above expression will have no zeroes if, like in the one-soliton case above, we assume that $2|\mu_{01}| > |\mu_{02}$ and require in addition $\sigma_{01} = 0$ or $\sigma_{01} = \pi$ Then the corresponding two-soliton solutions will be regular.

6   Conclusions

We have constructed the Fundamental Analytic Solutions for nonlocal reductions of 3- and 4-wave systems (related to the algebras $sl(3, \mathbb{C})$ and $so(5, \mathbb{C})$ respectively) having $\mathcal{PT}$-symmetry and briefly outlined the spectral properties of the associated Lax operators. Furthermore, we have derived the one- and two-soliton solutions of the 3-wave equations with complex J.

Analysis of the one soliton solutions shows that they can be regular for all x and t provided the polarization vectors satisfy certain constraints, see Subsection 5.1. We also find specific set of parameters for the two-soliton solutions of the $\mathcal{PT}$ symmetric 3-wave equations that ensure their regularity.

These examples demonstrate that the $\mathcal{PT}$ symmetric 3-wave equations may have regular multi-soliton solutions for some specific choices of their parameters. Similar analysis can be applied also to the 4-wave $\mathcal{PT}$ symmetric equations. Details about it will be published elsewhere.

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