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$K(\pi, 1)$ 'S FOR ARTIN GROUPS OF FINITE TYPE

THOMAS BRADY AND COLUM WATT

1. INTRODUCTION.

This paper is a continuation of a programme to construct new $K(\pi, 1)$'s for Artin groups of finite type which began in [4] with Artin groups on 2 and 3 generators and was extended to braid groups in [3]. These $K(\pi, 1)$'s differ from those in [6] in that their universal covers are simplicial complexes.

In [4] a complex is constructed whose top-dimensional cells correspond to minimal factorizations of a Coxeter element as a product of reflections in a finite Coxeter group. Asphericity is established in low dimensions using a metric of non-positive curvature. Since the non-positive curvature condition is difficult to check in higher dimensions a combinatorial approach is used in [3] in the case of the braid groups.

It is clear from [3] that the techniques used can be applied to any finite Coxeter group W . When W is equipped with the partial order given by reflection length and γ is a Coxeter element in W , the construction of the $K(\pi, 1)$'s is exactly analogous provided that the interval $[I, \gamma]$ forms a lattice. In dimension 3, see [4], establishing this condition amounts to observing that two planes through the origin meet in a unique line. In the braid group case, see [3], where the reflections are transpositions and the Coxeter element is an n -cycle this lattice property is established by identifying $[I, \gamma]$ with the lattice of non-crossing partitions of $\{1, 2, \dots, n\}$.

In this paper, we consider the Artin groups of type C_n and D_n . Thus, for each finite reflection group W of type C_n or D_n , partially ordered by reflection length, we identify a lattice inside W and use it to construct a finite aspherical complex $K(W)$. In the C_n case this lattice coincides with the lattice of noncrossing partitions of $\{1, 2, \dots, n, -1, \dots, -n\}$ studied in [8]. The final ingredient is to prove that $\pi_1(K(W))$ is isomorphic to $A(W)$, the associated finite type Artin group. As in [4] and [3] this involves a lengthy check that the obvious maps between the two presentations are well-defined.

David Bessis has independently obtained similar results which can be seen at [1]. His approach exploits in a clever way the extra structure given by viewing these groups as complex reflection groups. In addition, he has verified that in the exceptional cases that the interval $[I, \gamma]$ forms a lattice and that the corresponding poset groups are isomorphic to the respective Artin groups of finite type. Combined with the results of our section 5 below this provides the new $K(\pi, 1)$'s in these cases and we thank him for drawing our attention to this fact.

In section 2 we collect some general facts about the reflection length function on finite reflection groups and the induced partial order. In section 3 we study the cube group C_n and its index two subgroup D_n . In section 4 we identify the subposets of interest in C_n and D_n and show that they are lattices. In section 5 we define the poset group $\Gamma(W, \alpha)$ associated to the interval $[I, \alpha]$ for $\alpha \in W$. In the case where $[I, \alpha]$ is a lattice we construct the complexes $K(W, \alpha)$ and show that they are $K(\pi, 1)$'s. Section 6 shows that the groups $\Gamma(C_n, \gamma_C)$ and $\Gamma(D_n, \gamma_D)$ are indeed the Artin groups of the appropriate type when γ_C and γ_D are the respective Coxeter elements.

2. A PARTIAL ORDER ON FINITE REFLECTION GROUPS.

Let W be a finite reflection group with reflection set \mathcal{R} and identity element I . We let $d : W \times W \rightarrow \mathbf{Z}$ be the distance function in the Cayley graph of W with generating set \mathcal{R} and define the *reflection length function* $l : W \rightarrow \mathbf{Z}$ by $l(w) = d(I, w)$. So $l(w)$ is the length of the shortest product of reflections yielding the element w . It follows from the triangle inequality for d that $l(w) \leq l(u) + l(u^{-1}w)$ for any $u, w \in W$.

Definition 2.1. *We introduce the relation \leq on W by declaring*

$$u \leq w \quad \Leftrightarrow \quad l(w) = l(u) + l(u^{-1}w).$$

Thus $u \leq w$ if and only if there is a geodesic in the Cayley graph from I to w which passes through u . Alternatively, equality occurs if and only if there is a shortest factorisation of u as a product of reflections which is a prefix of a shortest factorisation of w . It is readily shown that \leq is reflexive, antisymmetric and transitive so that (W, \leq) becomes a partially ordered set.

Since $(u^{-1}w)^{-1}w = w^{-1}uw$ is conjugate to u it follows that $u^{-1}w \leq w$ whenever $u \leq w$. Furthermore, whenever $\alpha \leq \beta \leq \gamma$ we have

$$l(\gamma) = l(\alpha) + (l(\alpha^{-1}\beta) + l(\beta^{-1}\gamma)),$$

so that $\alpha^{-1}\beta \leq \alpha^{-1}\gamma$.

We recall some general facts about orthogonal transformations from [5]. If $A \in O(n)$, we associate to A two subspaces of \mathbf{R}^n , namely

$$M(A) = \text{im}(A - I) \quad \text{and} \quad F(A) = \text{ker}(A - I).$$

We recall that $M(A)^\perp = F(A)$. We use the notation $|V|$ for $\dim(V)$ when V is a subspace of \mathbf{R}^n . It is shown in [5] that

$$|M(AC)| \leq |M(A)| + |M(C)|$$

We define a partial order on $O(n)$ by

$$A \leq_o B \quad \Leftrightarrow \quad |M(B)| = |M(A)| + |M(A^{-1}B)|$$

and we note that $A \leq_o B$ if and only if $M(B) = M(A) \oplus M(A^{-1}B)$. In particular $A \leq_o B$ implies that $M(A) \subseteq M(B)$ or equivalently $F(B) \subseteq F(A)$. The main result we will use from [5] is that for each $A \in O(n)$ and each subspace V of $M(A)$ there exists a unique $B \in O(n)$ with $B \leq_o A$ and $M(B) = V$.

Our finite reflection group W is a subgroup of $O(n)$, so the results of [5] can be applied to the elements of W . We begin with a geometric interpretation of the length function l on W .

Proposition 2.2. $l(\alpha) = |M(\alpha)| = n - |F(\alpha)|$, for $\alpha \in W$.

Proof. First note that the proposition holds when $\alpha = I$ so we will assume $\alpha \neq I$ and let $k = |M(\alpha)| > 0$.

To establish the inequality $l(\alpha) \leq k$ we show that α can be expressed as a product of k reflections. We will use induction on k noting that the case $k = 1$ is immediate. Consider the subspace $F(\alpha) \neq \mathbf{R}^n$. Recall from part (d) of Theorem 1.12 of [7] that the subgroup W' of W of elements which fix $F(\alpha)$ pointwise is generated by those reflections R in W satisfying $F(\alpha) \subset F(R)$. Since $\alpha \neq I$ there exists at least one such reflection R . Since $M(A) = F(A)^\perp$ we have $M(R) \subset M(\alpha)$. The unique orthogonal transformation induced on $M(R)$ by α must be R by Corollary 3 of [5]. Hence $R \leq_o \alpha$ and

$$|M(R\alpha)| = |M(\alpha)| - |M(R)| = k - 1.$$

By induction $R\alpha$ can be expressed as a product of $k - 1$ reflections and hence there is an expression $\alpha = R_1 \dots R_k$ for α as a product of k reflections. We note that by construction each of these reflections R_i satisfies $M(R_i) \subset M(\alpha)$.

To establish the other inequality suppose $\alpha = S_1 S_2 \dots S_m$ is an expression for α as a product of m reflections realizing $l(\alpha) = m$. Repeated

use of the identity $|M(AC)| \leq |M(A)| + |M(C)|$ gives

$$k = |M(\alpha)| \leq |M(S_1)| + \cdots + |M(S_m)| = m = l(\alpha). \quad \text{q.e.d.}$$

In particular the partial order \leq on W is a restriction of the partial order \leq_o on $O(n)$ and we will drop the subscript from \leq_o from now on. The following lemma is immediate.

Lemma 2.3. *Let W be a finite Coxeter group with reflection set \mathcal{R} and let W_1 be a subgroup generated by a subset \mathcal{R}_1 of \mathcal{R} . Then the length function for W_1 is equal to the restriction to W_1 of the length function for W .*

Definition 2.4. *For each $\delta \in W$ we define the reflection set of δ , S_δ , by $S_\delta = \{R \in \mathcal{R} \mid r \leq \delta\}$.*

Repeated application of $A \leq B \Rightarrow |M(B)| = |M(A)| + |M(A^{-1}B)|$ gives $M(\delta) = \text{Span}\{M(R) \mid R \leq \delta\}$ so that S_δ determines $M(\delta)$. However, in the case where $\delta \leq \gamma$, δ itself is determined by γ and S_δ since δ is the unique orthogonal transformation induced on $M(\delta)$ by γ . The following results are consequences of this fact.

Lemma 2.5. *If $\alpha, \beta \leq \gamma$ in W and $S_\alpha \subseteq S_\beta$ then $\alpha \leq \beta$.*

Proof. $M(\alpha) \subset M(\beta) \subset M(\gamma)$ and by uniqueness the transformation induced on $M(\alpha)$ by β is the same as the transformation induced by γ , namely α . q.e.d.

Lemma 2.6. *Suppose $\alpha, \beta \leq \gamma$ in W . If there is an element $\delta \in W$ with $\delta \leq \gamma$ and $S_\delta = S_\alpha \cap S_\beta$ then δ is the greatest lower bound of α and β in W , that is, if $\tau \in W$ satisfies $\tau \leq \alpha, \beta$ then $\tau \leq \delta$.*

3. THE CUBE GROUPS C_n AND D_n .

For general facts about the groups C_n and D_n see [2] or [7]. Let $I = [-1, 1]$ and let C_n denote the group of isometries of the cube I^n in \mathbb{R}^n . That is

$$C_n = \{\alpha \in O(n) : \alpha(I^n) = I^n\}$$

Let e_1, \dots, e_n denote the standard basis for \mathbb{R}^n and let x_1, \dots, x_n denote the corresponding coordinates. The set \mathcal{R}_c of all reflections in C_n consists of the following n^2 elements. For each $i = 1, \dots, n$, reflection in the hyperplane $x_i = 0$ is denoted $[i]$ and also by $[-i]$. For each $i \neq j$, reflection in the hyperplane $x_i = x_j$ is denoted by any one of the four expressions $((i, j))$, $((j, i))$, $((-i, -j))$ and $((-j, -i))$, while reflection in the plane $x_i = -x_j$ is denoted by any one of the four expressions $((i, -j))$, $((-i, j))$, $((j, -i))$, and $((-j, i))$. The set of these $n(n-1)$ reflections,

in hyperplanes of the form $x_i = \pm x_j$, is denoted \mathcal{R}_d and the subgroup they generate, D_n , is well known to be an index two subgroup of C_n . The group C_n acts on the set $\{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ in the obvious manner and this action satisfies $\alpha \cdot (-e_i) = -(\alpha \cdot e_i)$ for each i and each $\alpha \in C_n$. Thus we obtain an injective homomorphism p from C_n into the group Σ_{2n} of permutations of the set $\{1, 2, \dots, n, -1, -2, \dots, -n\}$. Note that for each i , $p([i])$ is a transposition in Σ_{2n} , while each element of \mathcal{R}_d is mapped to a product of two disjoint transpositions. Thus $p(D_n)$ is contained in the subgroup of even permutations.

For each cycle $c = (i_1, \dots, i_r)$ in Σ_{2n} , we define the cycle \bar{c} by

$$\bar{c} = (-i_1, \dots, -i_r)$$

Note that $\bar{c} = z_0 c z_0$ where $z_0 = (1, -1)(2, -2) \dots (n, -n)$ has order two. Note also that $z_0 = p(\zeta_0)$ where $\zeta_0 = [1][2] \dots [n]$ is the nontrivial element in the centre of C_n .

Proposition 3.1. *The image $p(C_n)$ is the centraliser $Z(z_0)$ of z_0 in Σ_{2n} . It consists of all products of disjoint cycles of the form*

$$(1) \quad c_1 \bar{c}_1 \dots c_k \bar{c}_k \gamma_1 \dots \gamma_r \quad \text{where } \gamma_j = \bar{\gamma}_j \quad \forall j = 1, \dots, r.$$

The image $p(D_n)$ consists of all elements of the form (1) with r even.

Proof. Since z_0 has order 2 and $z_0 c_1 c_2 \dots c_k z_0 = \bar{c}_1 \bar{c}_2 \dots \bar{c}_k$ for any product of cycles in Σ_{2n} , it follows that the centraliser $Z(z_0)$ consists of those products of disjoint cycles $c_1 c_2 \dots c_k$ for which

$$c_1 c_2 \dots c_k = \bar{c}_1 \bar{c}_2 \dots \bar{c}_k$$

By uniqueness (up to reordering) of cycle decomposition in Σ_{2n} , for each i either $c_i = \bar{c}_j$ for some $j \neq i$ or else $c_i = \bar{c}_i$. It follows that the centraliser of z_0 is precisely the set of elements in Σ_{2n} of the form (1). For each $\alpha \in C_n$, the identity $\zeta_0 \alpha \zeta_0 = \alpha$ implies that $p(\alpha)$ lies in the centraliser of z_0 . Thus $p(C_n) \subset Z(z_0)$. In the reverse direction, if $c = (i_1, \dots, i_k)$ is disjoint from \bar{c} , one may readily verify that

$$(2) \quad c \bar{c} = p((i_1, i_2)(i_2, i_3) \dots (i_{q-1}, i_q))$$

Likewise, if $c = \bar{c}$ then c must be the form $c = (i_1, \dots, i_k, -i_1, \dots, -i_k)$ for some $-n \leq i_1, i_2, \dots, i_k \leq n$ and one may verify that

$$(3) \quad c = (i_1, -i_1)(i_1, i_2)(-i_1, -i_2) \dots (i_{k-1}, i_k)(-i_{k-1}, -i_k)$$

$$(4) \quad = p([i_1](i_1, i_2) \dots (i_{k-1}, i_k))$$

It follows that any element of the form (1) lies in $p(C_n)$ and hence $p(C_n) = Z(z_0)$.

Let $\alpha \in D_n$ and write $p(\alpha) = c_1 \bar{c}_1 \dots c_k \bar{c}_k \gamma_1 \dots \gamma_r$. Since $p(\alpha)$ and each $c_i \bar{c}_i$ is an even permutation while each γ_j is an odd permutation, r must

be even. To show that every element of the form (1) with r even is in $p(D_n)$, we need only note the following facts.

- If the cycle c is disjoint from \bar{c} then equation (2) implies that $c\bar{c} \in p(D_n)$.
- If $i \neq j$ then $[i][j] = (i, j)(i, -j)$ and hence is an element of $p(D_n)$. It now follows from equation (3) that if $c_1 = \bar{c}_1$ and $c_2 = \bar{c}_2$ are disjoint cycles then $c_1c_2 \in p(D_n)$. q.e.d.

Notation. From now on we will identify C_n and D_n with their respective images in Σ_{2n} . If a cycle $c = (i_1, \dots, i_k)$ is disjoint from \bar{c} then we write

$$((i_1, \dots, i_k)) = c\bar{c} = (i_1, \dots, i_k)(-i_1, \dots, -i_k)$$

and we call $c\bar{c}$ a *paired cycle*. If $k = 1$ then $c = (i_1)$ and the paired cycle $c\bar{c} = ((i_1))$ fixes the vector e_{i_1} . If $c = \bar{c} = (i_1, \dots, i_r, -i_1, \dots, -i_r)$ then we say that c is a *balanced cycle* and we write

$$c = [i_1, \dots, i_k].$$

This notation is consistent with that introduced earlier for the elements of the generating set \mathcal{R}_c . With these conventions, proposition 3.1 states that each element of C_n may be written as a product of disjoint paired cycles and balanced cycles. If $\alpha \in C_n$ fixes the standard basis vector e_i then we will assume that the paired cycle $((i))$ appears in the corresponding expression (1) for α .

Denote the length function for C_n with respect to the generating set \mathcal{R}_c by l . Lemma 2.3 allows us to use the same symbol l for the length function of D_n with respect to the set \mathcal{R}_d . The length function for Σ_{2n} with respect to the set T of all transpositions is denoted by L .

Lemma 3.2. *The fixed space $F(((i_1, \dots, i_k)))$ has dimension $n - k + 1$ and is given by*

$$\{x \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \dots = x_{i_k}\}$$

where x_i means $-x_{|i|}$ for $i < 0$. *The fixed space $F([i_1, \dots, i_k])$ has dimension $n - k$ and is given by*

$$\{x \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \dots = x_{i_k} = 0\}$$

Proof. By inspection. q.e.d.

Lemma 3.3. *The l -length of a paired cycle $c\bar{c} = ((i_1, \dots, i_k))$ is $k - 1$. Moreover, no minimal length factorisation of $c\bar{c}$ as a product of elements of \mathcal{R}_c contains a generator of the form $[i]$.*

Proof. The fixed space $F(c\bar{c})$ has dimension $n - k + 1$ by lemma 3.2 and thus $l(c\bar{c}) = n - (n - k + 1) = k - 1$.

If a minimal l -length factorisation of $c\bar{c}$ contained a term of the form $[i]$, we would obtain a factorisation of $c\bar{c}$ as a product of fewer than $2(k - 2) + 1 = 2k - 3$ transpositions. As $L(c\bar{c}) = 2k - 2$ this is impossible. q.e.d.

Lemma 3.4. *The l -length of $\gamma = [j_1, \dots, j_r]$ as a product of elements of \mathcal{R}_c is r . Moreover any minimal length factorisation of γ as a product of elements of \mathcal{R}_c contains exactly one generator of the form $[i]$.*

Proof. As the fixed space $F(\gamma)$ is $(n - r)$ -dimensional by lemma 3.2, we find $l(\gamma) = n - (n - r) = r$.

As $L(\gamma) = 2r - 1$, any factorisation of γ as a product of r elements of \mathcal{R}_c can contain at most one generator of the form $[i]$. If such a factorisation contained no element of this form, we would have an expression for γ as a product of an even number of transpositions. But this contradicts the fact that the $2r$ -cycle γ has odd parity in Σ_{2n} . q.e.d.

Proposition 3.5. *If $\alpha = c_1\bar{c}_1 \dots c_a\bar{c}_a\gamma_1 \dots \gamma_b \in C_n$ is a product of disjoint cycles then*

$$l(\alpha) = \sum_{i=1}^a l(c_i\bar{c}_i) + \sum_{j=1}^b l(\gamma_j)$$

Proof. By choosing a new basis from $\{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ if necessary, we may assume that $c_i = (j_{i-1} + 1, j_{i-1} + 2, \dots, j_i)$ and $\gamma_i = [k_{i-1} + 1, k_{i-1} + 2, \dots, k_i]$ where $1 = j_0 < j_1 < \dots < j_a < j_a + 1 = k_0 < k_1 < \dots < k_b = n$. Then $c_i\bar{c}_i$ (resp. γ_j) maps $U_i = \text{span}(e_{j_{i-1}+1}, e_{j_{i-1}+2}, \dots, e_{j_i})$ (resp. $V_i = \text{span}(e_{k_{i-1}+1}, e_{k_{i-1}+2}, \dots, e_{k_i})$) to itself and leaves all the other U 's and V 's pointwise fixed. As $c_i\bar{c}_i$ (resp. γ_j) fixes a 1 (resp. 0) dimensional subspace of U_i (resp. V_j), we see that α fixes an a -dimensional subspace of \mathbb{R}^n . Therefore $l(\alpha) = n - a$. Since $\sum(1 + l(c_i\bar{c}_i)) + \sum l(\gamma_j) = n$ by lemmas 3.3 and 3.4, the result follows. q.e.d.

Consider now the effect of multiplying $\alpha \in C_n$ on the right by a reflection $R = ((i, j))$ or $R = [i]$. It is clear that only those cycles which contain an integer of R will be affected. The following example lists the possibilities and the corresponding changes in lengths.

Example 3.6. *The following four identities can be verified directly.*

$$\begin{aligned} [i_1, i_2, \dots, i_k][i_k] &= ((i_1, i_2, \dots, i_k)) \\ [i_1, i_2, \dots, i_k)((i_j, i_k)) &= [i_1, \dots, i_j)((i_{j+1}, i_{j+2}, \dots, i_k)) \\ ((i_1, i_2, \dots, i_k))((i_j, i_k)) &= ((i_1, \dots, i_j))((i_{j+1}, i_{j+2}, \dots, i_k)) \\ [i_1, \dots, i_j][i_{j+1}, \dots, i_k)((-i_j, i_k)) &= ((i_1, i_2, \dots, i_k)) \end{aligned}$$

Since each reflection has order 2, the following identities are immediate.

$$\begin{aligned} [i_1, i_2, \dots, i_k] &= ((i_1, i_2, \dots, i_k))[i_k] \\ [i_1, i_2, \dots, i_k] &= [i_1, \dots, i_j)((i_{j+1}, i_{j+2}, \dots, i_k))((i_j, i_k)) \\ ((i_1, i_2, \dots, i_k)) &= ((i_1, \dots, i_j))((i_{j+1}, i_{j+2}, \dots, i_k))((i_j, i_k)) \\ [i_1, \dots, i_j][i_{j+1}, \dots, i_k] &= ((i_1, i_2, \dots, i_k))(-i_j, i_k) \end{aligned}$$

By proposition 3.5, we see that

$$\begin{aligned} l([i_1, i_2, \dots, i_n]) &= l(((i_1, i_2, \dots, i_n))) + 1 \\ l([i_1, i_2, \dots, i_n]) &= l([i_1, \dots, i_j][(i_{j+1}, i_{j+2}, \dots, i_n)]) + 1 \\ l(((i_1, i_2, \dots, i_n))) &= l(((i_1, \dots, i_j))((i_{j+1}, i_{j+2}, \dots, i_n))) + 1 \\ l([i_1, \dots, i_j][i_{j+1}, \dots, i_k]) &= l(((i_1, i_2, \dots, i_k))) + 1 \end{aligned}$$

Definition 3.7. *Let $\sigma = c_1 c_2 \cdots c_k$ and $\tau = d_1 d_2 \cdots d_l$ be two products of disjoint cycles in Σ_{2n} . We say that σ is contained in τ (and write $\sigma \subset \tau$) if for each i we can find j such that the set of integers in the cycle c_i is a subset of the set of integers in the cycle d_j . This notion restricts to give a notion of containment for elements of C_n .*

A reflection $((i, j))$ is s -contained in $\alpha = c_1 \bar{c}_1 \cdots c_a \bar{c}_a \gamma_1 \cdots \gamma_b \in C_n$ (and we write $((i, j)) \sqsubset \alpha$) if i is contained in γ_k and j is contained in γ_l for some $k \neq l$.

Lemma 3.8. *Let $\alpha \in C_n$ and $R \in \mathcal{R}_c$. Then $R \leq \alpha$ if and only if $R \subset \alpha$ or $R \sqsubset \alpha$.*

Proof. By proposition 3.5 and the calculations in example 3.6 we see that $l(\alpha R) < l(\alpha)$ if and only if $R \subset \alpha$ or $R \sqsubset \alpha$. Since $R \leq \alpha$ if and only if $l(\alpha R) < l(\alpha)$, the lemma follows. q.e.d.

4. THE LATTICE PROPERTY

In this section we show that the interval $[1, \gamma]$ in $(W \leq)$ is a lattice for $W = C_n, D_n$ and γ a Coxeter element in W . Since all Coxeter elements in W are conjugate we can choose our favourite one in each case.

Definition 4.1. *We choose the Coxeter elements γ_C in C_n and γ_D in D_n given by $\gamma_C = [1, 2, \dots, n]$ and $\gamma_D = [1][2, 3, \dots, n]$.*

Proposition 4.2. *Write the Coxeter element $\gamma_C \in C_n$ (resp. $\gamma_D \in D_n$) as $\gamma_C = R_1 R_2 \dots R_n$ (resp. $\gamma_D = R_1 R_2 \dots R_n$) for reflections R_1, \dots, R_n in \mathcal{R}_c (resp. \mathcal{R}_d) and let b_i denote the number of balanced cycles in $R_1 R_2 \dots R_i$. Then there exists i_0 such that $b_i = 0$ for $i < i_0$ and $b_i = 1$ (resp. $b_i = 2$) for $i \geq i_0$. In the D_n case, if $b_i = 2$ then one of the balanced cycles in $R_1 \dots R_i$ must be [1].*

Proof. By example 3.6, if the multiplication of $\alpha \in C_n$ by $R \in \mathcal{R}_c$ increases the number of balanced cycles then $l(\alpha R) = l(\alpha) + 1$ and αR contains either 1 or 2 balanced cycles more than α . Conversely, if multiplication of α by R decreases either the number of balanced cycles or the size of a balanced cycle, then $l(\alpha R) = l(\alpha) - 1$. Since $l(R_1 \dots R_i) + 1 = l(R_1 \dots R_{i+1})$ it follows that $b_{i+1} - b_i \in \{0, 1, 2\}$. As γ_C consists of a single balanced cycle, the claim for C_n is immediate. For γ_D , none of the R_i can be of the form $[j]$ and hence $b_{i+1} - b_i$ cannot be 1. As the passage from $R_1 \dots R_i$ to $R_1 \dots R_{i+1}$ cannot decrease the size of any balanced cycle and as γ_D contains the balanced cycle [1], this cycle must be present in $R_1 \dots R_i$ for each $i \geq i_0$. q.e.d.

Corollary 4.3. *If $\alpha \leq \gamma_C$ in C_n then α has at most one balanced cycle. If $\beta \leq \gamma_D$ in D_n then β has either no balanced cycles or two balanced cycles. In the latter case, one of these balanced cycles is [1].*

4.1. The C_n lattice. Set $\gamma = \gamma_C = [1, 2, \dots, n]$.

Definition 4.4. *The action of γ defines a cyclic order on the set $A = \{1, \dots, n, -1, \dots, -n\}$ in which the successor of i is $\gamma(i)$ (thus 1 is the successor of $-n$). An ordered set of elements i_1, i_2, \dots, i_s in A is oriented consistently (with the cyclic order on A) if there exist integers $0 < r_2 < \dots < r_s \leq 2n - 1$ such that $i_j = \gamma^{r_j}(i_1)$ for $j = 2, \dots, s$. A cycle (i_1, \dots, i_s) or $[i_1, \dots, i_s]$ is oriented consistently if the ordered set $i_1, \dots, i_s, -i_1, \dots, -i_s$ in A is oriented consistently.*

Definition 4.5. *Two disjoint reflections $R_1 = ((i, j))$ and $R_2 = ((k, l))$ (resp. $R_2 = [k]$) are said to cross if one of the following four ordered sets is oriented consistently in A : i, k, j, l or $i, -k, j, -l$ or k, i, l, j or $k, -i, l, -j$ (resp. $i, k, j, -k$ or $i, -k, j, k$ or $k, i, -k, j$ or $-k, i, k, j$). Two disjoint cycles ζ_1 and ζ_2 in C_n are said to cross if there exist crossing reflections R_1 and R_2 which are contained in ζ_1 and ζ_2 respectively. An element $\sigma \in C_n$ is called crossing if some pair of disjoint cycles of σ cross. Otherwise σ is non-crossing.*

Proposition 4.6. *If $\sigma \in C_n$ satisfies $\sigma \leq \gamma$ then the cycles of σ are oriented consistently and are noncrossing.*

Proof. We will proceed by induction on $n - l(\sigma)$. If $l(\sigma) = n$ then $\sigma = \gamma$ and the two conditions of the conclusion are satisfied.

We assume therefore that the proposition is true for $\tau \in C_n$ with $n - l(\tau) = 0, 1, \dots, k - 1$ and that $\sigma \leq \gamma$ satisfies $l(\sigma) = n - k$. By definition there is an expression for γ as a product of n reflections $\gamma = R_1 R_2 \dots R_{n-k} R R_{n-k+2} \dots R_n$ with $\sigma = R_1 R_2 \dots R_{n-k}$. We define $\tau = \sigma R$ so that $l(\tau) = l(\sigma) + 1$ and $\tau \leq \gamma$. By induction, the cycles of τ are noncrossing and oriented consistently with γ .

We know that R is either of the form $((i, j))$ or $[i]$ and that $R \leq \tau \leq \gamma$. Lemma 3.8 thus implies that R is contained in some paired cycle or some balanced cycle of τ . The effect of multiplying this cycle by R is thus described by one of the first three equations in Example 3.6. Since the cycles of τ are noncrossing and oriented consistently with γ , we see that the same is true for σ . q.e.d.

Proposition 4.7. *Let $\sigma \in C_n$. If the cycles of σ are oriented consistently and are noncrossing then $\sigma \leq \gamma$.*

Proof. Assume that $\sigma \in C_n$ satisfies the two hypotheses of the proposition. Write $\sigma = c_1 \bar{c}_1 \dots c_a \bar{c}_a \gamma_1 \dots \gamma_b$ and set $t(\sigma) = a + b$. We proceed by induction on $t(\sigma)$. If $t(\sigma) = 1$ then either σ consists of a single balanced cycle or a single paired cycle. In the former case, consistent orientation implies that $\sigma \leq \gamma$. In the latter case, consistent orientation implies that $\sigma = ((i, i + 1, \dots, n, -1, \dots, -i + 1))$ for some i . As $l(\sigma) = n - 1$ and $\sigma[i - 1] = \gamma$, we see that $\sigma \leq \gamma$.

Assume now that $t(\sigma) \geq 2$ and that the proposition is true for each element $\theta \in C_n$ with $t(\theta) < t(\sigma)$. If σ contains a balanced cycle, the non-crossing hypothesis implies that there can be only one which we denote $\tau = [i_1, \dots, i_r]$. Otherwise let $\tau = ((i_1, \dots, i_r))$ be some paired cycle of σ . As $\sigma \neq \tau$, there exists an i_k whose successor does not lie in $\{\pm i_1, \dots, \pm i_r\}$. By choosing one of the other $2r - 1$ cycle expressions for τ if necessary, we may assume that the successor j_1 of i_r does not lie in $\{\pm i_1, \dots, \pm i_r\}$. Let $\rho = ((j_1, \dots, j_s))$ be the paired cycle of σ which contains j_1 and let $R = ((i_r, j_s))$. Then $\sigma = \tau \rho \sigma_1 \dots \sigma_k$ for some disjoint paired cycles $\sigma_1, \dots, \sigma_k$ (some $k \geq 0$) and

$$\sigma R = \begin{cases} [i_1, \dots, i_r, j_1, \dots, j_s] \sigma_1 \dots \sigma_k & \text{or} \\ ((i_1, \dots, i_r, j_1, \dots, j_s)) \sigma_1 \dots \sigma_k. \end{cases}$$

Note that $t(\sigma R) = t(\sigma) - 1$. As the cycles τ and ρ do not cross and each is oriented consistently, our choice of j_1 ensures that the ordered set $i_1, \dots, i_r, j_1, \dots, j_s, -i_1, \dots, -i_r, -j_1, \dots, -j_s$ is also oriented consistently.

Assume now that one of the cycles σ_e crosses the cycle $\tau\rho R$ of σR . Then there exist crossing reflections R_1 and R_2 contained in $\tau\rho R$ and σ_e respectively. Since σ_e is paired, R_2 is necessarily paired; $R_2 = \langle\langle c, d \rangle\rangle$ say. Since σ is non-crossing, R_1 cannot be contained in τ or in ρ . There are three cases to consider

- (1) $R_1 = \langle\langle i_a, j_b \rangle\rangle$ for some $1 \leq a \leq r$ and $1 \leq b \leq s$.
- (2) $R_1 = \langle\langle j_b, -j_b \rangle\rangle$ for some $1 \leq b \leq s$ (τ is necessarily balanced).
- (3) $R_1 = \langle\langle i_a, -j_b \rangle\rangle$ for some $1 \leq b \leq s$ (τ is necessarily balanced).

By a suitable choice of the representative $R = \langle\langle c, d \rangle\rangle = \langle\langle d, c \rangle\rangle = \langle\langle -c, -d \rangle\rangle = \langle\langle -d, -c \rangle\rangle$, the first case splits into two essential subcases: (a) the ordered set i_a, c, j_b, d is oriented consistently and (b) the ordered set c, i_a, d, j_b is oriented consistently. We know that c is not in $\{\pm i_1, \dots, \pm i_r, \pm j_1, \dots, \pm j_s\}$. In particular $c \neq i_r, j_1$. In case (1a), if c precedes i_r , then $S = \langle\langle i_1, i_r \rangle\rangle$ is contained in τ and crosses R_2 , contradicting the fact that σ is non-crossing. Likewise, if c follows i_r then c follows j_1 and $S = \langle\langle j_1, j_b \rangle\rangle$ is contained in ρ and crosses R_2 , again contradicting the fact that σ is non-crossing. Thus case (1a) is impossible. A similar argument shows that case (1b) is also impossible.

As in case 1, case 2 splits into two subcases: (a) the ordered set $j_b, c, -j_b, d$ is oriented consistently and (b) the ordered set $c, j_b, d, -j_b$ is oriented consistently. In case (2a), if c precedes $-i_r$ then the ordered set $i_r, j_b, c, -i_r, d$ is oriented consistently and hence $\langle\langle c, d \rangle\rangle$ crosses $[-i_r] \subset \tau$. But this contradicts the fact that σ is non-crossing. If c follows $-i_r$, then c necessarily succeeds $-j_1$ and we find that the ordered set $-j_1, c, -j_b, d$ is consistently oriented. Thus $\langle\langle c, d \rangle\rangle$ crosses $\langle\langle -j_1, -j_b \rangle\rangle \subset \rho$, again contradicting the fact that σ is non-crossing. Thus case (2a) is impossible. A similar argument shows that case (2b) is also impossible.

Finally, case 3 also splits into two subcases: (a) the ordered set $i_a, c, -j_b, d$ is oriented consistently and (b) the ordered set $c, i_a, d, -j_b$ is oriented consistently. We show that (3b) is impossible (the proof that case (3a) is impossible is similar). We are given that the ordered set $c, i_a, d, -j_b$ is oriented consistently. If d precedes $-i_a$ then $\langle\langle c, d \rangle\rangle$ crosses $[i_a]$ in σ , a contradiction. Therefore d follows $-i_a$. If d now precedes $-i_r$, then the ordered set $c, -i_a, d, -i_r$ is oriented consistently. Hence $\langle\langle -i_a, -i_r \rangle\rangle$ crosses $\langle\langle c, d \rangle\rangle$ in σ , a contradiction. Therefore d follows $-i_r$ and hence $-j_1$. But now $\langle\langle -j_1, -j_b \rangle\rangle$ crosses $\langle\langle c, d \rangle\rangle$ in σ , a contradiction. Thus case (3b) is impossible.

We conclude that the cycles $\tau\rho R$ and σ_e do not cross. Since no two distinct elements of $\sigma_1, \dots, \sigma_k$ cross (because σ is assumed non-crossing), it follows that σR is non-crossing. As $t(\sigma R) = t(\sigma) - 1$ and the cycles

of σR are oriented consistently, it follows by induction that $\sigma R \leq \gamma$. Thus there exist reflections R_1, \dots, R_k with $k = n - l(\sigma R)$ and

$$(5) \quad \sigma R R_1 \dots R_k = \gamma$$

As $l(\sigma R) = l(\sigma) + 1$ by lemmas 3.3 and 3.4 and proposition 3.5, we see that $k + 1 = n - l(\sigma)$. Hence equation (5) also implies that $\sigma \leq \gamma$. q.e.d.

Lemma 4.8. *If $\sigma \leq \gamma$ and $\tau \leq \gamma$ then $\sigma \leq \tau$ if and only if $\sigma \subset \tau$.*

Proof. Follows from Lemma 2.5 and lemma 3.8. q.e.d.

Combining the previous three results yields the following Theorem.

Theorem 4.9. *Let NCP denote Reiner's non-crossing partition lattice for the C_n group from [8]. The mapping*

$$: \{\alpha \in C_n : \alpha \leq \gamma\} \longrightarrow NCP$$

which takes α to the noncrossing partition defined by its cycle structure is a bijective poset map. In particular, $\{\alpha \in C_n : \alpha \leq \gamma\}$ is a lattice.

4.2. The D_n lattice. Set $\gamma = \gamma_D = [1][2, 3, \dots, n]$ and suppose $\alpha \leq \gamma$. Recall from Corollary 4.3 that for such an α either $[1][k] \leq \alpha$ for some $k \in \{2, 3, \dots, n\}$ or l and $-l$ are in different α orbits for all $l \in \{1, 2, \dots, n\}$. In the former case we will call α *balanced* and in the latter case we will call α *paired*.

We note that lattices are associated to the groups C_n and D_n in [8]. We have shown the Reiner C_n lattices are isomorphic to ours. However the Reiner D_n lattices are not the same as the ones we consider. In particular, the Reiner D_n lattices are subposets of the Reiner C_n lattices.

To show that the interval $[I, \gamma]$ in D_n is a lattice we will compute $\alpha \wedge \beta$ for $\alpha, \beta \leq \gamma$. Since the poset is finite the existence of least upper bounds follows. We will consider different cases depending on the types of α and β . In all cases we will construct a candidate σ for $\alpha \wedge \beta$ and show that $\sigma \in D_n$, $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$. Since the reverse inclusion is immediate it follows from Lemma 2.6 that $\sigma = \alpha \wedge \beta$.

Note 4.10. *In this section we will frequently pass between the posets determined by C_n , D_n and several other finite reflection subgroups of C_n . As the partial order on each of these groups is the restriction of the partial order on $O(n)$, we can use the same symbol \leq to denote the partial order in each case. The reflection subgroup in question should be clear from the context.*

Suppose first that both α and β are balanced. Since $D_n \subset C_n$ and C_{n-1} can be identified with the subgroup of C_n which fixes 1, each balanced element of D_n can be used to define a balanced element of C_{n-1} , that is, an element containing a balanced cycle. Thus we define the balanced C_{n-1} elements α' and β' by

$$\alpha = [1]\alpha' \quad \text{and} \quad \beta = [1]\beta'$$

and the C_{n-1} element $\sigma' = \alpha' \wedge \beta'$, where the meet is taken in C_{n-1} . Now σ' may or may not be balanced. If σ' is balanced define the C_n element σ by $\sigma = [1]\sigma'$. If σ' is not balanced set $\sigma = \sigma'$.

Proposition 4.11. *If α and β are balanced and σ is defined as above then $\sigma \in D_n$, $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.*

Proof. We show that $\sigma \in D_n$ and $\sigma \leq \alpha$. The proof that $\sigma \leq \beta$ is completely analogous. First consider the case where σ' is balanced. Thus $[k] \leq \sigma' \leq \alpha'$ in C_{n-1} for some k satisfying $2 \leq k \leq n$. So we can find reflections R_1, \dots, R_s in C_{n-1} with

$$\alpha' = R_1 R_2 \dots R_s, \quad \sigma' = R_1 R_2 \dots R_t, \quad R_1 = [k],$$

where $l(\alpha') = s \geq t = l(\sigma')$. Since $\alpha' \in C_{n-1}$, Lemma 3.4 gives R_2, \dots, R_s all of the form (i, j) or $(i, -j)$ for $2 \leq i < j \leq n$. In particular, these reflections lie in D_n . Now α is of length $s + 1$ in C_n and

$$\begin{aligned} \alpha &= [1]R_1 R_2 \dots R_t R_{t+1} \dots R_s \\ &= [1][k]R_2 \dots R_t R_{t+1} \dots R_s \\ &= (1, k)(1, -k)R_2 \dots R_t R_{t+1} \dots R_s. \end{aligned}$$

This last expression only uses D_n reflections so that

$$\sigma = (1, k)(1, -k)R_2 \dots R_t \leq \alpha \quad \text{in } D_n.$$

Next we consider the case where σ' is paired. Here $\sigma' \leq \alpha'$ and α' is balanced so we can find reflections R_1, \dots, R_s in C_{n-1} with

$$\alpha' = R_1 R_2 \dots R_s, \quad \sigma' = R_1 R_2 \dots R_t,$$

where $l(\alpha') = s > t = l(\sigma')$ and exactly one of R_{t+1}, \dots, R_s is of form $[k]$. Since $R[k] = [k]([k]R[k])$, we can assume $R_{t+1} = [k]$. Note also that R_1, \dots, R_t are each of the form (i, j) or $(i, -j)$ for $2 \leq i < j \leq n$ and hence commute with $[1]$ in C_n . Thus we can write the following identities in C_n .

$$\begin{aligned} \alpha &= [1]R_1 R_2 \dots R_t [k] R_{t+2} \dots R_s \\ &= R_1 \dots R_t [1][k] R_{t+2} \dots R_s \\ &= R_1 \dots R_t (1, k)(1, -k) R_{t+2} \dots R_s. \end{aligned}$$

This last expression only uses D_n reflections so that $\sigma \leq \alpha$ in D_n .

Finally we show that $S_\alpha \cap S_\beta \subset S_\sigma$. First suppose σ' is balanced and $R \in S_\alpha \cap S_\beta$. Thus R is a reflection satisfying $R \leq \alpha, \beta$. If R is of the form $((1, k))$, then $[1][k] \leq \alpha, \beta$ since k must belong to a balanced cycle of both α and β . Thus $[k] \leq \alpha', \beta'$ so that $[k] \leq \sigma'$ and $[1][k] \leq \sigma$, which gives $((1, k)) \leq \sigma$ as required. If R is not of form $((1, k))$ then $R \leq \alpha, \beta$ implies $R \leq \alpha', \beta'$ so that $R \leq \sigma'$ and $R \leq \sigma$.

In the case where σ' is paired, $R \leq \alpha, \beta$ implies R must be of form $((i, j))$ or $((i, -j))$ for $2 \leq i < j \leq n$ so that $R \leq \alpha', \beta'$ giving $R \leq \sigma' = \sigma$. q.e.d.

Since we have completed the case where both α and β are balanced we will assume from now on that α is paired. We note some consequences of this fact which will apply in the remaining cases. The fact that α is paired means that $\alpha \leq ((1, k))\gamma$ or $\alpha \leq ((1, -k))\gamma$ for some $k \in \{2, 3, \dots, n\}$. Since conjugation by the C_{n-1} element $[2, \dots, n]$ is a poset isomorphism of the interval $[I, \gamma]$ in D_n , we may assume for convenience of notation that $k = -2$ so that

$$\alpha \leq ((1, -2))[1][2, \dots, n] = ((1, 2, \dots, n)).$$

If we let $\delta = ((1, 2, \dots, n))$ then a reflection R in D_n satisfies $R \leq \delta$ if and only if $R \subset \delta$. Thus we can identify the interval $[I, \delta]$ in D_n with the set of non-crossing partitions of $\{1, 2, \dots, n\}$. Recall that a non-crossing partition of the ordered set $\{a_1, a_2, \dots, a_n\}$ is a partition with the property that whenever

$$1 \leq i < j < k < l \leq n$$

with a_i, a_k belonging to the same block B_1 and a_j, a_l belonging to the same block B_2 we have $B_1 = B_2$. If $\alpha \wedge \beta$ exists, it will satisfy

$$\alpha \wedge \beta \leq \alpha \leq ((1, 2, \dots, n))$$

and so will correspond to a noncrossing partition of $\{1, 2, \dots, n\}$. Accordingly, we define a reflexive, symmetric relation on $\{1, 2, \dots, n\}$ by

$$i \sim j \iff i = j \quad \text{or} \quad ((i, j)) \leq \alpha, \beta.$$

We need to show that \sim is transitive and hence is an equivalence relation. We then show that the resulting partition of $\{1, 2, \dots, n\}$ is non-crossing and determines an element σ of D_n which satisfies $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.

Suppose that α is paired and β is balanced. Recall that β has two balanced cycles, one of which is $[1]$. For convenience of terminology we will call the other balanced cycle the second balanced cycle of β . As

above we will have occasion to use the balanced element $\beta' \leq [2, \dots, n]$ in C_{n-1} defined by $\beta = [1]\beta'$.

Proposition 4.12. *If α is paired and β is balanced then the relation \sim above determines an element σ of D_n satisfying $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.*

Proof. First we establish the transitivity of the \sim relation. Suppose i, j, k are distinct elements of $\{1, 2, \dots, n\}$ with $i \sim j$ and $j \sim k$. Since $((i, j), (j, k)) \leq \alpha$ we get $((i, k)) \leq \alpha$ since α corresponds to a partition of $\{1, 2, \dots, n\}$. If $1 \notin \{i, j, k\}$ then $((i, j), (j, k)) \subset \beta$ (s-containment cannot arise) and it follows that $((i, k)) \leq \beta$. If $i = 1$, then $((i, j)) \leq \beta$ means that $((i, j)) \sqsubset \beta$ so that j belongs to the second balanced cycle of β . Since $j \sim k \neq 1$, k also belongs to this second balanced cycle and $((i, k)) \leq [1][j, k] \leq \beta$. If $j = 1$, then both i and k belong to the second balanced cycle of β . Hence $((i, k)) \leq \beta$. The case $k = 1$ is analogous to the case $i = 1$.

To show that the partition of $\{1, \dots, n\}$ defined by \sim is non-crossing suppose $1 \leq i < j < k < l \leq n$ with

$$((i, k), (j, l)) \leq \alpha, \beta.$$

Since α corresponds to a noncrossing partition we have $((i, j, k, l)) \leq \alpha$. If $i = 1$, then k belongs to the second balanced cycle and $[k] \leq \beta'$ in C_{n-1} . Since $1 < j < k < l$, $((j, l)) \leq \beta'$ and $\beta' \leq [2, \dots, n]$ in C_{n-1} , the crossing pair consisting of (j, l) and $(k, -k)$ must lie in the same β' cycle. Thus $[j, k, l] \leq \beta'$ and $((1, j, k, l)) \leq [1][j, k, l] \leq \beta$. If $i \neq 1$, then $((i, k), (j, l)) \leq \beta'$ and since $\beta' \leq [2, \dots, n]$ in C_{n-1} , $((i, j, k, l)) \leq \beta'$ by proposition 4.6, giving $((i, j, k, l)) \leq \beta$.

Thus the relation \sim defines a noncrossing partition of $\{1, 2, \dots, n\}$ and hence determines an element σ of D_n . By the definition of \sim the element σ satisfies $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$. q.e.d.

Finally we consider the case where both α and β are paired.

Proposition 4.13. *If α and β are paired then the relation \sim above determines an element σ of D_n satisfying $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$.*

Proof. To establish the transitivity of \sim in this case let i, j, k be distinct elements of $\{1, 2, \dots, n\}$ with $i \sim j$ and $j \sim k$. As in the previous proposition, $((i, k)) \leq \alpha$ follows immediately. Since β is paired, $i \sim j$ and $j \sim k$ mean that i, j, k belong to the same cycle of β so that $((i, k)) \leq \beta$ also.

To show that the partition of $\{1, \dots, n\}$ defined by \sim is noncrossing suppose $1 \leq i < j < k < l \leq n$ with

$$((i, k), (j, l)) \leq \alpha, \beta.$$

Since α corresponds to a noncrossing partition we have $((i, j, k, l)) \leq \alpha$. The element β is paired so we can assume $\beta \leq \tau = ((1, m))\gamma$ or $\beta \leq \tau = ((1, -m))\gamma$, for some $m \in \{2, 3, \dots, n\}$. Looking at the case $\tau = ((1, m))\gamma$ first we get

$$\tau = ((1, -m, -m - 1, \dots, -n, 2, 3, \dots, m - 1)).$$

Since $\beta \leq \tau$ the element β corresponds to a noncrossing partition of the ordered set $\{1, -m, -m - 1, \dots, -n, 2, 3, \dots, m - 1\}$. Since $1 \leq i < j < k < l \leq n$, we deduce that either

$$1 \leq i < j < k < l \leq m - 1 \quad \text{or} \quad m \leq i < j < k < l \leq n.$$

Since β corresponds to a noncrossing partition of the ordered set

$$\{1, -m, -m - 1, \dots, -n, 2, 3, \dots, m - 1\}$$

and $((i, k), (j, l)) \leq \beta$ it follows in either case that $((i, j, k, l)) \leq \beta$. The case $\tau = ((1, -m))\gamma$ is similar. Here

$$\tau = ((1, m, m + 1, \dots, n, -2, -3, \dots, -m + 1)),$$

and again we can deduce $((i, j, k, l)) \leq \beta$.

Thus \sim defines a noncrossing partition of $\{1, 2, \dots, n\}$ and hence an element σ in D_n satisfying $\sigma \leq \alpha, \beta$ and $S_\alpha \cap S_\beta \subset S_\sigma$ as in previous proposition. q.e.d.

Combining the results of this subsection we obtain the following theorem.

Theorem 4.14. *The interval $[I, \gamma]$ in D_n is a lattice.*

5. POSET GROUPS AND $K(\pi, 1)$ 'S.

Definition 5.1. *If W is a finite Coxeter group and $\gamma \in W$ we define the poset group $\Gamma = \Gamma(W, \gamma)$ to be the group with the following presentation. The generating set for Γ consists of a copy of the set of non-identity elements in $[I, \gamma]$. We will denote by $\{w\}$ the generator of Γ corresponding the element $w \in (I, \gamma]$. The relations in Γ are all identities of the form $\{w_1\}\{w_2\} = \{w_3\}$, where w_1, w_2 and w_3 lie in $(I, \gamma]$ with $w_1 \leq w_3$ and $w_2 = w_1^{-1}w_3$.*

Since none of the relations involve inverses of the generators, there is a semigroup, which we will denote by $\Gamma_+ = \Gamma_+(W, \gamma)$, with the same presentation. As in section 5 of [3], we define a *positive* word in Γ to

be a word in the generators that does not involve the inverses of the generators. We say two positive words A and B are *positively equal*, and we write $A \doteq B$, if A can be transformed to B through a sequence of positive words, where each word in the sequence is obtained from the previous one by replacing one side of a defining relator by the other side. Since the interval $(I, \gamma]$ inherits the reflection length from W we use this to associate a *length* to each generator of $\Gamma(W, \gamma)$ and hence a length $l(A)$ to each positive word A . It is immediate that positively equal words have the same length.

From now on we only consider those pairs (W, γ) with the property that *the interval $[I, \gamma]$ in W forms a lattice*. It is clear that the results stated for the braid group in sections 5 and 6 of [3] apply to poset groups under this extra assumption. We will review them briefly below.

In [3] it is shown that this lattice condition is satisfied when W is a Coxeter group of type A_n and γ is a Coxeter element. In section 4 above we have shown that the lattice condition is satisfied when W is a Coxeter group of type C_n or D_n and γ is a Coxeter element. When the Coxeter group is generated by two reflections the lattice condition is automatic for any γ . When the Coxeter group is generated by three reflections the lattice condition reduces to checking the only case where

$$\alpha \wedge \beta \notin \{\alpha, \beta, \gamma\}.$$

This occurs when α and β are distinct reflections and have at least one common upper bound of length 2. Any such length 2 element δ must have $F(\delta)$ coinciding with the unique line of intersection of the two reflection planes. Hence δ is unique. This is precisely the ingredient which makes the metric constructed in [4] have non-positive curvature. The following result is taken from [3]. Its proof is the same.

Lemma 5.2. *Assume that the interval $[I, \gamma]$ forms a lattice and suppose $a, b, c \leq \gamma$. We define nine elements d, e, f, g, h, k, l, m and n in $[I, \gamma]$ by the equations*

$$a \vee b = ad = be, \quad b \vee c = bf = cg, \quad c \vee a = ch = ak$$

and

$$a \vee b \vee c = (a \vee b)l = (b \vee c)m = (c \vee a)n.$$

Then we can deduce

$$e \vee f = el = fm, \quad d \vee k = dl = kn, \quad h \vee g = hn = gm.$$

The statements and proofs of the results of section 5 and section 6 of [3] generalize in a straightforward manner to the current setting. In particular, we have the following definitions and results.

Lemma 5.3. *The semigroup associated to Γ has right and left cancellation properties.*

Lemma 5.4. *Suppose $a_1, a_2, \dots, a_k \leq \gamma$ in W , P is positive and*

$$P \doteq X_1\{a_1\} \doteq \dots \doteq X_k\{a_k\}$$

with X_i all positive. Then there is a positive word Z satisfying

$$P \doteq Z\{a_1 \vee \dots \vee a_k\}.$$

Theorem 5.5. *In Γ , if two positive words are equal they are positively equal. In other words, the semigroup Γ_+ embeds in Γ .*

As in [3] we define an abstract simplicial complex $X(W, \gamma)$ for each $\Gamma(W, \gamma)$.

Definition 5.6. *We let $X = X(W, \gamma)$ be the abstract simplicial complex with vertex set Γ , which has a k -simplex on the subset $\{g_0, g_1, \dots, g_k\}$ if and only if $g_i = g_0\{w_i\}$ for $i = 1, 2, \dots, k$ where*

$$I < w_1 < \dots < w_k \leq \gamma \quad \text{in } W.$$

There is an obvious simplicial action of Γ on X given by

$$g \cdot \{g_0, g_1, \dots, g_k\} = \{gg_0, gg_1, \dots, gg_k\}.$$

The main result of section 6 of [3] also holds for these poset groups.

Theorem 5.7. *$X(W, \gamma)$ is contractible.*

If we define $K = K(W, \gamma)$ to be the quotient space $K = \Gamma \backslash X$, then K is a $K(\Gamma, 1)$.

We finish this section with an example of a poset group $\Gamma(W, \gamma)$, with $[I, \gamma]$ a lattice but γ not a Coxeter element in W .

Example 5.8. *Let $W = C_2$ and $\gamma = [1][2]$. The group $\Gamma(C_2, \gamma)$ has presentation*

$$\langle a, b, c, d, x \mid x = ab = ba = cd = dc \rangle$$

where $a = \{[1]\}$, $b = \{[2]\}$, $c = \{(1, 2)\}$, $d = \{(1, -2)\}$ and $x = \{[1][2]\}$. From the presentation we see that Γ is an amalgamated free product of a copy $\mathbb{Z} \times \mathbb{Z}$ generated by a and b with a copy $\mathbb{Z} \times \mathbb{Z}$ generated by c and d over the infinite cyclic subgroup generated by x . The above construction gives a two-dimensional contractible universal cover for the presentation 2-complex which can be shown to be simplicially isomorphic to $X(C_2, [1, 2])$.

6. GROUP PRESENTATIONS.

In this section we prove that the poset groups $\Gamma(W, \gamma)$ of section 5 are isomorphic to the Artin groups $A(W)$ for W of type C_n or D_n and γ the appropriate Coxeter element. The proof is based on the following surprising property that these Artin groups share with the braid group. If $X = x_1 x_2 \dots x_n$ is the product of the standard Artin generators then there is a finite set of elements in $A(W)$ which is invariant under conjugation by X . Moreover under the canonical surjection from $A(W)$ to W this set is taken bijectively to the set of reflections in W . The following lemma is a straightforward generalisation of Lemma 4.5 of [3].

Lemma 6.1. *The poset group $\Gamma(W, \gamma)$ is isomorphic to the abstract group generated by the set of all $\{R\}$, for R a reflection in $[I, \gamma]$, subject to the relations*

$$\{R_1\}\{R_2\} \dots \{R_n\} = \{S_1\}\{S_2\} \dots \{S_n\},$$

for R_i, S_j reflections satisfying

$$\gamma = R_1 R_2 \dots R_n \quad \text{and} \quad \gamma = S_1 S_2 \dots S_n,$$

where $n = l(\gamma)$.

We will refer to $\{w\} \in \Gamma(W, \gamma)$ as the lift of $w \in W$ whenever $w \leq \gamma$. In particular, we will refer to $\{w\}$ as a reflection lift whenever w is a reflection.

Since the Artin groups of type C_n and D_n both contain copies of the n -strand braid group B_n we collect here some facts about the braid group which will be useful. We recall that B_n is the group with generating set x_2, x_3, \dots, x_n and defining relations

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \quad \text{for} \quad 2 \leq i \leq n-1,$$

$$x_i x_j = x_j x_i \quad \text{for} \quad |j - i| \geq 2.$$

We define $x_{i,j}$ and $Y_{i,j}$, for $1 \leq i < j \leq n$ by

$$Y_{i,j} = x_{i+1} \dots x_j, \quad \text{and} \quad Y_{i,j} = Y_{i+1,j} x_{i,j}.$$

Then Lemma 4.2 of [3] gives, for $1 \leq i < j < k \leq n$,

$$x_{i,j} x_{j,k} = x_{j,k} x_{i,k} = x_{i,k} x_{i,j}.$$

Since $x_k = x_{k-1,k}$ it follows that $x_{i,j} Y_{i,j-1} = Y_{i,j}$ and that

$$x_k Y_{i,j} = Y_{i,j} x_{k-1} \quad \text{for} \quad i+2 \leq k \leq j.$$

When $k = i+1$ we have $x_{i+1} Y_{i,j} = x_{i+1} Y_{i+1,j} x_{i,j} = Y_{i,j} x_{i,j}$.

6.1. The C_n case. The Artin group $A(C_n)$ has a presentation with generating set x_1, x_2, \dots, x_n , subject to the relations

$$x_1 x_2 x_1 x_2 = x_2 x_1 x_2 x_1$$

$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$$

whenever $1 < i < n$ and

$$x_i x_j = x_j x_i$$

whenever $|j - i| \geq 2$.

Definition 6.2. We define a function ϕ from the generators of $A(C_n)$ to $\Gamma(C_n, \gamma)$ by

$$x_1 \mapsto \{[1]\}, x_2 \mapsto \{(1, 2)\}, x_3 \mapsto \{(2, 3)\}, \dots, x_n \mapsto \{(n-1, n)\}$$

Lemma 6.3. The function ϕ determines a well-defined and surjective homomorphism.

Proof: The relations involving $\phi(x_1)$ hold in $\Gamma(C_n, \gamma)$ by virtue of the following identities in $\Gamma(C_n, \gamma)$.

$$\begin{aligned} \{[1]\}\{(1, 2)\}\{[1]\}\{(1, 2)\} &= \{[1, 2]\}\{[1, 2]\} \\ &= \{(1, 2)\}\{[2]\}\{(1, -2)\}\{[1]\} \\ &= \{(1, 2)\}\{[1, 2]\}\{[1]\} \\ &= \{(1, 2)\}\{[1]\}\{(1, 2)\}\{[1]\} \end{aligned}$$

$$\{[1]\}\{(i, i+1)\} = \{(i, i+1)\}\{[1]\}, \quad \text{for } i \geq 2.$$

The image of the subgroup generated by $\{x_2, \dots, x_n\}$ lies in the copy of the braid group corresponding to $\Sigma_n < C_n$ so that the relations not involving $\phi(x_1)$ hold by Lemma 4.2 and Lemma 4.4 of [3]. Thus ϕ is well-defined.

To establish surjectivity, first note that

$$\{(i, i+1, \dots, j)\} = \phi(Y_{i,j}) \quad \text{and} \quad \{(i, j)\} = \phi(x_{i,j})$$

for $1 \leq i < j \leq n$ all lie in $\text{im}(\phi)$. Next $\{[j]\} \in \text{im}(\phi)$ since

$$\phi(x_1 x_{1,j}) = \{[1]\}\{(1, j)\} = \{[1, j]\} = \{(1, j)\}\{[j]\}.$$

Finally, $\{(i, -j)\} \in \text{im}(\phi)$ for $1 \leq i < j \leq n$ since

$$\{(i, j)\}\{[j]\} = \{[i, j]\} = \{[j]\}\{(i, -j)\}.$$

q.e.d.

To construct an inverse to ϕ we will use the presentation for $\Gamma(C_n, \gamma)$ given by lemma 6.1.

Definition 6.4. We define a function θ from the generators of $\Gamma(C_n, \gamma)$ to $A(C_n)$ by

$$\begin{aligned} \{[1]\} &\mapsto x_1, \{(i, j)\} \mapsto x_{i,j} \quad \text{for } 1 \leq i < j \leq n, \\ \{[j]\} &\mapsto y_j \quad \text{for } 2 \leq j \leq n, \quad \{(i, -j)\} \mapsto z_{i,j} \quad \text{for } 1 \leq i < j \leq n, \end{aligned}$$

where y_j is the unique element of $A(C_n)$ satisfying

$$x_1 x_2 \dots x_j = x_2 \dots x_j y_j$$

and $z_{i,j}$ is the unique element of $A(C_n)$ satisfying

$$z_{i,j} y_i = y_i x_{i,j}.$$

The homomorphism determined by θ will be surjective since each x_i is the image of some reflection lift. We note that $Y_{i,j} y_j = y_i Y_{i,j}$ for $1 \leq i < j \leq n$ if we define $y_1 = x_1$. To show that θ determines a well-defined homomorphism we first define the special element $X = x_1 x_2 \dots x_n$ in $A(C_n)$ and establish the following result.

Proposition 6.5. For any reflection R in C_n ,

$$X\theta(\{R\})X^{-1} = \theta(\{\gamma R \gamma^{-1}\}).$$

Proof. Since $X = x_1 Y_{1,n}$ and x_1 commutes with x_3, \dots, x_n , it follows that $Xx_i = x_{i+1}X$ for $2 \leq i < n$ and $Xx_{i,j} = x_{i+1,j+1}X$ for $1 \leq i < j < n$. This establishes the proposition for R of the form $\{(i, j)\}$ for $1 \leq i < j < n$.

The identity $Xy_j = y_{j+1}X$ for $1 \leq j < n$ is a consequence of the following calculation.

$$\begin{aligned} Y_{2,j+1}Xy_j &= x_2 Y_{3,j+1}Xy_j = x_2 X Y_{2,j} y_j = x_2 X x_1 Y_{2,j} \\ &= x_2 x_1 x_2 Y_{3,n} x_1 Y_{2,j} = x_2 x_1 x_2 x_1 Y_{3,n} Y_{2,j} \\ &= x_1 x_2 x_1 x_2 Y_{3,n} Y_{2,j} = x_1 x_2 X Y_{2,j} = x_1 x_2 Y_{3,j+1} X \\ &= x_1 Y_{2,j+1} X = Y_{2,j+1} y_{j+1} X \end{aligned}$$

This establishes the proposition for R of the form $[j]$ for $1 \leq i < n$.

Conjugating y_n by X gives x_1 , since

$$Xy_n = (x_1 x_2 \dots x_n) y_n = x_1 (x_2 \dots x_n y_n) = x_1 (x_1 \dots x_n).$$

This establishes the proposition for the reflection $[n]$.

Next we show $Xx_{i,n} = z_{1,i+1}X$.

$$\begin{aligned} z_{1,i+1}X &= z_{1,i+1} x_1 Y_{1,n} = x_1 x_{1,i+1} Y_{1,n} = x_1 x_{1,i+1} Y_{1,i} Y_{i,n} \\ &= x_1 Y_{1,i} x_{i+1} Y_{i,n} + x_1 Y_{1,i} x_{i+1} Y_{i+1,n} x_{i,n} = x_1 Y_{1,n} x_{i,n} = Xx_{i,n} \end{aligned}$$

This establishes the proposition for R of the form $\{(i, n)\}$ for $1 \leq i < n$.

The identity $Xz_{i,j} = z_{i+1,j+1}X$ for $1 \leq i < j < n$ follows from the definition of $z_{i,j}$ and the corresponding identities for $x_{i,j}$ and y_i , which establishes the proposition for R of the form $((i, -j))$ for $1 \leq i < j < n$.

Next we observe that, for $3 \leq j \leq n$, $y_j z_{1,j} = x_{1,j} y_j$ because

$$\begin{aligned} Y_{1,j} y_j z_{1,j} x_1 &= x_1 Y_{1,j} z_{1,j} x_1 = x_1 Y_{1,j} x_1 x_{1,j} = x_1 x_2 Y_{2,j} x_1 x_{1,j} \\ &= x_1 x_2 x_1 Y_{2,j} x_{1,j} = x_1 x_2 x_1 Y_{1,j} = x_1 x_2 x_1 x_2 Y_{2,j} \\ &= x_2 x_1 x_2 x_1 Y_{2,j} = x_2 x_1 x_2 Y_{2,j} x_1 = x_2 x_1 Y_{1,j} x_1 \\ &= x_2 Y_{1,j} y_j x_1 = Y_{1,j} x_{1,j} y_j x_1. \end{aligned}$$

Since $Xz_{i,n} y_i = X y_i x_{i,n} = y_{i+1} z_{1,i+1} X = x_{1,i+1} y_{i+1} X = x_{1,i+1} X y_i$, it follows that $Xz_{i,n} = x_{1,i+1} X$ and hence the proposition is established for the final case, R of the form $((i, -n))$ for $1 \leq i < n$. q.e.d.

Definition 6.6. We define a lift of γ to $A(C_n)$ to be an element of the form

$$E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\}),$$

where the R_i are reflections in C_n satisfying $R_1 R_2 \dots R_n = [1, 2, 3, \dots, n]$.

We note that one lift of γ to $A(C_n)$ is

$$X = x_1 x_2 \dots x_n = \theta(\{[1]\})\theta(\{(1, 2)\}) \dots \theta(\{(n-1, n)\}).$$

To show that θ is well-defined it suffices, by Lemma 6.1, to prove the following.

Proposition 6.7. For any lift E of γ to $A(C_n)$ we have $E = X$.

Proof. Given a lift $E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\})$ of γ to $A(C_n)$, we know that $R_1 R_2 \dots R_n = [1, 2, \dots, n]$ and by Lemma 3.4 exactly one of the R_k is of the form $[j]$. Since $E = X$ if and only if $X^l E X^{-l} = X$ for any integer l , we may assume by the previous proposition that $R_k = [1]$. We will construct a new lift E' of γ satisfying $E' = E$ and

$$E' = \theta(\{R_1\}) \dots \theta(\{R_{k-2}\})\theta(\{[1]\})\theta(\{R'\})\theta(\{R_{k+1}\}) \dots \theta(\{R_n\}),$$

for some reflection R' .

To simplify notation we set $R_{k-1} = T$ so that $R_{k-1} R_k = T[1]$. Since $T[1] \leq \gamma$ we know that $T \leq \gamma[1]$ or

$$T \leq (1, -2, -3, \dots, -n)$$

so that T has the form $((1, -p))$ for $2 \leq p \leq n$ or T has the form $((i, j))$ with $2 \leq i < j \leq n$. In the latter case $\theta(\{T\})$ lies in the subgroup of $A(C_n)$ generated by $\{x_3, x_4, \dots, x_n\}$ and so commutes with $\theta(\{[1]\}) = x_1$.

Thus we can use $R' = T$. In the former case, $\theta(\{T\}) = z_{1,p}$ and E' can be constructed using

$$\theta(\{T\})\theta(\{[1]\}) = z_{1,p}x_1 = x_1x_{1,p} = \theta(\{[1]\})\theta(\{(1,p)\}).$$

After $k - 1$ such steps we get $E = x_1\theta(\{S_2\})\dots\theta(\{S_n\})$, where the product on the right is a lift of γ to $A(C_n)$. However, this means $S_2S_3\dots S_n = (1, 2, \dots, n)$ in C_n so that $S_i \in \Sigma_n < C_n$ and

$$\theta(\{S_2\})\dots\theta(\{S_n\}) = x_2x_3\dots x_n,$$

by Lemma 4.6 of [3].

q.e.d.

Combining the results in this subsection we get the following theorem.

Theorem 6.8. *The poset group $\Gamma(C_n, \gamma)$ is isomorphic to the Artin group $A(C_n)$ for γ a Coxeter element in C_n .*

6.2. The D_n case. In this case our approach will be exactly as in the C_n case. However, the computations are more numerous and more complicated. The Artin group $A(D_n)$ has a presentation with generating set x_1, x_2, \dots, x_n , subject to the relations

$$\begin{aligned} x_1x_2 &= x_2x_1, \\ x_1x_3x_1 &= x_3x_1x_3, \\ x_1x_i &= x_ix_1, \quad \text{for } i \geq 4 \\ x_ix_{i+1}x_i &= x_{i+1}x_ix_{i+1}, \quad \text{for } 1 < i < n \quad \text{and} \\ x_ix_j &= x_jx_i, \quad \text{for } |j - i| \geq 2 \quad \text{and } i, j \neq 1. \end{aligned}$$

Definition 6.9. *We define a function ϕ from the generators of $A(D_n)$ to $\Gamma(D_n, \gamma)$ by*

$$x_1 \mapsto \{(1, -2)\}, x_2 \mapsto \{(1, 2)\}, x_3 \mapsto \{(2, 3)\}, \dots, x_n \mapsto \{(n-1, n)\}$$

Lemma 6.10. *The function ϕ determines a well-defined surjective homomorphism.*

Proof: The relations involving $\phi(x_1)$ hold in $\Gamma(D_n, \gamma)$ by virtue of the following identities in $\Gamma(D_n, \gamma)$.

$$\begin{aligned} \{(1, -2)\}\{(1, 2)\} &= \{[1][2]\} = \{(1, 2)\}\{(1, -2)\} \\ \{(1, -2)\}\{(2, 3)\}\{(1, -2)\} &= \{(1, -2, -3)\}\{(1, -2)\} \\ &= \{(2, 3)\}\{(1, -3)\}\{(1, -2)\} \\ &= \{(2, 3)\}\{(1, -2, -3)\} \\ &= \{(2, 3)\}\{(1, -2)\}\{(2, 3)\} \\ \{(1, -2)\}\{(i, i+1)\} &= \{(i, i+1)\}\{(1, -2)\}, \quad \text{for } i \geq 3. \end{aligned}$$

The image of the subgroup generated by $\{x_2, \dots, x_n\}$ again lies in the copy of the braid group corresponding to $\Sigma_n < D_n$ so that the relations not involving $\phi(x_1)$ hold by Lemma 4.2 and Lemma 4.4 of [3]. Thus ϕ is well-defined.

To establish surjectivity, note that both $\{(i, j)\}$ and $\{(i, i+1, \dots, j)\}$ lie in $\text{im}(\phi)$, for $1 \leq i < j \leq n$ as in the C_n case. To find the other reflection lifts in $\text{im}(\phi)$ first note that

$$\phi(x_1 x_2 \dots x_j) = \{[1][2, 3 \dots, j]\} = \{(1, -2)\}\{(1, 2, \dots, j)\} \in \text{im}(\phi),$$

and $\{(1, -j)\} \in \text{im}(\phi)$ for $j \geq 3$ since

$$\{(1, 2, \dots, j)\}\{(1, -j)\} = \{[1][2, \dots, j]\}.$$

Reflection lifts of the form $\{(2, -j)\}$ for $j \geq 3$ lie in $\text{im}(\phi)$ since

$$\{(1, -2)\}\{(1, j)\} = \{(1, j, -2)\} = \{(2, -j)\}\{(1, -2)\}$$

and reflection lifts of the form $\{(i, -j)\}$ for $3 \leq i < j \leq n$ lie in $\text{im}(\phi)$ since

$$\{(i, -j)\}\{(1, i)\}\{(1, -i)\} = \{[1][i, j]\} = \{(1, i)\}\{(1, -i)\}\{(i, j)\}.$$

q.e.d.

To construct an inverse to ϕ we will use the presentation for $\Gamma(D_n, \gamma)$ given by Lemma 6.1.

Definition 6.11. *We define a function θ from the generators of $\Gamma(D_n, \gamma)$ to $A(D_n)$ by*

$$\{(1, -2)\} \mapsto x_1, \quad \{(i, j)\} \mapsto x_{i,j} \quad \text{and} \quad \{(i, -j)\} \mapsto z_{i,j},$$

for $1 \leq i < j \leq n$, where $z_{i,j}$ is the unique element of $A(D_n)$ satisfying

$$\begin{aligned} z_{1,j}x_1 &= x_1x_{2,j} & \text{when } j &\geq 3 \\ z_{2,j}x_1 &= x_1x_{1,j} & \text{when } j &\geq 3 \\ z_{i,j}x_{1,i}z_{1,i} &= x_{1,i}z_{1,i}x_{i,j} & \text{when } 3 \leq i < j &\leq n \end{aligned}$$

We note that $z_{1,2} = x_1$. Since each $x_{i,j}$ lies in the copy of B_n generated by $\{x_2, \dots, x_n\}$ the elements $x_{i,j}$ satisfy the same identities as in the C_n case. The homomorphism determined by θ will be surjective since each x_i is the image of some reflection lift. To show that θ determines a well-defined homomorphism we define the special element $X = x_1x_2 \dots x_n$ in $A(D_n)$ and establish the D_n analogue of Proposition 6.5 .

Proposition 6.12. *For any reflection R in D_n ,*

$$X\theta(\{R\})X^{-1} = \theta(\{\gamma R \gamma^{-1}\}).$$

Proof. Since $X = x_1 Y_{1,n}$ and x_1 commutes with x_4, \dots, x_n it follows that $Xx_i = x_{i+1}X$ for $3 \leq i < n$ and $Xx_{i,j} = x_{i+1,j+1}X$ for $3 \leq i < j < n$. This establishes the proposition in the case $R = ((i, j))$ for $3 \leq i < j < n$.

For some of the later cases we will require the identities $x_{2,j}z_{1,j} = x_1x_{2,j}$ and $x_{1,j}z_{2,j} = x_1x_{1,j}$ for $3 \leq j \leq n$. The first follows from

$$\begin{aligned} Y_{3,j}x_{2,j}z_{1,j}x_1 &= x_3Y_{3,j}z_{1,j}x_1 = x_3Y_{3,j}x_1x_{2,j} = x_3x_1Y_{3,j}x_{2,j} \\ &= x_3x_1x_3Y_{3,j} = x_1x_3x_1Y_{3,j} = x_1x_3Y_{3,j}x_1 = x_1Y_{3,j}x_{2,j}x_1 \\ &= Y_{3,j}x_1x_{2,j}x_1, \end{aligned}$$

while the second follows from

$$\begin{aligned} x_1Y_{2,j}x_{1,j}z_{2,j}x_1 &= x_1x_2Y_{2,j}z_{2,j}x_1 = x_1x_2Y_{2,j}x_1x_{1,j} = x_1x_2x_3Y_{3,j}x_1x_{1,j} \\ &= x_1x_2x_3x_1Y_{3,j}x_{1,j} = x_2x_1x_3x_1Y_{3,j}x_{1,j} = x_2x_3x_1x_3Y_{3,j}x_{1,j} \\ &= x_2x_3x_1Y_{2,j}x_{1,j} = x_2x_3x_1x_2Y_{2,j} = x_2x_3x_1x_2Y_{2,j} \\ &= x_2x_3x_2x_1Y_{2,j} = x_3x_2x_3x_1Y_{2,j} = x_3x_2x_3x_1x_3Y_{3,j} \\ &= x_3x_2x_1x_3x_1Y_{3,j} = x_3x_2x_1x_3Y_{3,j}x_1 = x_3x_2x_1Y_{2,j}x_1 \\ &= x_3x_1x_2Y_{2,j}x_1 = x_3x_1Y_{2,j}x_{1,j}x_1 = x_3x_1x_3Y_{3,j}x_{1,j}x_1 \\ &= x_1x_3x_1Y_{3,j}x_{1,j}x_1 = x_1x_3Y_{3,j}x_{1,j}x_1. \end{aligned}$$

The conjugation action of X on x_1 is given by $Xx_1 = x_{1,3}X$ since

$$\begin{aligned} x_3Xx_1 &= x_3x_1x_2x_3Y_{3,n}x_1 = x_3x_1x_2x_3x_1Y_{3,n} = x_3x_2x_1x_3x_1Y_{3,n} \\ &= x_3x_2x_3x_1x_3Y_{3,n} = x_2x_3x_2x_1x_3Y_{3,n} = x_2x_3x_1x_2x_3Y_{3,n} \\ &= Y_{1,3}X = x_3x_{1,3}X. \end{aligned}$$

A similar calculation gives $x_3Xx_2 = x_1x_3X$. Since

$$x_1x_3X = x_1x_{2,3}X = x_{2,3}z_{1,3}X$$

we get $Xx_2 = z_{1,3}X$. This establishes the proposition in the cases $R = ((1, -2))$ and $R = ((1, 2))$.

Next we establish $Xx_n = z_{2,n}X$.

$$Xx_n = x_1Y_{1,n}x_n = x_1x_{1,n}Y_{1,n-1}x_n = z_{2,n}x_1Y_{1,n} = z_{2,n}X$$

which takes care of the case $R = ((n-1, n))$. To obtain the identity $Xx_{1,j} = z_{1,j+1}X$ we note that

$$Y_{1,n}x_{1,j}Y_{1,j-1} = Y_{1,n}Y_{1,j} = Y_{2,j+1}Y_{1,n} = x_{2,j+1}Y_{2,j}Y_{1,n} = x_{2,j+1}Y_{1,n}Y_{1,j-1}$$

giving $Y_{1,n}x_{1,j} = x_{2,j+1}Y_{1,n}$ so that

$$Xx_{1,j} = x_1Y_{1,n}x_{1,j} = x_1x_{2,j+1}Y_{1,n} = z_{1,j+1}x_1Y_{1,n} = z_{1,j+1}X.$$

This completes the case $R = ((1, j))$ for $2 \leq j < n$.

For the identity $Xx_{1,n} = x_2X$ we compute

$$Xx_{1,n} = x_1x_2(x_3 \dots x_n)x_{1,n} = x_1x_2(x_2x_3 \dots x_n) = x_2X,$$

which establishes the case $R = (1, n)$.

For $2 \leq i < n$ we have

$$\begin{aligned} Xx_{i,n} &= x_1Y_{1,i+1}Y_{i+1,n}x_{i,n} = x_1Y_{1,i+1}x_{i+1}Y_{i+1,n} \\ &= x_1x_{1,i+1}Y_{1,i}x_{i+1}Y_{i+1,n} = z_{2,i+1}x_1Y_{1,n} = z_{2,i+1}X \end{aligned}$$

and hence the proposition is true for $R = (i, n)$ with $2 \leq i < n$.

The identity $Xz_{1,j} = x_{1,j+1}X$ for $3 \leq j < n$ follows from

$$Xz_{1,j}x_1 = Xx_1x_{2,j} = x_{1,3}x_{3,j+1}X = x_{1,j+1}x_{1,3}X = x_{1,j+1}Xx_1,$$

while the identity $Xz_{1,n} = z_{1,2}X = x_1X$ follows from

$$Xz_{1,n}x_1 = Xx_1x_{2,n} = x_{1,3}z_{2,3}X = x_1x_{1,3}X = x_1Xx_1.$$

This establishes the proposition for $R = (1, -j)$ with $2 \leq j \leq n$.

The identity $Xz_{i,n} = x_{2,i+1}X$ for $2 \leq i < n$ follows from

$$\begin{aligned} Xz_{i,n}x_{1,i}z_{1,i} &= Xx_{1,i}z_{1,i}x_{i,n} = z_{1,i+1}x_{1,i+1}z_{2,i+1}X \\ &= z_{1,i+1}x_1x_{1,i+1}X = x_1x_{2,i+1}x_{1,i+1}X \\ &= x_{2,i+1}z_{1,i+1}x_{1,i+1}X = x_{2,i+1}Xx_{1,i}z_{1,i}. \end{aligned}$$

This establishes the proposition for $R = (i, -n)$ with $2 \leq i < n$.

Finally we note that $x_{1,i}z_{1,i} = z_{1,i}x_{1,i}$ since

$$\begin{aligned} x_{2,i}x_{1,i}z_{1,i} &= x_2x_{2,i}z_{1,i} = x_2x_1x_{2,i} = x_1x_2x_{2,i} \\ &= x_1x_{2,i}x_{1,i} = x_{2,i}z_{1,i}x_{1,i}. \end{aligned}$$

From this we deduce that $Xz_{i,j} = z_{i+1,j+1}X$ for $2 \leq i < j < n$ since

$$\begin{aligned} Xz_{i,j}x_{1,i}z_{1,i} &= Xx_{1,i}z_{1,i}x_{i,j} = z_{1,i+1}x_{1,i+1}x_{i+1,j+1}X \\ &= z_{i+1,j+1}z_{1,i+1}x_{1,i+1}X = z_{i+1,j+1}Xx_{1,i}z_{1,i}. \end{aligned}$$

This establishes the proposition for the remaining cases $R = (i, -j)$ with $2 \leq i < j < n$. q.e.d.

Definition 6.13. We define a lift of γ to $A(D_n)$ to be an element of the form

$$E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\}),$$

where the R_i are reflections in D_n satisfying $R_1R_2 \dots R_n = [1][2, 3, \dots, n]$.

We note that one lift of γ to $A(D_n)$ is

$$X = x_1 x_2 \dots x_n = \theta(\{(1, -2)\})\theta(\{(1, 2)\}) \dots \theta(\{(n-1, n)\}).$$

To show that θ determines a well-defined homomorphism it suffices, by Lemma 6.1, to prove the following.

Proposition 6.14. *For any lift E of γ to $A(D_n)$ we have $E = X$.*

Proof: Given a lift E of γ to $A(D_n)$, where

$$E = \theta(\{R_1\})\theta(\{R_2\}) \dots \theta(\{R_n\}),$$

we know that $R_1 R_2 \dots R_n = [1][2, \dots, n]$. It follows for the proof of proposition 4.2 that one of the R_k is of the form $(1, \pm j)$. Since $E = X$ if and only if $X^l E X^{-l} = X$ for any integer l , we may assume $R_k = (1, \pm 2)$. We treat these two cases separately.

Suppose that $R_k = (1, -2)$. We will construct a new lift E' of γ satisfying $E' = E$ and

$$E' = \theta(\{R_1\}) \dots \theta(\{R_{k-2}\})\theta(\{(1, -2)\})\theta(\{R'\})\theta(\{R_{k+1}\}) \dots \theta(\{R_n\}),$$

for some reflection R' .

To simplify notation we set $R_{k-1} = T$ so that $R_{k-1} R_k = T(1, -2)$. Since $T(1, -2) \leq [1][2, \dots, n]$ we know that

$$T \leq (1, -3, -4, \dots, -n, 2)$$

so that T has one of the forms

- (1) $(1, 2)$,
- (2) (i, j) for $3 \leq i < j \leq n$,
- (3) $(1, -p)$ for $3 \leq p \leq n$ or
- (4) $(2, -p)$ for $3 \leq p \leq n$.

In the first case $\theta(\{T\}) = x_2$, which commutes with $\theta(\{(1, -2)\}) = x_1$. In the second case, $\theta(\{T\}) = x_{i,j}$ lies in the subgroup generated by $\{x_4, \dots, x_n\}$ and hence also commutes with x_1 . In the third case E' can be constructed using

$$\theta(\{T\})\theta(\{(1, -2)\}) = z_{1,p}x_1 = x_1x_{2,p} = \theta(\{(1, -2)\})\theta(\{(2, p)\})$$

and in the fourth case using

$$\theta(\{T\})\theta(\{(1, -2)\}) = z_{2,p}x_1 = x_1x_{1,p} = \theta(\{(1, -2)\})\theta(\{(1, p)\}).$$

After $k-1$ such steps we get $E = x_1\theta(\{S_2\}) \dots \theta(\{S_n\})$, where the product on the right is a lift of γ to $A(D_n)$. However, this means $S_2 S_3 \dots S_n = (1, 2, \dots, n)$ in C_n so that $S_i \in \Sigma_n < C_n$ and

$$\theta(\{S_2\}) \dots \theta(\{S_n\}) = x_2 x_3 \dots x_n,$$

by Lemma 4.6 of [3].

Next suppose $R_k = \langle\langle 1, 2 \rangle\rangle$. As in the previous case, we will construct a new lift E' of γ satisfying $E' = E$ and

$$E' = \theta(\{R_1\}) \dots \theta(\{R_{k-2}\})\theta(\langle\langle 1, 2 \rangle\rangle)\theta(\{R'\})\theta(\{R_{k+1}\}) \dots \theta(\{R_n\}),$$

for some reflection R' . To simplify notation we again set $R_{k-1} = T$ so that $R_{k-1}R_k = T\langle\langle 1, 2 \rangle\rangle$. Since $T\langle\langle 1, 2 \rangle\rangle \leq [1][2, \dots, n]$ we know that

$$T \leq \langle\langle 1, 3, 4, \dots, n, -2 \rangle\rangle$$

so that T has one of the forms

- (1) $\langle\langle 1, -2 \rangle\rangle$,
- (2) $\langle\langle i, j \rangle\rangle$ for $3 \leq i < j \leq n$,
- (3) $\langle\langle 1, p \rangle\rangle$ for $3 \leq p \leq n$ or
- (4) $\langle\langle 2, -p \rangle\rangle$ for $3 \leq p \leq n$.

In the first case $\theta(\{T\}) = x_1$, which commutes with $\theta(\langle\langle 1, 2 \rangle\rangle) = x_2$. In the second case, $\theta(\{T\}) = x_{i,j}$ lies in the subgroup generated by $\{x_4, \dots, x_n\}$ and hence also commutes with x_2 . In the third case E' can be constructed using

$$\theta(\{T\})\theta(\langle\langle 1, 2 \rangle\rangle) = x_{1,p}x_{1,2} = x_{1,2}x_{2,p} = \theta(\langle\langle 1, 2 \rangle\rangle)\theta(\langle\langle 2, p \rangle\rangle).$$

In the fourth case E' is constructed using

$$\theta(\{T\})\theta(\langle\langle 1, 2 \rangle\rangle) = z_{2,p}x_2 = x_2z_{1,p} = \theta(\langle\langle 1, 2 \rangle\rangle)\theta(\langle\langle 1, -p \rangle\rangle).$$

The middle equality holds since

$$z_{2,p}x_2x_1 = z_{2,p}x_1x_2 = x_1x_{1,p}x_2 = x_1x_2x_{2,p} = x_2x_1x_{2,p} = x_2z_{1,p}x_1.$$

After $k - 1$ such steps we get $E = x_2\theta(\{S_2\}) \dots \theta(\{S_n\})$, where the product on the right is a lift of γ to $A(D_n)$. However, this means $S_2S_3 \dots S_n = \langle\langle 1, -2, \dots, -n \rangle\rangle$ in C_n so that S_i lie in the copy of Σ_n generated $\{\langle\langle 1, -2 \rangle\rangle, \langle\langle 2, 3 \rangle\rangle, \dots, \langle\langle n-1, n \rangle\rangle\}$ and

$$\theta(\{S_2\}) \dots \theta(\{S_n\}) = x_1x_3 \dots x_n,$$

by Lemma 4.6 of [3]. Finally

$$E = x_2x_1x_3 \dots x_n = x_1x_2x_3 \dots x_n.$$

q.e.d.

Combining the results in this subsection we get the following theorem.

Theorem 6.15. *The poset group $\Gamma(D_n, \gamma)$ is isomorphic to the Artin group $A(D_n)$ for γ a Coxeter element in D_n .*

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