2013

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Daphne Gilbert
Technological University Dublin, daphne.gilbert@dit.ie

Recommended Citation

Gilbert, D. (2013) Eigenfunction Expansions Associated with the One-Dimensional Schrödinger Operator, *Operator Methods in Mathematical Physics,* Vol. 227, pp. 89-105. DOI 10.1007/978-3-0348-0531-5_4

Eigenfunction expansions associated with the one-dimensional Schrödinger operator

D. J. Gilbert

Abstract. We consider the form of eigenfunction expansions associated with the time-independent Schrödinger operator on the line, under the assumption that the limit point case holds at both of the infinite endpoints. It is well known that in this situation the multiplicity of the operator may be one or two, depending on properties of the potential function. Moreover, for values of the spectral parameter in the upper half complex plane, there exist Weyl solutions associated with the restrictions of the operator to the negative and positive half-lines respectively, together with corresponding Titchmarsh-Weyl functions.

In this paper, we establish some alternative forms of the eigenfunction expansion which exhibit the underlying structure of the spectrum and the asymptotic behaviour of the corresponding eigenfunctions. We focus in particular on cases where some or all of the spectrum is simple and absolutely continuous. It will be shown that in this situation, the form of the relevant part of the expansion is similar to that of the singular half line case, in which the origin is a regular endpoint and the limit point case holds at infinity. Our results demonstrate the key role of real solutions of the differential equation which are pointwise limits of the Weyl solutions on one of the half-lines, while all solutions are of comparable asymptotic size at infinity on the other half-line.

Mathematics Subject Classification (2000). 34L10, 47E05, 47B25.

Keywords. Eigenfunction expansions, Sturm-Liouville problems, unbounded selfadjoint operators.

1 Introduction

The study of eigenfunction expansions associated with singular differential operators of the Sturm-Liouville type was initiated by Weyl [6], [17], and subsequently generalised by Stone, Kodaira, Titchmarsh, and others [14], [12], [15], [16]. Particularly well-known are the Fourier, Hermite and Legendre expansions, the Laguerre polynomials and the Fourier-Bessel series, all of which have widespread applications in engineering and the physical sciences. It therefore seems worthwhile to explore the general behaviour of the eigenfunctions which contribute to such expansions, to investigate the relationship between the eigenfunctions and spectral properties, and to consider under what circumstances the formal structure of the standard expansions can be improved.

In the case of singular Sturm-Liouville operators, the eigenfunctions are themselves solutions of the differential equations. If the multiplicity of the spectrum is two then every solution of $Lu = \lambda u$ may be regarded as an eigenfunction, but since the dimension of the solution space is two, we expect not more than two two linearly independent solutions to feature as integral kernels in the expansions. In a similar way, it seems reasonable to expect that when the spectrum, or a part of the spectrum, is simple then for each relevant value of λ, precisely one linearly independent solution should feature in the expansion.

It is the purpose of this paper to demonstrate that, in the case where part or all of the absolutely continuous spectrum is simple, the corresponding part of the expansion can be reformulated in such a way that the integral kernels are, up to scalar multiples, the eigenfunctions themselves. To achieve this result we start from the Weyl-Kodaira expansion formula [12], [3], and following the method of Kac [10], [11], diagonalise the spectral density matrix in order to identify the eigenfunctions and simplify the expansion. It turns out that in the process of reformulating the part of the expansion where the spectral multiplicity is one, the contribution of the half-line operators $H_{-\infty}$ and H_{∞} is reflected through their respective Titchmarsh-Weyl m-functions and corresponding spectral densities. This information enables the asymptotic behaviour of the eigenfunctions at $\pm\infty$ to be determined in terms of the theory of subordinacy [8], [9], and details of this process will be demonstrated through worked examples.

Note Throughout the paper the use of the term "eigenfunctions" is not restricted to specific real solutions of the differential equation which are in $L_2(\mathbf{R})$, but also includes "eigendifferentials" in the terminology of Weyl [4], [17], as well as other relevant solutions associated with the essential spectrum. Where appropriate we will distinguish between singular eigenfunctions and absolutely continuous eigenfunctions depending on whether the corresponding λ-values are in the minimal supports of the singular, respectively absolutely continuous, parts of the spectral measure, and the term generalised eigenfunction will be used to refer to eigenfunctions of either type.

2 Mathematical Background

In this section we briefly summarize the relevant underlying theory from which the main results of this paper are obtained.

Consider the differential operator H on $L_2(-\infty, \infty)$ associated with

$$Lu := -u'' + q(r)u = \lambda u, \qquad -\infty < r < \infty,$$

where $q(r) : \mathbf{R} \to \mathbf{R}$ is locally integrable, $\lambda \in \mathbf{R}$ is the spectral parameter, and the differential expression L is in Weyl's limit point case at $\pm\infty$. In this case the unique self-adjoint operator H is defined by

$$Hf = Lf, \qquad f \in \mathcal{D}(H),$$

where $f \in \mathcal{D}(H)$ if

(i) $f, Lf \in L_2(\mathbf{R})$,

(ii) f, f' are locally absolutely continuous on $\mathbf{R}$.

We refer to H as the Schrödinger operator on the line.

It is convenient in this context to use the so-called splitting method to analyse the spectrum of H and derive appropriate formulations of the associated eigenfunction expansion. We first define the half-line operator H_∞ to be the restriction of H to $L_2([0,\infty))$ with a Dirichlet boundary condition at $r = 0$, and choose a fundamental set of solutions $\{u_1(r,z), u_2(r,z)\}$ of $Lu = zu$, $z \in \mathbf{C}$, to satisfy

$$u_1(0,z) = u_2^{'}(0,z) = 0, \qquad u_2(0,z) = u_1^{'}(0,z) = 1, \tag{2.1}$$

where $'$ denotes differentiation with respect to r. Associated with H_∞, there exists a Herglotz function $m_\infty(z) : \mathbf{C}^+ \to \mathbf{C}^+$ known as the Titchmarsh-Weyl function, such that the solution $u_2(r,z) + m_\infty(z)\, u_1(r,z)$ of $Lu = zu$ is in $L_2([0,\infty))$ for all $z \in \mathbf{C}^+$. The related spectral function $\rho_\infty(\lambda) : \mathbf{R} \to \mathbf{R}$ is non-decreasing, continuous on the right and generates a non-negative Borel-Stieltjes measure μ_∞ on $\mathbf{R}$. Its derivative, the spectral density function $\rho_\infty^{'}(\lambda)$, exists and satisfies

$$\rho_\infty^{'}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi}\, \mathrm{Im}\, m_\infty(z), \quad z = \lambda + \mathrm{i}\epsilon, \tag{2.2}$$

for Lebesgue and μ_∞-almost all $\lambda \in \mathbf{R}$, and the spectrum $\sigma(H_\infty)$ may be defined by

$$\sigma(H_\infty) := \mathbf{R} \setminus \{\lambda \in \mathbf{R} : \rho_\infty(\lambda) \text{ is constant in a neighbourhood } N(\lambda) \text{ of } \lambda\},$$

which is the smallest closed set containing the points of increase of $\rho_\infty(\lambda)$.

The half-line operator $H_{-\infty}$ on $L_2((-\infty,0])$ is defined in a similar way, the principal difference being that the Titchmarsh-Weyl function $m_{-\infty}(z)$ has negative imaginary part on $\mathbf{C}^+$, so that the spectral density $\rho_{-\infty}^{'}(\lambda)$ satisfies

$$\rho_{-\infty}^{'}(\lambda) = -\lim_{z \uparrow \lambda} \frac{1}{\pi}\, \mathrm{Im}\, m_{-\infty}(z), \quad z = \lambda + \mathrm{i}\epsilon, \tag{2.3}$$

for Lebesgue and $\mu_{-\infty}$-almost all $\lambda \in \mathbf{R}$. We note that the spectrum of both half-line operators is simple, that is to say, both H_∞ and $H_{-\infty}$ have spectral multiplicity one.

The standard form of the eigenfunction expansion associated with the self-adjoint operator H_∞ is as follows for $f(r) \in L_2([0,\infty))$:

$$f(r) = \underset{\omega \to \infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} u_1(r,\lambda)\, G(\lambda)\, d\rho_\infty(\lambda), \tag{2.4}$$

with $u_1(r,\lambda)$ as in (2.1) for $\lambda = z \in \mathbf{R}$, and

$$G(\lambda) = \underset{\eta \to \infty}{\text{l.i.m.}} \int_0^{\eta} u_1(r,\lambda)\, f(r)\, dr,$$

where convergence in the mean is in $L_2([0,\infty))$ and $L_2(\mathbf{R}; d\rho_\infty(\lambda))$ respectively [2]. We see that the integral kernel of both transform and inverse transform is a solution of $Lu = \lambda u$ satisfying the Dirichlet boundary condition at $r = 0$, namely $u_1(r,\lambda)$, and it follows that $u_1(r,\lambda)$ is an eigenfunction of H_∞ for those λ contributing to the spectrum of the operator. Note that the spectral function $\rho_\infty(\lambda)$ is constant on each open interval of the resolvent set, so that there is no contribution to the integral in (2.4) when λ is in the resolvent set. The expansion associated with the operator $H_{-\infty}$ has a similar form, with obvious adjustments.

In the case of the full line operator H, the analogue of the Titchmarsh-Weyl m-function is a 2×2 M-matrix, which is defined in terms of the scalar m-functions associated with $H_{-\infty}$ and H_∞ by

$$M(z) := \frac{1}{m_{-\infty} - m_\infty} \begin{pmatrix} m_{-\infty} m_\infty & \frac{1}{2}(m_{-\infty} + m_\infty) \\ \frac{1}{2}(m_{-\infty} + m_\infty) & 1 \end{pmatrix} \tag{2.5}$$

for $z \in \mathbf{C}^+$. The associated matrix function for $\lambda \in \mathbf{R}$,

$$(\rho_{ij}(\lambda)) := \begin{pmatrix} \rho_{11}(\lambda) & \rho_{12}(\lambda) \\ \rho_{21}(\lambda) & \rho_{22}(\lambda) \end{pmatrix},$$

is continuous on the right, has bounded variation on compact subintervals of the real line, and generates a Borel-Steltjes measure (μ_{ij}) which is positive semi-definite [12]. The components of the corresponding density matrix $\left(\rho'_{ij}(\lambda)\right)$ satisfy

$$\rho'_{ij}(\lambda) = \lim_{z\downarrow\lambda} \frac{1}{\pi} \operatorname{Im} M_{ij}(z), \quad z \in \lambda + i\epsilon, \tag{2.6}$$

Lebesgue and μ_{ij}-almost everywhere on $\mathbf{R}$, for each $i,j = 1,2$, and the spectrum of H is given by

$$\sigma(H) := \mathbf{R} \setminus \{\lambda \in \mathbf{R} : (\rho_{ij}(\lambda)) \text{ is constant in a neighbourhood } N(\lambda) \text{ of } \lambda\}.$$

As noted above, both the half-line operators, $H_{-\infty}$ and H_∞, have spectral multiplicity one; however, in the case of the full line operator H where both endpoints are limit point, some or all of the spectrum may have multiplicity two. Here the standard formulation of the expansion in its most general form is given by the Weyl-Kodaira formula [3], [12], from which we have for $f \in L_2(\mathbf{R})$,

$$f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} \sum_{i=1}^{2} \sum_{j=1}^{2} u_i(r,\lambda)\, F_j(\lambda)\, d\rho_{ij}(\lambda), \tag{2.7}$$

where

$$F(\lambda) = \{F_1(\lambda), F_2(\lambda)\} = \underset{\eta\to\infty}{\text{l.i.m.}} \left\{ \int_{-\eta}^{\eta} u_1(r,\lambda) f(r)dr, \int_{-\eta}^{\eta} u_2(r,\lambda) f(r)dr \right\},$$

and $u_1(r,\lambda)$, $u_2(r,\lambda)$, are as in (2.1) with $z = \lambda \in \mathbf{R}$, convergence of the integrals being in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_{ij}(\lambda))$, respectively. An advantage of this form of the eigenfunction expansion is that it has very general application, so that with suitable adjustments it may also be applied to cases where o the endpoints $-\infty$, ∞, are replaced by a, b, respectively, and the decomposition point 0 by c, where $-\infty \leq a < c < b \leq \infty$,

o the Dirichlet boundary condition at the decomposition point c is replaced by $\cos(\alpha)u(c,\lambda) + \sin(\alpha)u'(c,\lambda) = 0$ for some $\alpha \in (0,\pi)$,

o one or both of the endpoints is in the limit circle case.

For further details, see [8].

However, a significant drawback of the Weyl-Kodaira formula is that in the case of simple spectrum those solutions of the differential equation which are eigenfunctions of the operator H cannot be identified directly from the expansion as it stands. We note that the formulation (2.7) contains four terms, whereas the multiplicity of the spectrum, and hence the dimension of the eigenspaces, is at most two. It will be shown that whenever the spectrum of H has multiplicity one, the expansion can be reduced to a much simpler form which replicates many of the features of (2.4) in the half-line case. In this situation the eigenfunctions feature as integral kernels in the simplified expansion and can be completely characterised in terms of their subordinacy properties.

To clarify the significance of the theory of subordinancy in this context, we briefly introduce some key relevant features. A subordinate solution of $Lu = \lambda u$, $r \geq 0$, when L is regular at 0 and in the limit point case at infinity, is defined as follows:

Definition: A solution $u_s(r,\lambda)$ of $Lu = \lambda u$, $-\infty < \lambda < \infty$, $0 \leq r < \infty$, is said to be *subordinate* at infinity if

$$\lim_{N\to\infty} \frac{\| u_s(r,\lambda) \|_N}{\| u(r,\lambda) \|_N} = 0,$$

where $\| \, . \, \|_N$ denotes $(\int_0^N | \, . \, |^2 dr)^{1/2}$, and $u(r,\lambda)$ denotes any solution of $Lu = \lambda u$ which is linearly independent from $u_s(r,\lambda)$.

Thus a subordinate solution is unique up to λ-multiples, and is asymptotically smaller than any linearly independent solution of the same equation, in the sense of limiting ratios of Hilbert space norms. To contribute to the eigenfunction expansion associated with H_∞, a solution $u(r,\lambda)$ of $Lu = \lambda u$ must either

(a) satisfy the boundary condition at $r = 0$ in the case where no solution is subordinate at infinity, or

(b) satisfy the boundary condition at $r = 0$ and be subordinate at infinity,

since the set of all $\lambda \in \mathbf{R}$ for which neither condition holds has μ_∞-measure zero [9]. It will be demonstrated in Section 4 that solutions of $Lu = \lambda u$, $r \geq 0$ which satisfy (a) are absolutely continuous eigenfunctions of H_∞, and solutions which satisfy (b) are singular eigenfunctions of H_∞. The definition of subordinate solutions and the distinguishing properties of the eigenfunctions are entirely analogous in the case of $H_{-\infty}$.

The relationship between the eigenfunctions and the corresponding parts of the spectrum of H can be made more precise using the concept of a minimal support of a Borel-Stieltjes measure.

Definition: A subset S of $\mathbf{R}$ is said to be a *minimal support* of a measure ν on $\mathbf{R}$ if the following conditions hold:

(i) $\nu(\mathbf{R} \setminus S) = 0$,

(ii) if S_0 is a subset of S such that $\nu(S_0) = 0$, then $\mid S_0 \mid = 0$,

where $\mid . \mid$ denotes Lebesgue measure.

Note that a minimal support of a measure ν is unique up to ν- and Lebesgue measure zero and, in the case of a spectral measure, provides an indication of where the spectrum is concentrated (for further details, (see [9]). Corresponding to the decomposition of a Borel-Stieltjes measure ν into absolutely continuous and singular parts, $\nu_{a.c.}$ and $\nu_{s.}$, there exist minimal supports, $S(\nu_{a.c.})$ and $S(\nu_{s.})$ respectively such that $S(\nu_{a.c.}) \cup S(\nu_{s.}) = S(\nu)$ and $S(\nu_{a.c.}) \cap S(\nu_{s.}) = \emptyset$. For H_∞, as shown in [9], minimal supports $\mathcal{M}_{a.c.}(H_\infty)$, $\mathcal{M}_{s.}(H_\infty)$ of the absolutely continuous and singular parts respectively of μ_∞ are as follows:

$$\begin{aligned}
\mathcal{M}_{a.c}(H_\infty) &= \{\lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at } \infty\}, \\
\mathcal{M}_{s.}(H_\infty) &= \{\lambda \in \mathbf{R} : \text{there exists a solution of } Lu = \lambda u \text{ which satisfies the} \\
&\quad \text{Dirichlet boundary condition at 0 and is subordinate at } \infty\}
\end{aligned}$$

Minimal supports, $\mathcal{M}_{a.c}(H_{-\infty})$ and $\mathcal{M}_{s.}(H_{-\infty})$, of the absolutely continuous and singular parts of $\mu_{-\infty}$ are obtained by replacing ∞ with $-\infty$ in the above equations. It is important to note that in general, the Lebesgue measure of a minimal support of a spectral measure and the Lebesgue measure of the corresponding spectrum are not in general equal. For example, in the case where there is dense singular spectrum on a real interval $[c, d]$, the singular spectrum, being a closed set, will contain $[c, d]$, so that the Lebesgue measure of the interval is $c - d$; however, the minimal support of the singular part of a spectral measure will always have Lebesgue measure zero. A similar situation can also arise in relation to the absolutely continuous spectrum.

In the case of the full-line operator H on $L_2(\mathbf{R})$ with two limit point endpoints, $\mathcal{M} = \mathcal{M}_{a.c.}(H) \cup \mathcal{M}_{s.}(H)$ is a minimal support of the spectral measure μ_τ, where

$$\begin{aligned}
\mathcal{M}_{a.c.}(H) &= \{\lambda \in \mathbf{R} : \text{either no solution of } Lu = \lambda u \text{ exists which is} \\
&\quad \text{subordinate at } -\infty, \text{ or no solution of } Lu = \lambda u \text{ exists} \\
&\quad \text{which is subordinate at } +\infty, \text{ or both}\} \\
\mathcal{M}_{s.}(H) &= \{\lambda \in \mathbf{R} : \text{a solution of } Lu = \lambda u \text{ exists which is} \\
&\quad \text{subordinate at both} \pm \infty\}
\end{aligned}$$

(see [7]). The nature of the eigenfunctions when H has simple spectrum will be considered in Section 4.

3 Diagonalising the spectral density matrix

In order to simplify the eigenfunction expansion in the case where both endpoints are limit point, we follow the method of I. S. Kac [10], [11], and begin by introducing a spectral density matrix. Let $M_\tau(z)$ denote the trace of the M-matrix in (2.5), so that for $z \in \mathbf{C}^+$

$$\begin{aligned} M_\tau(z) &:= M_{11}(z) + M_{22}(z) \\ &= \frac{m_{-\infty}(z)\, m_\infty(z) + 1}{m_{-\infty}(z) - m_\infty(z)}. \end{aligned} \tag{3.1}$$

Since $m_{-\infty}$ and m_∞ are anti-Herglotz and Herglotz functions respectively, it is straightforward to check that $M_\tau(z)$ is Herglotz, so that $\operatorname{Im} M_\tau(z) > 0$ for $z \in \mathbf{C}^+$. It follows that a non-decreasing function $\rho_\tau(\lambda) : \mathbf{R} \to \mathbf{R}$ exists such that

$$\rho_\tau^{'}(\lambda) = \lim_{z \downarrow \lambda} \frac{1}{\pi} \operatorname{Im} M_\tau(z), \quad z = \lambda + i\epsilon, \tag{3.2}$$

exists and is satisfied for Lebesgue and μ_τ-almost all $\lambda \in \mathbf{R}$, where μ_τ is the non-negative Borel-Stieltjes measure generated by $\rho_\tau(\lambda)$. Moreover, since $(\rho_{ij}(\lambda))$ is positive semi-definite, μ_{ij} is absolutely continuous with respect to μ_τ for each $i, j = 1, 2$, from which it may be inferred that for $\lim_{z\downarrow\lambda} \operatorname{Im} M_\tau(z) \neq 0$,

$$d\rho_{ij}(\lambda) = \frac{d\rho_{ij}}{d\rho_\tau}(\lambda) d\rho_\tau(\lambda) = \lim_{z\downarrow\lambda} \frac{\operatorname{Im} M_{ij}(z)}{\operatorname{Im} M_\tau(z)} d\rho_\tau(\lambda), \quad z = \lambda + \mathrm{i}\epsilon, \tag{3.3}$$

Lebesgue and μ_τ-almost everywhere on $\mathbf{R}$ (see [8], Lemma 5.3). We refer to

$$\left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) = \begin{pmatrix} \frac{d\rho_{11}}{d\rho_\tau}(\lambda) & \frac{d\rho_{12}}{d\rho_\tau}(\lambda) \\ \frac{d\rho_{21}}{d\rho_\tau}(\lambda) & \frac{d\rho_{22}}{d\rho_\tau}(\lambda) \end{pmatrix} \tag{3.4}$$

as the spectral density matrix for H. Since the set

$$S_0 := \{\lambda \in \mathbf{R} : \lim_{z\downarrow\lambda} \operatorname{Im} M_\tau(z) = 0\}$$

has μ_τ-measure zero, we may take

$$\left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for all } \lambda \in S_0. \tag{3.5}$$

Also, noting that the limits as $z \downarrow \lambda$ in (3.3) exist for Lebesgue and μ_τ-almost all $\lambda \in \mathbf{R} \setminus S_0$, we have that

$$\frac{d\rho_{11}}{d\rho_\tau}(\lambda) = \frac{\operatorname{Im} m_{-\infty}(\lambda) \mid m_\infty(\lambda) \mid^2 - \operatorname{Im} m_\infty(\lambda) \mid m_{-\infty}(\lambda) \mid^2}{D(\lambda)}, \tag{3.6}$$

$$\frac{d\rho_{12}}{d\rho_\tau}(\lambda) = \frac{d\rho_{21}}{d\rho_\tau}(\lambda) = \frac{\mathrm{Im}\, m_{-\infty}(\lambda)\ \mathrm{Re}\, m_{\infty}(\lambda) - \mathrm{Im}\, m_{\infty}(\lambda)\ \mathrm{Re}\, m_{-\infty}(\lambda)}{D(\lambda)}, \tag{3.7}$$

$$\frac{d\rho_{22}}{d\rho_\tau}(\lambda) = \frac{\mathrm{Im}\, m_{-\infty}(\lambda) - \mathrm{Im}\, m_{\infty}(\lambda)}{D(\lambda)}, \tag{3.8}$$

almost everywhere on $\mathbf{R} \setminus S_0$, where

$$D(\lambda) = \mathrm{Im}\, m_{-\infty}(\lambda) \left(1 + \mid m_\infty(\lambda) \mid^2\right) - \mathrm{Im}\, m_\infty(\lambda) \left(1 + \mid m_{-\infty}(\lambda) \mid^2\right),$$

and $m_{-\infty}(\lambda)$, $m_\infty(\lambda)$, denote the normal limits of $m_{-\infty}(z)$, $m_\infty(z)$, respectively as $z \downarrow \lambda \in \mathbf{R}$.

The following theorem establishes a rigorous correlation between the rank of the spectral density matrix and the multiplicity of the spectrum of H.

Theorem 1 (Kac) *The spectral multiplicity of H is two if and only if the μ_τ-measure of the set*

$$\mathcal{M}_2 := \left\{ \lambda \in E : \mathrm{rank}\left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) = 2 \right\}$$

is strictly positive, where

$$E := \left\{ \lambda \in \mathcal{M} : \left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) \text{ exists} \right\}$$

for some minimal support $\mathcal{M}$ of μ_τ. The set $\mathcal{M}_2$ is a maximal set of multiplicity 2, and for μ_τ-almost all $\lambda \in (\mathbf{R} \setminus \mathcal{M}_2)$, H has multiplicity one with

$$\mathrm{rank}\left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) = 1.$$

An important consequence of the theorem, as recognized by Kac, is the following.

Corollary 1 *Let*

$$S_d := \{ \lambda \in \mathbf{R} : \mathrm{Im}\, m_{-\infty}(\lambda) < 0,\ \mathrm{Im}\, m_\infty(\lambda) > 0 \}.$$

Then $\mu_\tau(\mathcal{M}_2 \triangle S_d) = 0$, where $\triangle$ denotes the symmetric difference.

The corollary implies that only the absolutely continuous part of the spectrum can have multiplicity two, or contain a non-trivial subset with multiplicity two. It also implies that the degenerate spectrum is supported on

$$S' := \{ \lambda \in \mathbf{R} : \text{no solution of } Lu = \lambda u \text{ is subordinate at either } -\infty \text{ or at } \infty \},$$

where $\mu_\tau(S_d \triangle S') = 0$ (see [8] for further details).

We now use the spectral density matrix to rearrange the Weyl-Kodaira formulation (2.7) in such a way that the generalised eigenfunctions of H are exhibited explicitly in the expansion. Using (3.3) - (3.5), it is straightforward to see that the right hand side of (2.7) may be expressed as follows:

$$\int_{-\omega}^{\omega} \sum_{i=1}^{2} \sum_{j=1}^{2} u_i(r,\lambda) F_j(\lambda) d\rho_\tau(\lambda) = \int_{-\omega}^{\omega} (U(r,\lambda))^T \left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) F(\lambda) d\rho_\tau(\lambda) \quad (3.9)$$

where the superfix T denotes the transpose, and

$$U(r,\lambda) = \begin{pmatrix} u_1(r,\lambda) \\ u_2(r,\lambda) \end{pmatrix}, \qquad F(\lambda) = \begin{pmatrix} F_1(\lambda) \\ F_2(\lambda) \end{pmatrix},$$

with $F_1(\lambda)$ and $F_2\lambda)$ as in (2.7). Since the spectral density matrix (3.4) is real and symmetric, we may decompose it in such a way that

$$\left(\frac{d\rho_{ij}}{d\rho_T}(\lambda) \right) = P^T \, D \, P, \quad (3.10)$$

where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

according as the rank of the spectral density matrix is 1 or 2 respectively, and P is a 2×2 matrix given by

$$P = \begin{pmatrix} \sigma_{11}^{1/2} & \sigma_{11}^{-1/2} \sigma_{12} \\ 0 & \sigma_{11}^{-1/2} (\sigma_{11}\sigma_{22} - \sigma_{12}{}^2)^{1/2} \end{pmatrix} \quad (3.11)$$

for $\sigma_{11} \neq 0$, where for each $i, j = 1, 2$, σ_{ij} denotes $(d\rho_{ij}/d\rho_\tau)(\lambda)$. Note that $P^T \, P = (\sigma_{ij}(\lambda))$ and that P has full rank if and only if the determinant, $\sigma_{11}\sigma_{22} - \sigma_{12}^2$, of the spectral density matrix is non-zero. For non-trivial cases where $\sigma_{11} = 0$, we have

$$D = P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.12)$$

taking into account the positive semi-definite property of the spectral density matrix and the fact that $0 \leq | \sigma_{ij} | \leq 1$ for $i, j = 1, 2$ (see [8]).

Let $\mathcal{M}_1$ denote the set of all $\lambda \in E$ such that $((d\rho_{ij}/d\rho_\tau)(\lambda))$ has multiplicity one. From (2.7), (3.9) and (3.10), we have for $f(r) \in L_2(\mathbf{R})$,

$$\begin{aligned} f(r) &= \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} (U(r,\lambda))^T \; P^T \, D \, P \; F(\lambda) \; d\rho_\tau(\lambda) \\ &= \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} (V(r,\lambda))^T \; D \; G(\lambda) \; d\rho_\tau(\lambda), \end{aligned} \quad (3.13)$$

where

$$P \, U(r,\lambda) = V(r,\lambda) = \begin{pmatrix} v_1(r,\lambda) \\ v_2(r,\lambda) \end{pmatrix}, \quad P \, F(\lambda) = G(\lambda) = \begin{pmatrix} G_1(\lambda) \\ G_2(\lambda) \end{pmatrix}, \quad (3.14)$$

and convergence is in $L_2(\mathbf{R})$. Using (3.11) - (3.13), this leads to

$$\begin{aligned} f(r) &= \underset{\omega\to\infty}{\text{l.i.m.}} \left\{ \int_{(-\omega,\omega)\cap\mathcal{M}_1} v(r,\lambda)\ \ G(\lambda)\ \ d\rho_\tau(\lambda) \right. \\ &\left. \quad + \sum_{i=1}^{2} \int_{(-\omega,\omega)\cap\mathcal{M}_2} v_i(r,\lambda)\ \ G_i(\lambda)\ \ d\rho_\tau(\lambda) \right\}, \end{aligned} \tag{3.15}$$

where

$$G(\lambda) = \underset{\eta\to\infty}{\text{l.i.m.}} \int_{-\eta}^{\eta} v(r,\lambda)\ f(r)\ dr,$$

with

$$v(r,\lambda) = \begin{cases} \sigma_{11}^{1/2}\ u_1(r,\lambda) + \sigma_{11}^{-1/2}\sigma_{12}\ u_2(r,\lambda) & \sigma_{11} \neq 0 \\ u_2(r,\lambda) & \sigma_{11} = 0, \end{cases} \tag{3.16}$$

and for $i = 1, 2$,

$$G_i(\lambda) = \underset{\eta\to\infty}{\text{l.i.m.}} \int_{-\eta}^{\eta} v_i(r,\lambda)\ f(r)\ dr,$$

with

$$\begin{aligned} v_1(r,\lambda) &= \sigma_{11}^{1/2}\ u_1(r,\lambda) + \sigma_{11}^{-1/2}\,\sigma_{12}\ u_2(r,\lambda), \\ v_2(r,\lambda) &= \sigma_{11}^{-1/2}\ (\sigma_{11}\,\sigma_{22} - {\sigma_{12}}^2)^{1/2}\ u_2(r,\lambda). \end{aligned}$$

Also for $k = 1,\ 2$,

$$\mathcal{M}_k := \left\{ \lambda \in \mathbf{R} : \text{rank}\left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda) \right) = k \right\},$$

and convergence of the integrals is in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_\tau(\lambda))$, respectively.

From (3.15), $v(r,\lambda)$ is an eigenfunction of H for each $\lambda \in \mathcal{M}_1$, and for each $\lambda \in \mathcal{M}_2$, a basis for the eigenspace is $\{v_1(r,\lambda), v_2(r,\lambda)\}$. Note from Kac's theorem, that if $\mu_\tau(\mathcal{M}_2) = 0$ the spectrum of H is simple so that the second term in (3.15) is null and the expansion has a similar form to that of the half-line case in (2.4).

4 The case of simple spectrum

In this section, we focus on the particular case where the spectrum of H is simple and purely absolutely continuous. In this situation $\mu_\tau(\mathcal{M}_2) = 0$, and hence as shown in section 3, the expansion in (3.15) reduces to a simpler form, so that for $f(r) \in L_2(\mathbf{R})$

$$f(r) = \underset{\omega\to\infty}{\text{l.i.m.}} \int_{-\omega}^{\omega} v(r,\lambda)\ G(\lambda)\ d\rho_\tau(\lambda) \tag{4.1}$$

where $v(r, \lambda$ is as in (3.16), and

$$G(\lambda) = \underset{\eta\to\infty}{\text{l.i.m.}} \int_{-\eta}^{\eta} v(r,\lambda)\, f(r)\, dr,$$

with convergence in the mean in $L_2(\mathbf{R})$ and $L_2(\mathbf{R}; d\rho_\tau(\lambda))$ respectively.

We now investigate the structure of the eigenfunction $v(r, \lambda)$ in terms of its relationship to the Titchmarsh-Weyl functions, $m_{-\infty}$, m_∞, and the corresponding Weyl solutions associated with $H_{-\infty}$ and H_∞, respectively. This will enable the asymptotic behaviour of $v(r, \lambda)$ as $r \to \pm\infty$ to be ascertained in terms of the theory of subordinacy.

From (3.11) and (3.14), we have for $\sigma_{11} \neq 0$, $\sigma_{22} \neq 0$,

$$v(r,\lambda) = \begin{pmatrix} \sigma_{11}^{1/2} & \sigma_{11}^{-1/2}\,\sigma_{12} \end{pmatrix} \begin{pmatrix} u_1(r,\lambda) \\ u_2(r,\lambda) \end{pmatrix}, \tag{4.2}$$

since by Kac's theorem the assumption that H has simple spectrum implies that $\det(\sigma_{ij}) = 0$, and hence that the second row of P in (3.11) is zero. In cases where $\sigma_{11} = 0$ or $\sigma_{22} = 0$, it follows from (3.12) and (3.11) that $v(r, \lambda) = u_2(r, \lambda)$ or $v(r, \lambda) = u_1(r, \lambda)$, respectively.

From (4.2), $v(r, \lambda)$ is a linear combination of $u_1(r, \lambda)$ and $u_2(r, \lambda)$, whose coefficients are functions of λ which reflect the nature of the spectrum at λ. In order to determine the coefficients, we first introduce the following disjoint sets:

$$\begin{aligned} S_1(\lambda) &:= \{\lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{C}^-,\ \ m_\infty(\lambda) \in \mathbf{R} \cup \{\infty\}\} \\ S_2(\lambda) &:= \{\lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{R} \cup \{\infty\},\ \ m_\infty(\lambda) \in \mathbf{C}^+\} \\ S_3(\lambda) &:= \{\lambda \in \mathbf{R} : m_{-\infty}(\lambda) = m_\infty(\lambda) \in \mathbf{R} \cup \{\infty\}\} \\ S_4(\lambda) &:= \{\lambda \in \mathbf{R} : m_{-\infty}(\lambda) \in \mathbf{C}^-,\ m_\infty(\lambda) \in \mathbf{C}^+\} \end{aligned}$$

Note that $S_1(\lambda) \cup S_2(\lambda)$ is a minimal support of the absolutely continuous part of the spectral measure μ_τ in $\mathcal{M}_1$. The set $S_3(\lambda)$ is a minimal support of the singular part of μ_τ and always has Lebesgue measure zero. Altogether $\cup_{i=1}^3 S_i(\lambda)$ constitutes a minimal support of the part of the spectral measure μ_τ corresponding to the simple part of the spectrum, so is equal to $\mathcal{M}_1$ up to sets of μ_τ- and Lebesgue measure zero. Also from Corollary 1 $S_4(\lambda)$ and $\mathcal{M}_2$ are equal up to μ_τ- and Lebesgue null sets, so that the degenerate part of the spectrum is supported on those values of λ on which both $m_{-\infty}(\lambda)$ and $m_\infty(\lambda)$ are strictly complex (see also [8]). We note that the resolvent set is the largest open set in $\mathbf{R} \setminus \cup_{i=1}^4 S_i(\lambda)$.

In this section we concentrate particularly on $S_1(\lambda)$ and $S_2(\lambda)$ which are associated with the simple part of the absolutely continuous spectrum. In fact it is sufficient to consider $S_1(\lambda)$ in detail, since the derivations and results are almost entirely analogous for $S_2(\lambda)$.

For simplicity of exposition, suppose in the first instance that $S_1(\lambda) = \mathbf{R}$. If $\lambda \in S_1(\lambda)$, so that $m_\infty(\lambda) \in \mathbf{R} \cup \{\infty\}$, then since $m_{-\infty}(\lambda) \in \mathbf{C}^-$, we have from (3.6) and (3.7),

$$\sigma_{11} = \frac{(m_\infty(\lambda))^2}{1 + (m_\infty(\lambda))^2}, \qquad \sigma_{12} = \frac{m_\infty(\lambda)}{1 + (m_\infty(\lambda))^2} \tag{4.3}$$

and hence from (4.2),

$$v(r,\lambda) = \frac{u_2(r,\lambda) + m_\infty(\lambda)\; u_1(r,\lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}}, \tag{4.4}$$

which is a real solution of the differential equation and a scalar multiple of the pointwise limit of the Weyl solution for H_∞ (i.e. of $u_2(r,z)+m_\infty(z)\,u_1(r,z)$), as $z \downarrow \lambda$. Note also that when $m_\infty(z) \to \infty$ as $z \downarrow \lambda \in \mathbf{R}$, it follows from (4.3) that $\sigma_{11} = 1$ and $\sigma_{12} = 0$, so that $v(r,\lambda) = u_1(r,\lambda)$, and hence is an eigenfunction of H_∞; this is also directly evident from (4.4) above. In both cases we see that $v(r,\lambda)$ is a real solution of $Lu = \lambda u$ which is subordinate at ∞. Also, since $m_{-\infty}(\lambda)$ is strictly complex for $\lambda \in S_1(\lambda)$, we infer fom the theory of subordinacy that all solutions are of comparable asymptotic size as $r \to -\infty$. Substituting for $v(r,\lambda)$ from (4.4) into (4.1) now yields for $f(r) \in L_2(\mathbf{R})$,

$$f(r) = \operatorname*{l.i.m.}_{\omega\to\infty} \int_\omega^\omega \frac{u_2(r,\lambda) + m_\infty(\lambda)\, u_1(r,\lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}}\, G(\lambda)\; d\rho_\tau(\lambda), \tag{4.5}$$

where

$$G(\lambda) = \operatorname*{l.i.m.}_{\eta\to\infty} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_\infty(\lambda)\, u_1(r,\lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}}\; f(r)\, dr,$$

withe convergence as before.

We illustrate the process outlined above by considering two operators, both of which are in the limit point case at $\pm\infty$ and have purely absolutely continuous spectrum.

Example 1 Let

$$q(r) = \begin{cases} 0 & -\infty < r < 0, \\ 1 & 0 \le r < \infty, \end{cases} \tag{4.6}$$

and define a fundamental set of solutions $\{u_1(r,z), u_2(r,z)\}$ of $Lu = zu$ as in (2.1). Using the facts that for $\mathrm{Im} z > 0$,

$$\exp(-\mathrm{i}\sqrt{z}r) \in L_2(\mathbf{R}^-), \qquad \exp(\mathrm{i}\sqrt{z-1}r) \in L_2(\mathbf{R}^+),$$

it is straightforward to show that for $z \in \mathbf{C}^+$,

$$m_{-\infty}(z) = -\mathrm{i}\sqrt{z}, \qquad m_\infty(z) = \mathrm{i}\sqrt{z-1},$$

so that the boundary values for $\lambda \in \mathbf{R}$ are given by

$$m_{-\infty}(\lambda) = -\mathrm{i}\sqrt{\lambda}, \qquad m_\infty(\lambda) = \mathrm{i}\sqrt{\lambda - 1}, \tag{4.7}$$

respectively. We remark that $m_{-\infty}(\lambda)$ is real for $\lambda \le 0$ and complex for $\lambda > 0$, while $m_\infty(\lambda)$ is real for $\lambda \le 1$ and complex for $\lambda > 1$. Thus we have $S_1(\lambda) = (0,1]$, $S_2(\lambda) = S_3(\lambda) = \emptyset$ and $S_4(\lambda) = S_d = (1,\infty)$, where S_d is as in Corollary 1. Since both $m_{-\infty}(\lambda)$ and $m_\infty(\lambda)$ are real for $\lambda \le 0$, we may take $(\sigma_{ij}(\lambda))$ to be the zero matrix as in (3.5) for $\lambda \le 0]$, and we note that the resolvent set is $(-\infty, 0)$.

It is straightforward to determine the spectral density matrix explicitly for $\lambda > 0$, using (3.6) - (3.8) and noting that $m_\infty(\lambda) = -\sqrt{1-\lambda}$ for $\lambda \in (0, 1]$. The derivation of the spectral density matrix for $\lambda > 1$, which has full rank, may be found in [8]. In the case where $0 < \lambda \leq 1$, we construct the spectral density matrix (3.4) using (3.6) - (3.8), to give

$$\left(\frac{d\rho_{ij}}{d\rho_\tau}(\lambda)\right) = \frac{1}{2-\lambda}\begin{pmatrix} 1-\lambda & -\sqrt{1-\lambda} \\ -\sqrt{1-\lambda} & 1 \end{pmatrix}, \tag{4.8}$$

which has rank 1, from which it follows from Theorem 1 that spectrum of H is simple on $(0, 1]$. Moreover, the spectral density matrix in (4.8) may be decomposed as in (3.10), to give

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2-\lambda}}\begin{pmatrix} -\sqrt{1-\lambda} & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.9}$$

where $\lambda \in (0, 1]$. Thus from (4.7) and (4.9),

$$\begin{aligned} PU &= \left(-\frac{\sqrt{1-\lambda}}{\sqrt{2-\lambda}}u_1(r,\lambda) + \frac{1}{\sqrt{2-\lambda}}u_2(r,\lambda)\right) \\ &= \frac{u_2(r,\lambda) + m_\infty(\lambda)u_1(r,\lambda)}{(1+(m_\infty(\lambda))^2)^{\frac{1}{2}}} \\ &= v(r,\lambda), \end{aligned} \tag{4.10}$$

for $\lambda \in (0, 1]$, where $u_1(r,\lambda)$, $u_2(r,\lambda)$ satisfy

$$u_1(r,\lambda) = \begin{cases} \frac{1}{2\sqrt{1-\lambda}}\left(e^{\sqrt{1-\lambda}r} - e^{-\sqrt{1-\lambda}r}\right), & r \geq 0 \\ \frac{\sin(\sqrt{\lambda}r)}{\sqrt{\lambda}}, & r < 0 \end{cases} \tag{4.11}$$

$$u_2(r,\lambda) = \begin{cases} \frac{1}{2}\left(e^{\sqrt{1-\lambda}r} + e^{-\sqrt{1-\lambda}r}\right), & r \geq 0 \\ \cos\left(\sqrt{\lambda}r\right), & r < 0 \end{cases} \tag{4.12}$$

for $\lambda \in (0, 1]$.

Note from (4.11) and (4.12) that no solution of $Lu = \lambda u$ is subordinate at $-\infty$, for $\lambda \in (0, 1]$, and that the eigenfunction $v(r,\lambda)$ satisfies (4.5) and is subordinate at ∞. Furthermore, since $m_\infty(\lambda)$ is real for $\lambda \in (0, 1]$, we have from (3.1) and (3.2),

$$\begin{aligned} \rho'_\tau(\lambda) &= \frac{1}{\pi}\frac{-\text{Im }m_{-\infty}(\lambda)\left(1+(m_\infty(\lambda))^2\right)}{\mid m_{-\infty}(\lambda) - m_\infty(\lambda)\mid^2}, \qquad 0 < \lambda \leq 1, \\ &= \frac{1}{\pi}\sqrt{\lambda}\,(2-\lambda). \end{aligned} \tag{4.13}$$

In this example, since $m_\infty(\lambda)$ is finite for all values of λ, we may multiply (4.10) by $(1+(m_\infty(\lambda))^2)^{\frac{1}{2}}$ and divide (4.13) by $(1+(m_\infty(\lambda))^2)$ to obtain a simpler form of the expansion which holds on the λ-region $S_1 = (0, 1]$. The part of the expansion corresponding to the simple spectrum of H on the λ-set $(0, 1]$ is given by the following spectral projection:

$$E_\lambda((0,1])f(r) = \frac{1}{\pi}\int_0^1 e^{-\sqrt{1-\lambda}\,r}\, G(\lambda)\,\sqrt{\lambda}\, d\lambda,$$

where

$$G(\lambda) = \int_{-\infty}^{\infty} e^{-\sqrt{1-\lambda}\,r}\, f(r)\, dr,$$

and the integrals converge absolutely. Note that the full expansion would also include the λ-region $S_4 = (1, \infty)$, on which the spectrum is purely absolutely continuous with multiplicity 2, and the resolvent set is $(-\infty, 0)$.

Example 2 Consider the Airy operator associated with the singular Sturm-Liouville equation,

$$-u''(r,\lambda) + r\,u(r,\lambda) = \lambda u(r,\lambda), \qquad -\infty < r < \infty. \tag{4.14}$$

It is well known that a fundamental set of solutions for (4.9) is given by $\{A\mathrm{i}(r-\lambda), B\mathrm{i}(r-\lambda)\}$, and that the differential equation is in the limit point case at both endpoints [5]. Moreover, Ai and Bi can be expressed in terms of Bessel functions [1], [13], and the solution $A\mathrm{i}(r-\lambda)$, $\lambda \in \mathbf{R}$, is real valued and square integrable at infinity with respect to r for all $\lambda \in \mathbf{R}$. It follows that $A\mathrm{i}(r-\lambda)$ is subordinate at ∞ [9], from which it may be inferred that

$$m_\infty(\lambda) = \lim_{z\downarrow\lambda} \left.\frac{\mathrm{Ai}'(r-\lambda)}{\mathrm{Ai}(r-\lambda)}\right|_{r=0}, \tag{4.15}$$

for all $\lambda \in \mathbf{R}$, where $'$ denotes differentiation with respect to r.

To investigate the asymptotic behaviour of solutions of (4.14) at $-\infty$, we first note that the conditions for validity of the Liouville-Green approximation are satisfied in this case (see [13], Chapter 6). Hence the asymptotic behaviour of a fundamental set, $\{u_+(r,\lambda), u_-(r,\lambda)\}$, of solutions of (4.14) as $r \to -\infty$ is given by:

$$u_\pm(r,\lambda) = \frac{1}{(r-\lambda)^{\frac{1}{4}}} \exp\left(\int^r \pm(r-\lambda)^{\frac{1}{2}}\, dr\right)(1 + o(1)). \tag{4.16}$$

Since for each fixed $\lambda \in \mathbf{R}$, $(r-\lambda)$ is eventually negative as $r \to -\infty$, we may write

$$u_\pm(r,\lambda) = K\,\frac{1}{(\lambda-r)^{\frac{1}{4}}} \exp\left(\int^r \pm\mathrm{i}(\lambda-r)^{\frac{1}{2}}\, dr\right)(1 + o(1)), \tag{4.17}$$

as $r \to -\infty$ for each $\lambda \in \mathbf{R}$, where K is a constant which is independent of r and λ. It follows from (4.17) that for every real value of λ, no solution of (4.14) is subordinate at $-\infty$, so that $m_{-\infty}(\lambda) \in \mathbf{C}^-$ for all $\lambda \in \mathbf{R}$ [9]. This, together with (4.15), which implies that $m_\infty(\lambda) \in \mathbf{R} \cup \{\infty\}$ for all $\lambda \in \mathbf{R}$, shows that $S_1 = \mathbf{R}$.

Thus we have shown that in the case of the Airy operator, the spectrum of H is purely absolutely continuous with multiplicity one on the whole real line, and that the absolutely continuous eigenfunctions which feature in the expansion (4.1) are given by

$$v(r,\lambda) = \frac{u_2(r,\lambda) + m_\infty(\lambda) u_1(r,\lambda)}{(1 + (m_\infty(\lambda))^2)^{1/2}}, \tag{4.18}$$

where $m_\infty(\lambda)$ is as in (4.15). In this example, the form of the eigenfunction $v(r,\lambda)$ in (4.1) is convenient as it stands, since it accommodates the case where λ is an eigenvalue of H_∞, that is, when $m_\infty(z) \to \infty$ as $z \downarrow \lambda$, so that $v(r,\lambda) = u_1(r,\lambda)$.

To determine the scalar Herglotz function, $M(z)$ explicitly, it is first necessary to identify (up to a scalar multiple) the Weyl solution of $Lu = z\,u$, which is in $L_2(\mathbf{R}^-)$ for $z \in \mathbf{C}^+$. Then $m_{-\infty}$ can be obtained by evaluating the logarithmic derivative of the Weyl solution at $r = 0$, as in (4.14).

These examples confirm that when part or all of the the spectrum is absolutely continuous with multiplicity one, the corresponding eigenfunctions in the expansion (4.1) are solutions of $L\,u = \lambda u$ which are subordinate at one of the limit point endpoints, and are of comparable asymptotic size to all linearly independent solutions of the same equation at the other endpoint. In cases where both $S_1(\lambda)$ and $S_2(\lambda)$ have positive μ_τ-measure, but $S_3(\lambda)$ and $S_4(\lambda)$ are empty, the simplified expansions will still valid on the relevant λ-set, but non-overlapping spectral projections are needed to separate the corresponding parts of the expansion. Thus we expect the absolutely continuous eigenfunctions associated with $S_1(\lambda)$ and $S_2(\lambda)$ respectively to be real λ-multiples of $u_2(r,\lambda) + m_\infty\, u_1(r,\lambda)$ and $u_2(r,\lambda) + m_{-\infty}\, u_1(r,\lambda)$, and the corresponding spectral densities to be positive λ-multiples of $\rho_\infty(\lambda)$ and $\rho_{-\infty}(\lambda)$, so that for $f(r) \in L_2(-\infty,\infty)$,

$$\begin{aligned} f(r) \quad &= \quad \underset{\omega\to\infty}{\text{l.i.m.}} \left\{ \int_{(-\omega,\omega)\cap S_1} \frac{u_2(r,\lambda) + m_\infty(\lambda) u_1(r,\lambda)}{|\, m_{-\infty}(\lambda) - m_\infty(\lambda)\,|}\, G_+(\lambda)\, d\rho_{-\infty}(\lambda) \right. \\ &\quad + \left. \int_{(-\omega,\omega)\cap S_2} \frac{u_2(r,\lambda) + m_{-\infty}(\lambda)\, u_1(r,\lambda)}{|\, m_{-\infty}(\lambda) - m_\infty(\lambda)\,|}\, G_-(\lambda)\, d\rho_\infty(\lambda) \right\}, \end{aligned}$$

where

$$\begin{aligned} G_+(r,\lambda) \quad &= \quad \underset{\eta\to\infty}{\text{l.i.m.}} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_\infty(\lambda) u_1(r,\lambda)}{|\, m_{-\infty}(\lambda) - m_\infty(\lambda)\,|} f(r)\, dr, \\ G_-(r,\lambda) \quad &= \quad \underset{\eta\to\infty}{\text{l.i.m.}} \int_{-\eta}^{\eta} \frac{u_2(r,\lambda) + m_{-\infty}(\lambda) u_1(r,\lambda)}{|\, m_{-\infty}(\lambda) - m_\infty(\lambda)\,|} f(r)\, dr. \end{aligned}$$

with convergence in $L_2(\mathbf{R})$, $L_2(\mathbf{R}; d\rho_{-\infty})$ and $L_2(\mathbf{R}; d\rho_\infty)$ respectively. This form of the eigenfunction expansion clearly demonstrates the relationships between properties of the spectrum and the asymptotic behaviour of the absolutely continuous eigenfunctions, and indicates the contribution of the Titchmarsh-Weyl m-functions and associated spectral densities of $H_{-\infty}$ and H_{∞} to the structure of the full-line expansion for H.

Acknowledgements The author wishes to thank Hubert Kalf for useful discussions and Murray Edwards College (formerly New Hall), University of Cambridge, for a Visiting Scholarship since 2008.

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Daphne J. Gilbert
School of Mathematical Sciences
Dublin Institute of Technology
Ireland
email: daphne.gilbert@dit.ie