2000

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Recommended Citation

R. Ivanov and M. Tuite, Rational Generalised Moonshine from Orbifolds, Balkan Physics Letters, BPU-4 Supplement, (2000) 523-526.

BALKAN PHYSICS LETTERS
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BPL (Proc. Suppl.) BPU-4 (2000)

RATIONAL GENERALISED MOONSHINE FROM ORBIFOLDS

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Abstract

Frenkel, Lepowsky and Meurman constructed the Moonshine Module as a $\mathbb{Z}_2$ orbifold of the Leech lattice Meromorphic Conformal Field Theory. The group of automorphisms of this theory is the 'Monster Group' $\mathbb{M}$ - the largest finite sporadic simple group (with order $\sim 8.10^{53}$). 'Monstrous Moonshine' is the famous observation that the Thompson series, corresponding to each class of $\mathbb{M}$, is a hauptmodul for some genus zero fixing group. Norton considered Generalised Moonshine Functions (GMF), depending on two commuting Monster elements and suggested that they are also hauptmoduls. Using meromorphic abelian orbifoldings of the Moonshine Module we identify the singularity structure of the GMF in some nontrivial cases so that the genus zero property is demonstrated and the corresponding genus zero fixing group is identified.

1.Self-Dual Meromorphic Conformal Field Theories

Let $\mathcal{V}$ denote a Self-Dual Meromorphic Conformal Field Theory (MCFT) [Go] of central charge $C = 24$. We use the same letter $\mathcal{V}$ to denote the corresponding Hilbert space of the MCFT. The characteristic function (or genus one partition function) for $\mathcal{V}$ is given by $Z(\tau) = Tr_{\mathcal{V}}(q^{L_0-1})$, $q = e^{2\pi i\tau}$, where $\tau \in \mathbb{H}$, the upper half complex plane, is the usual elliptic modular parameter. $\mathcal{V}$ has integral grading (the conformal weight) with respect to L_0, the Virasoro level operator so that $Z(\tau)$ is invariant under $T : \tau \to \tau + 1$. Furthermore, self-duality of $\mathcal{V}$ implies invariance under $S : \tau \to -1/\tau$ so that $Z(\tau)$ is invariant under the modular group $\Gamma = PSL(2,\mathbb{Z})$, generated by S, T . Hence $Z(\tau)$ is given up to an additive constant by $J(\tau)$, the hauptmodul for Γ,[Se]: $Z(\tau) = J(\tau) + N_0$,

$$J(\tau) = \frac{E_4^3(\tau)}{\eta^{24}(\tau)} - 744 = \frac{1}{q} + 0 + 196884q + 21493760q^2 + ... \qquad (1)$$

where $\eta(\tau)$ is the Dedekind eta function, $E_n(\tau)$ is the Eisenstein form of weight n [Se] and N_0 is the number of conformal weight 1 operators in $\mathcal{V}$. For the Leech lattice MCFT $N_0 = 24$ and for the FLM Moonshine Module $\mathcal{V}^\natural$ [FLM] $N_0 = 0$, which means that the latter does not contain conformal dimension 1 operators i.e. there is an absence of the usual Kac-Moody symmetry.

2. $\mathbb{Z}_n$ Orbifoldings of Self-Dual MCFTs

Let $G = \mathrm{Aut}(\mathcal{V})$, denote the automorphism group for $\mathcal{V}$ and $\mathcal{V}_g$ the g-twisted sector of the Hilbert space for $g \in G$ of finite order n. Then $\mathrm{Aut}(\mathcal{V}_g)$ is an extension of $C_g = \{h \in$

$G|gh = hg\}$, the centraliser of g in G. Since $\mathcal{V}_g$ is unique for a self-dual MCFT we have $\mathrm{Aut}(\mathcal{V}_g) = U(1).C_g$. If the twisted vacuum is unique then $\mathrm{Aut}(\mathcal{V}_g) = U(1) \times C_g$. We can define the general trace function for $\hat{h} \in \mathrm{Aut}(\mathcal{V}_g)$ lifted from $h \in C_g$ by

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) \equiv \mathrm{Tr}_{\mathcal{V}_g}(\hat{h}q^{L_0-1}) = \chi_g^0(\hat{h})q^{E_0^g} + ..., \tag{2}$$

where χ_g^0 denotes a character of $\hat{h}$ for ρ_g^0 a representation for $\mathrm{Aut}(\mathcal{V}_g)$ corresponding to the vacuum subspace, E_0^g is the vacuum energy of $\mathcal{V}_g$. We say that g is a Normal element of G if $nE_0^g \in \mathbb{Z}$ so that $\hat{g}$ is of order n, otherwise we say that g is an Anomalous element of G. We assume below that a particular choice h for $\hat{h}$ can be made which resolves the ambiguity inherent in the notation $Z\left[\begin{array}{c} h \\ g \end{array}\right]$. When $\dim(\rho_g^0) = 1$, then up to a trivial $U(1)$ phase, $\chi_g^0(\hat{h})$ is a character for C_g. For general commuting elements g, h, and for the appropriate lifting the trace function transforms under $\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2,\mathbb{Z})$ where $\gamma\tau = \frac{a\tau+b}{c\tau+d}$ as follows:

$$Z\left[\begin{array}{c} h \\ g \end{array}\right](\gamma\tau) = Z\left[\begin{array}{c} h^d g^{-b} \\ h^{-c}g^a \end{array}\right](\tau), \tag{3}$$

Assume that all elements of $\langle g \rangle \simeq \mathbb{Z}_n$, the abelian group of order n generated by g, are normal elements of $\mathrm{Aut}(\mathcal{V})$. Let $\mathcal{P}_H \equiv \frac{1}{|H|}\sum\limits_{u\in H} u$ denotes the projection with respect to the group H of order $|H|$. The $\langle g \rangle$ orbifold MCFT $\mathcal{V}_{\mathrm{orb}}^{\langle g \rangle}$ is the MCFT with Hilbert space $\mathcal{V}_{\mathrm{orb}}^{\langle g \rangle} \equiv \mathcal{P}_{\langle g \rangle}(\oplus_{k=0}^{n-1}\mathcal{V}_{g^k})$. The characteristic function of $\mathcal{V}_{\mathrm{orb}}^{\langle g \rangle}$ is $Z_{\mathrm{orb}}^{\langle g \rangle} = \frac{1}{n}\sum_{l,k=0}^{n-1} Z\left[\begin{array}{c} g^l \\ g^k \end{array}\right]$ which is a modular invariant from (3). Hence $\mathcal{V}_{\mathrm{orb}}^{\langle g \rangle}$ is a self-dual MCFT and so $Z_{\mathrm{orb}}^{\langle g \rangle}(\tau) = J(\tau) + N_0^{\langle g \rangle}$.

Similarly, $\mathcal{V}_{\mathrm{orb}}^{\langle g,h \rangle}$ is the orbifolding of $\mathcal{V}$ with respect to the abelian group $\langle g, h \rangle$ generated by two commuting elements g, h where all the elements of $\langle g, h \rangle$ are assumed to be normal. The Hilbert space is $\mathcal{V}_{\mathrm{orb}}^{\langle g,h \rangle} = \mathcal{P}_{\langle g,h \rangle}(\oplus_{u\in\langle g,h \rangle}\mathcal{V}_u)$ and the characteristic function is similarly modular invariant. $\mathcal{V}_{\mathrm{orb}}^{\langle g,h \rangle}$ is therefore a self-dual MCFT with $Z_{\mathrm{orb}}^{\langle g,h \rangle}(\tau) = J(\tau) + N_0^{\langle g,h \rangle}$.

For independent generators g, h of $\langle g, h \rangle$ (i.e. $g^A \neq h^B$ for all $(A, B) \neq (0, 0)$) $|\langle g, h \rangle| = |\langle g \rangle|.|\langle h \rangle|$ and $\mathcal{P}_{\langle g,h \rangle} = \mathcal{P}_{\langle g \rangle}\mathcal{P}_{\langle h \rangle}$. Then $\mathcal{V}_{\mathrm{orb}}^{\langle g,h \rangle}$ can be considered as a composition of orbifoldings: $\mathcal{V}_{\mathrm{orb}}^{\langle g,h \rangle} = (\mathcal{V}_{\mathrm{orb}}^{\langle h \rangle})_{\mathrm{orb}}^{\langle g \rangle} = (\mathcal{V}_{\mathrm{orb}}^{\langle g \rangle})_{\mathrm{orb}}^{\langle h \rangle}$.

3. Monstrous Moonshine

In the case where $\mathcal{V} = \mathcal{V}^\natural$, the Moonshine Module, $G = \mathbb{M}$, the Monster group [FLM]. Conway and Norton [CN] suggested and Borcherds proved the following famous result known as **Monstrous Moonshine**: *For each $g \in \mathbb{M}$, each Thompson series $T_g(\tau) \equiv Z\left[\begin{array}{c} g \\ 1 \end{array}\right](\tau) \equiv Tr_{\mathcal{V}^\natural}(gq^{L_0-1})$ is the hauptmodul for a genus zero fixing modular group Γ_g.*

For each $g \in \mathbb{M}$, $T_g(\tau) = \frac{1}{q} + 0 + [1 + \chi_A(g)]q + ...$ where $\chi_A(g)$ is the character of the 196883 dimensional adjoint representation for $\mathbb{M}$. The Thompson series for the identity element is $J(\tau)$, (1) which is the hauptmodul for the genus zero modular group $\Gamma = PSL(2,\mathbb{Z})$ as already stated.

In general, for g of order n, $T_g(\tau)$ is found to be $\Gamma_0(n)$ invariant up to m^{th} roots of unity where $m|n$ and $m|24$. $T_g(\tau)$ is fixed by some Γ_g with $\Gamma_0(N) \subseteq \Gamma_g$ and contained in the normalizer of $\Gamma_0(N)$ in $SL(2,\mathbb{R})$ where $N = nm$ [CN]. This normalizer always contains the Fricke involution $W_N : \tau \to -1/N\tau$. The classes with $m = 1$ are normal and those with $m \neq 1$ are anomalous.

All the classes of $\mathbb{M}$ can be divided into Fricke and non-Fricke classes according to whether or not $T_g(\tau)$ is invariant under W_N. There are a total of 51 non-Fricke classes of which 38 are normal and there are a total of 120 Fricke classes of which 82 are normal.

Assuming the FLM uniqueness conjecture for $\mathcal{V}^\natural$ (i.e. that $\mathcal{V}^\natural$ is the unique self-dual $C = 24$ MCFT with characteristic function $J(\tau)$) Monstrous Moonshine is equivalent to the fact that for all normal elements of $\mathbb{M}$ $(\mathcal{V}^\natural)^{\langle g\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ when g is Fricke and $(\mathcal{V}^\natural)^{\langle g\rangle}_{\text{orb}} \simeq \mathcal{V}^\Lambda$ (the Leech lattice MCFT) when g is non-Fricke, [T1], [T2].

4. Generalised Moonshine

Norton has proposed the following [N] **Generalised Moonshine Conjecture:** $Z\begin{bmatrix} h \\ g \end{bmatrix}$ *is either constant or is a hauptmodul for some genus zero fixing group for all* $g \in \mathbb{M}$ and $h \in C_g$.

Using orbifold methods we can analyse Generalised Moonshine Functions (GMFs) $Z\begin{bmatrix} h \\ g \end{bmatrix}$ where g is assumed to be Fricke and of prime order p for all possible primes $p = 2, 3, 5, ..., 31, 41, 47, 59, 71$ (when all elements of $\langle g, h\rangle$ are non-Fricke the GMF is constant).

If $\langle g, h\rangle \simeq \mathbb{Z}_n$ then the corresponding GMF can always be related to a regular Thompson series via an appropriate modular transformation [T2] and in these cases, the genus zero property trivially follows from that for a regular Thompson series. So we consider only non $\mathbb{Z}_n$ cases. It is not hard to see that it is sufficient to consider only independent pairs g, h. The GMFs with non-trivial genus zero behaviour then occur for $h \in C_g$ of order pk for $k \geq 1$ where $p \leq 13$. For simplicity we consider only rational GMFs i.e. with q expansion with rational (and therefore integral) coefficients. This implies that the GMF has leading q expansion

$$Z\begin{bmatrix} h \\ g \end{bmatrix}(\tau) = \frac{1}{q^{1/p}} + 0 + O(q^{1/p}), \tag{4}$$

i.e. $\rho^0_g(h) = \chi^0_g(h) = 1$ since for g-Fricke, $\dim(\rho^0_g) = 1$. From (3) it is clear that (4) is always $\Gamma^0_0(pk, p) \sim \Gamma_0(p^2k)$ invariant. The value of (4) at any parabolic cusp $\tau = a/c$ (a and c co-prime) is determined by the vacuum energy of the $u = g^a h^{-c}$ twisted sector. In particular, only the Fricke classes are responsible for singular cusp points [N]. The residue of these cusps is determined by $\rho^0_u(h^d g^{-b})$. Now we will illustrate through the following example how using orbifoldings we can find these residues, and therefore we can demonstrate the invariance of the GMF with respect to a genus zero group.

Example. Let g and h be the only Fricke elements of $\langle g, h\rangle$. Let $\mathcal{V} = \mathcal{V}^\natural$. Then $\mathcal{V}^{\langle h\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ and by assumption $\rho^0_g(h) = 1$ so that $(\mathcal{V}^{\langle h\rangle}_{\text{orb}})_g \equiv \mathcal{P}_h(\mathcal{V}^\natural_g \oplus \mathcal{V}^\natural_{gh} \oplus \mathcal{V}^\natural_{gh^2} \oplus ...)$ is tachyonic (i.e. with negative vacuum energy). Therefore g acts as a Fricke element on $\mathcal{V}^{\langle h\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ and orbifolding the latter with respect to g we find that $\mathcal{V}^{\langle g,h\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$. Consider $\mathcal{V}^\natural \simeq \mathcal{V}^{\langle g,h\rangle}_{\text{orb}} \equiv \mathcal{P}_h(\mathcal{V}^{\langle g\rangle}_{\text{orb}} \oplus (\mathcal{V}^{\langle g\rangle}_{\text{orb}})_h \oplus (\mathcal{V}^{\langle g\rangle}_{\text{orb}})_{h^2} \oplus ... \oplus (\mathcal{V}^{\langle g\rangle}_{\text{orb}})_{h^{pk-1}})$. Since g is Fricke $\mathcal{V}^{\langle g\rangle}_{\text{orb}} \simeq \mathcal{V}^\natural$ and therefore h acts as a Fricke element on $\mathcal{V}^{(g)}_{\rm orb}$ of the same order pk. Hence $(\mathcal{V}^{(g)}_{\rm orb})_h = \mathcal{P}_g(\mathcal{V}^\natural_h \oplus \mathcal{V}^\natural_{gh} \oplus \mathcal{V}^\natural_{g^2h} \oplus ...)$ is tachyonic which is possible iff $\mathcal{P}_g = 1$ on precisely one of the tachyonic vacuum sectors $(\mathcal{V}^\natural_{g^sh})^{(0)}$ for some unique s. Then g^sh is a Fricke element and $\rho^0_{g^sh}(g) = 1$. But by assumption g^sh is a Fricke element iff it is in one of the classes g or h. The only possibility is $g^sh = h$ $(s = 0 \mod p)$, so that $\rho^0_h(g) = 1$. Therefore having in mind $\dim(\rho^0_h) = 1$ we have

$$Z\left[\begin{array}{c} g^{-1} \\ h \end{array}\right](\tau) = \chi^0_h(g^{-1})q^{-1/pk} + 0 + O(q^{1/pk}) = q^{-1/pk} + 0 + O(q^{1/pk}).$$

Consider the transformation $\hat{W}_{p^2k}$: $\tau \to -1/k\tau$ which is conjugate to W_{p^2k}. The function $f(\tau) = Z\left[\begin{array}{c} h \\ g \end{array}\right](\hat{W}_{p^2k}\tau) - Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = Z\left[\begin{array}{c} g^{-1} \\ h \end{array}\right](k\tau) - Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) = 0 + O(q^{1/p})$ is $\Gamma^0_0(pk,p)$ invariant, without poles on $\mathbb{H}\,/\Gamma^0_0(pk,p)$ and hence is constant and equal to zero. Therefore $Z\left[\begin{array}{c} h \\ g \end{array}\right](p\tau)$ has a unique pole and therefore is a hauptmodul for the genus zero invariance group $\Gamma_0(p^2k) + W_{p^2k}$. Such a modular group exists only when $p = 2,3,5,7$ and $k = 1$; $p = 2$ and $k = 3,5$; $p = 3,5$ and $k = 2$; $p = k = 2,3$; $p = 2$ and $k = 6,9$; $p = 3$ and $k = 4$. Thus there are non-trivial constraints on the possible class structure of $\mathbb{M}$. □

Similarly, in [IT] we consider in detail all possible cases for $k = 1$, $k = p$ and k prime with $k \neq p$. In all these cases the GMF is shown to be a hauptmodul for a genus zero invariance group verifying the Generalized Moonshine conjecture in these cases.

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