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FROM PERMUTAHEDRON TO ASSOCIAHEDRON

THOMAS BRADY AND COLUM WATT

ABSTRACT. For each finite real reflection group W , we identify a copy of the type- W simplicial generalised associahedron inside the corresponding simplicial permutahedron. This defines a bijection between the facets of the generalised associahedron and the elements of the type W non-crossing partition lattice which is more tractable than previous such bijections. We show that the simplicial fan determined by this associahedron coincides with the Cambrian fan for W .

1. INTRODUCTION

Let W be an irreducible finite real reflection group of rank n acting on \mathbf{R}^n . The *type- W simplicial permutahedron* is the simplicial complex obtained by intersecting the unit sphere \mathbf{S}^{n-1} with the fan defined by the reflecting hyperplanes of W . The *type- W simplicial generalised associahedron* is obtained by intersecting the unit sphere \mathbf{S}^{n-1} with the cluster fan associated to a chosen Coxeter element c of W (see [3]). Since its introduction, similarities have been noticed between the local structures of the generalised associahedron and the corresponding permutahedron. In the $W = A_n$ case, this relationship was investigated in [8]. In [5], a combinatorial isomorphism (which is linear for bipartite factorisations of c) is constructed between the cluster fan and the Cambrian fan, a certain coarsening of the fan defined by the reflecting hyperplanes of W . In [4] it is shown that the Cambrian fan is the normal fan of a simple polytope.

This paper constructs the c -Cambrian fan for a bipartite Coxeter element c , without using Coxeter-sortable elements. Our approach exhibits the c -Cambrian fan as the fan determined by the image $\mu(AX(c))$ of an isometric copy $AX(c)$ of the simplicial generalised associahedron under the linear isomorphism $\mu = 2(I - c)^{-1}$ from [2]. The vertex set of the complex $AX(c)$ consists of the positive roots and the first n negative roots relative to the total ordering on roots defined in [2]. We show that the codimension one simplices of $\mu(AX(c))$ are pieces of the

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original reflecting hyperplanes and that each facet is a union of permutahedron facets. Thus the fan defined by $\mu(\text{AX}(c))$ is a coarsening of the fan determined by the reflection hyperplanes and we show that this fan coincides with the c -Cambrian fan.

The set of facets of $\mu(\text{AX}(c))$ and the non-crossing partition lattice, NCP_c , are equinumerous (see, for example, [1]). In the current setting, this can be shown with the following easily described bijection.

For each $w \in \text{NCP}_c$, define a region $F(w)$ in \mathbf{R}^n as follows. Let $\{\delta_1, \dots, \delta_k\}$ be the simple system for the parabolic subgroup determined by w (with reflection set consisting of those reflections in W whose fixed hyperplanes contain the fixed subspace of w) and let $\{\theta_1, \dots, \theta_{n-k}\}$ be the simple system for the parabolic subgroup determined by cw^{-1} . Now set

$$F(w) = \{x \in \mathbf{R}^n \mid x \cdot \delta_i \leq 0 \quad \text{and} \quad x \cdot \theta_j \geq 0\}.$$

Our main theorem is the following.

Theorem 1.1. *The collection $\{F(w) \mid w \in \text{NCP}_c\}$ is the set of facets of a complete simplicial fan. Moreover, this fan is linearly isomorphic to the corresponding cluster fan.*

Note: The recent paper [6] defines cones for a general (not necessarily finite) W via Coxeter-sortable elements. We expect that these cones should coincide with the facets $F(w)$ for finite W and bipartite c , but this has not been shown.

2. PRELIMINARIES

Fix a fundamental chamber C for the action of W on \mathbf{R}^n , denote the inward unit normals by $\alpha_1, \dots, \alpha_n$ and let R_1, \dots, R_n be the corresponding reflections. Assume that $S_1 = \{\alpha_1, \dots, \alpha_s\}$ and $S_2 = \{\alpha_{s+1}, \dots, \alpha_n\}$ are orthonormal sets. Let $c = R_1 R_2 \dots R_n$ be the corresponding bipartite Coxeter element. Letting \mathbf{T} be the set of all reflections in W , the total reflection length function on W is defined by

$$\ell(w) = \min\{k > 0 \mid w = T_1 T_2 \dots T_k, T_i \in \mathbf{T}\}.$$

We recall from [2] that $\ell(w)$ is the dimension of $M(w)$, the orthogonal complement of the fixed subspace of the orthogonal transformation w . The total reflection order on W is defined by

$$u \preceq w \quad \text{if and only if} \quad \ell(u) + \ell(u^{-1}w) = \ell(w)$$

and the set of W -noncrossing partitions, NCP_c , is defined to be the subset of W consisting of those elements w satisfying $w \preceq c$. Associated

to each $w \in \text{NCP}_c$ is a parabolic subgroup W_w , which is the finite reflection group with reflection set consisting of those $T \in \mathbf{T}$ with $T \preceq w$. The W fundamental domain C lies in a unique chamber for the action of W_w on \mathbf{R}^n and hence determines a simple system Π_w for W_w .

In [2], a total order, \leq , on the roots (vectors of the form $w(\alpha_i)$ for $w \in W$ and $1 \leq i \leq n$) is defined, following [7], by

$$\rho_i = R_1 R_2 \dots R_{i-1}(\alpha_i),$$

with R_j and α_i defined cyclically modulo n . Furthermore a simplicial complex $EX(c)$ is constructed with vertex set

$$\{\rho_{-n+s+1}, \dots, \rho_0, \rho_1, \dots, \rho_{nh/2}, \rho_{nh/2+1}, \dots, \rho_{nh/2+s}\}$$

(where $\rho_{-k} = \rho_{nh-k}$) and a simplex on each subset $\{\tau_1, \tau_2, \dots, \tau_k\}$ of the vertices satisfying

$$\tau_1 < \tau_2 < \dots < \tau_k \quad \text{and} \quad \ell(R(\tau_1) \dots R(\tau_k)\gamma) = n - k.$$

It is shown in [2] that $EX(c)$ coincides with the type- W generalised associahedron. We will continue to use the notation from [2]. In particular, $X(w)$ will denote the subcomplex of $EX(c)$ consisting of those simplices whose vertices are positive roots in the subspace $M(w)$ for $w \in \text{NCP}_c$ and μ will denote the linear operator $2(I - c)^{-1}$. We recall that if τ is a root of unit length then $\mu(\tau)$ is the unique vector in the fixed subspace of the length $n - 1$ element $R(\tau)c$ satisfying $\mu(\tau) \cdot \tau = 1$. Furthermore, $\{\mu(\rho_1), \dots, \mu(\rho_n)\}$ is the dual basis to $\{\alpha_1, \dots, \alpha_n\}$ and $c[\mu(\rho_i)] = \mu(\rho_{i+n})$.

3. THE INTERMEDIATE COMPLEX $AX(c)$.

Since S_1 and S_2 are orthonormal sets, c factors as a product of two involutions, $c = c_+ c_-$, where

$$c_+ = R(\alpha_1) \dots R(\alpha_s) \quad \text{and} \quad c_- = R(\alpha_{s+1}) \dots R(\alpha_n).$$

It follows that $c_+(S_1) = -S_1$ and that $c_+(S_2) = c_+ c_- c_-(S_2) = -c(S_2)$.

Definition 3.1. *We define the simplicial complex $AX(c)$ to be the result of applying the involution c_+ to $EX(c)$.*

The vertices and simplices of $AX(c)$ have the following characterisation.

Proposition 3.2. *The simplicial complex $AX(c)$ has vertex set*

$$\{\rho_1, \dots, \rho_{nh/2+n}\},$$

and a simplex on $\{\tau_1, \dots, \tau_k\}$ provided

$$\rho_1 \leq \tau_1 < \tau_2 < \dots < \tau_k \leq \rho_{nh/2+n} \quad \text{and} \quad \ell[R(\tau_1) \dots R(\tau_k)c] = n - k.$$

Proof. It follows from Sections 3 and 8 of [2] that the ordered set of roots

$$\{\rho_{-n+s+1}, \dots, \rho_0, \rho_1, \dots, \rho_{nh/2}, \rho_{nh/2+1}, \dots, \rho_{nh/2+s}, \dots, \rho_{nh/2+n}\}$$

is partitioned into the ordered sequence of subsets

$$-S_2, S_1, c(-S_2), c(S_1), \dots, c^{-1}(-S_1), S_2, -S_1, c(S_2).$$

Since c_+ and c_- are involutions, it follows that $c_+c_+ = c^{-1}$ and hence that

$$\begin{aligned} (1) \quad c_+(c^k(S_1)) &= c_+(c^k)c_+c_+(S_1) = -c^{-k}(S_1), \\ (2) \quad c_+(c^k(S_2)) &= c_+(c^k)c_+c_+(S_2) = -c^{1-k}(S_2). \end{aligned}$$

Thus the action of c_+ on the ordered sequence of subsets is

$$\begin{array}{ccccccccc} -S_2 & S_1 & -c(S_2) & c(S_1) & -c^2(S_2) & c^2(S_1) & \dots & & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & \\ c(S_2) & -S_1 & S_2 & -c^{-1}(S_1) & c^{-1}(S_2) & -c^{-2}(S_1) & \dots & & \end{array}$$

As the vertex set of $EX(c)$ is

$$-S_2 \cup S_1 \cup c(-S_2) \cup c(S_1) \cup \dots \cup c^{-1}(-S_1) \cup S_2 \cup -S_1$$

the vertex set of $AX(c)$ is

$$S_1 \cup c(-S_2) \cup c(S_1) \cup \dots \cup c^{-1}(-S_1) \cup S_2 \cup -S_1 \cup c(S_2)$$

which is equal to $\{\rho_1, \rho_2, \dots, \rho_{n+nh/2}\}$.

Next suppose τ and σ are vertices of $AX(c)$ with $\tau < \sigma$. We will show that an edge in $AX(c)$ joins τ and σ if and only if

$$c = R(\sigma)R(\tau)x \quad \text{for some } x \in W \text{ with } \ell(x) = n - 2.$$

Indeed, by definition, an edge in $AX(c)$ joins τ and σ if and only if an edge in $EX(c)$ joins $c_+(\tau)$ and $c_+(\sigma)$ and this holds if and only if either

- (i) $c_+(\sigma) < c_+(\tau)$ and $c = R(c_+(\tau))R(c_+(\sigma))y$ for some $y \in W$ with $\ell(y) = n - 2$, or
- (ii) $c_+(\tau) < c_+(\sigma)$ and $c = R(c_+(\sigma))R(c_+(\tau))y$ for some $y \in W$ with $\ell(y) = n - 2$.

Since the c_+ action inverts the order of the subsets $\pm c^k(S_i)$, the relation $c_+(\tau) < c_+(\sigma)$ can only occur when τ and σ belong to the same subset

$\pm c^k(S_j)$. Because these subsets are orthogonal, it follows that τ and σ are joined by an edge in $AX(c)$ if and only if

$$c = R(c_+(\tau))R(c_+(\sigma))y \quad \text{for some } y \text{ with } \ell(y) = n - 2.$$

However, using the fact that the set of reflections in W is closed under conjugation we deduce that $c = R(c_+(\tau))R(c_+(\sigma))y$ is equivalent to

$$c^{-1} = c_+cc_+ = R(\tau)R(\sigma)z = tR(\tau)R(\sigma)$$

which in turn is equivalent to $c = R(\sigma)R(\tau)x$, where y , z and t are length $n - 2$ elements in W , z is conjugate to y , t is conjugate to z and $x = t^{-1}$. In particular, $c = R(c_+(\tau))R(c_+(\sigma))y$ for some y with $\ell(y) = n - 2$ if and only if $c = R(\sigma)R(\tau)x$ for some x with $\ell(x) = n - 2$. This establishes the characterisation of edges in $AX(c)$. As both $EX(c)$ and $AX(c)$ are determined by their 1-skeletons, the proposition follows. \square

4. VERTEX TYPE REVISITED.

In this section we construct a bijection between facets of $AX(c)$ and elements of NCP_c by partitioning the vertices of each facet F of $AX(c)$ into forward and backward vertices in a manner similar to the way vertices of facets are partitioned into right and left vertices in [1]. The two notions of vertex type in a facet are different. We choose the one below because we can give a uniform characterisation of both forward vertices and backward vertices of facets. We recall that for $w \in \text{NCP}_c$, $X(w)$ is the subcomplex of $EX(c)$ (and hence also of $AX(c)$) consisting of those simplices whose vertices are positive roots in the subspace $M(w)$. The total order on the vertices of $EX(c)$ allows us to put a lexicographic order on simplices. In particular, the lexicographically first facets, which we will often refer to simply as the first facets, of the subcomplexes $X(w)$, for $w \in \text{NCP}_c$, will be used to define the bijection.

In this section F will be a facet of $AX(c)$ with ordered vertices $\tau_1 < \tau_2 < \dots < \tau_n$ so that $c = R(\tau_n)R(\tau_{n-1}) \cdots R(\tau_1)$.

Definition 4.1. For $1 \leq i \leq n$ we define the noncrossing partitions

$$\begin{aligned} u_i &= u_i(F) = R(\tau_n)R(\tau_{n-1}) \cdots R(\tau_i) \\ v_i &= v_i(F) = R(\tau_i) \cdots R(\tau_2)R(\tau_1) \end{aligned}$$

and say that τ_i is a **forward** vertex in F if τ_i is a vertex of the first facet of $X(v_i)$. Otherwise, we say that τ_i is a **backward** vertex in F .

Lemma 4.2. (i) If $\tau_i \in \{\rho_1, \dots, \rho_n\}$, then τ_i must be a forward vertex of F .

(ii) If $\tau_i \in \{\rho_{nh/2+1}, \dots, \rho_{nh/2+n}\}$, then τ_i must be a backward vertex of

F .

Proof. (i) Suppose $\tau_i = \rho_j$ is one of the first n roots. We claim that

$$M(v_i) \cap \{\rho_1, \rho_2, \dots, \rho_j = \tau_i\} = \{\tau_1, \dots, \tau_i\}.$$

Indeed the inclusion $\{\tau_1, \dots, \tau_i\} \subseteq M(v_i) \cap \{\rho_1, \rho_2, \dots, \rho_j = \tau_i\}$ follows from the definition of v_i and the ordering of the τ 's. If the reverse inclusion did not hold we would have $R(\rho_k) \preceq v_i$ for some ρ_k satisfying

$$\rho_k < \rho_j = \tau_i \quad \text{and} \quad \rho_k \notin \{\tau_1, \dots, \tau_i\}.$$

However, since the first n roots are linearly independent, $\{\tau_1, \dots, \tau_i, \rho_k\}$ would be a set of $i + 1$ linearly independent vectors in $M(v_i)$ contradicting $\ell(v_i) = i$. Thus the above equality of sets holds and the first facet of v_i is forced to have vertex set $\{\tau_1, \dots, \tau_i\}$. In particular, τ_i is a forward vertex of F .

(ii) The first facet of $X(w)$ is necessarily a set of positive roots for any $w \in \text{NCP}_c$. Thus any vertex of a facet F which is also a negative root must be a backward vertex of F . \square

The characterisation of forward vertices of a facet uses Lemma 3.3 of [1], which we now recall. Here $\delta_1, \dots, \delta_k$ is the ordered simple system for W_w .

Lemma 4.3 (Lemma 3.3 of [1]). *Let τ be a positive root in $M(w)$ and fix $1 \leq i \leq k$. Then $\tau \in \{\epsilon_1, \dots, \epsilon_i\}$ if and only if $R(\delta_i)R(\delta_{i-1}) \cdots R(\delta_1)\tau$ is a negative root. In particular, $\tau \in \{\epsilon_1, \dots, \epsilon_k\}$ if and only if $w^{-1}(\tau)$ is a negative root.*

Lemma 4.4. *The root τ_i is a forward vertex of F if and only if $v_i^{-1}(\tau_i)$ is a negative root.*

Proof. (\Rightarrow) By definition, if τ_i is a forward vertex of F , then τ_i is a vertex of the first facet of $X(v_i)$ and by Lemma 4.3, $v_i^{-1}(\tau_i)$ is a negative root.

(\Leftarrow) Conversely, assume that $v_i^{-1}(\tau_i)$ is a negative root. We deal separately with the two cases where τ_i is positive or negative. If τ_i is a positive root then τ_i is a vertex of the first facet of $X(v_i)$ by Lemma 4.3.

On the other hand, if τ_i is negative then

$$\tau_i \in \{\rho_{nh/2+1}, \dots, \rho_{nh/2+n}\} = (-S_1) \cup c(S_2).$$

Hence $-\tau_i$ belongs to $S_1 \cup c(-S_2)$ and is one of the first n roots. Since the first n roots form a linearly independent set, the vectors in

$$\{\rho_1, \dots, \rho_n\} \cap M(v_i)$$

must all lie in the first facet of $X(v_i)$. In particular, $-\tau_i$ lies in the first facet of $X(v_i)$. Thus $v_i^{-1}(-\tau_i)$ is a negative root by Lemma 4.3. However, this gives a contradiction since $v_i^{-1}(\tau_i)$ is assumed to be negative. \square

The following is an immediate consequence.

Corollary 4.5. *The root τ_i is a backward vertex of F if and only if $v_i^{-1}(\tau_i)$ is a positive root.*

We now turn to the characterisation of backward vertices of facets in $AX(c)$. We begin with an elementary observation.

Lemma 4.6. *If θ_i is the root defined by $\theta_i = c^{-1}(\tau_i)$ then*

$$v_i^{-1}(\tau_i) = -c^{-1}u_i c(\theta_i).$$

Proof. For convenience, let r denote the reflection $R(\tau_i)$ and note that $c = u_i r v_i$. Thus $v_i^{-1} = c^{-1}u_i r$ and hence

$$v_i^{-1}(\tau_i) = c^{-1}u_i r(\tau_i) = c^{-1}u_i(-\tau_i) = -c^{-1}u_i c(\theta_i).$$

\square

The characterisation of backward vertices also uses a result of [1], which we now recall.

Lemma 4.7 (Corollary 3.15 of [1]). *Let τ be a positive root in $M(w)$. Then $w(\tau)$ is a negative root if and only if τ is a vertex of the last facet of $X(w)$.*

Lemma 4.8. *The root τ_i is a backward vertex of F if and only if $\tau_i = c(\theta_i)$ for some vertex θ_i in the last facet of $X(c^{-1}u_i c)$.*

Proof. (\Leftarrow) Suppose that $\tau_i = c(\theta_i)$ for some vertex θ_i in the last facet of $X(c^{-1}u_i c)$. Then Lemma 4.6 gives $v_i^{-1}(\tau_i) = -c^{-1}u_i c(\theta_i)$. However the fact that θ_i is in the last facet of $X(c^{-1}u_i c)$ means that $c^{-1}u_i c(\theta_i)$ is negative by Lemma 4.7. Thus $v_i^{-1}(\tau_i)$ is positive and τ_i is a backward vertex of F by Corollary 4.5.

(\Rightarrow) Conversely, suppose that τ_i is a backward vertex of F . Then, by part (i) of Lemma 4.2, τ_i is not one of the first n roots. However, since $c(\rho_i) = \rho_{i+n}$, this means that $c^{-1}\tau_i$ is a positive root. Let θ_i be this positive root. By Lemma 4.7, it remains to show that $c^{-1}u_i c(\theta_i)$ is a negative root. However, by Lemma 4.6, $c^{-1}u_i c(\theta_i) = -v_i^{-1}(\tau_i)$ and this root is negative, by Corollary 4.5, since we are assuming τ_i is backward. Thus θ_i is a vertex of the last facet of $X(c^{-1}u_i c)$. \square

As in Lemma 5.3 of [1], forward and backward vertices of a facet F of $AX(c)$ are orthogonal if they appear in the wrong order in the factorisation of c determined by F . The induction proof of Lemma 5.3 of [1] could be adapted here but it is possible to give a more conceptual proof.

Lemma 4.9. *If τ_i is a backward vertex of F and τ_j is a forward vertex of F with $\tau_i < \tau_j$ then $\tau_i \cdot \tau_j = 0$.*

Proof. Let $\{\epsilon_1, \dots, \epsilon_j\}$ be the ordered vertex set of the first facet of $X(v_j)$, where v_j is the noncrossing partition

$$v_j = R(\tau_j) \dots R(\tau_1) = R(\epsilon_j) \dots R(\epsilon_1).$$

Since τ_j is forward, τ_j must, by definition, be one of $\epsilon_1, \dots, \epsilon_j$. Moreover, since the set $\{\tau_1, \tau_2, \dots, \tau_j\}$ is linearly independent we must have $\tau_j = \epsilon_j$. If $\tau_i \notin \{\epsilon_1, \dots, \epsilon_{j-1}\}$ then Lemma 3.4 of [1] gives $\tau_j \cdot \tau_i = 0$. Thus it remains to show that τ_i is not one of $\epsilon_1, \dots, \epsilon_{j-1}$.

In order to show this, let $\{\epsilon'_1, \dots, \epsilon'_i\}$ be the ordered vertex set of the first facet of $X(v_i)$, where v_i is the noncrossing partition

$$v_i = R(\tau_i) \dots R(\tau_1) = R(\epsilon'_i) \dots R(\epsilon'_1).$$

Since τ_i is backward, $\tau_i \notin \{\epsilon'_1, \dots, \epsilon'_i\}$ by definition. However, since $\{\epsilon'_1, \dots, \epsilon'_i\}$ is a basis for $M(v_i)$ and $\tau_i > \epsilon'_i$, it follows that the root τ_i lies in the linear span of the set

$$\{\rho \in M(v_i) \cap \{\rho_1, \dots, \rho_{nh/2}\} \mid \rho < \tau_i\}.$$

Since $v_i \preceq v_j$, we deduce that τ_i lies in the linear span of

$$\{\rho \in M(v_j) \cap \{\rho_1, \dots, \rho_{nh/2}\} \mid \rho < \tau_i\}.$$

However, the linear span of $\{\rho \in M(v_j) \cap \{\rho_1, \dots, \rho_{nh/2}\} \mid \rho < \epsilon_k\}$ does not contain ϵ_k for any $1 \leq k \leq j-1$ by Corollary 6.12 of [2]. Thus, $\tau_i \neq \epsilon_k$ for $1 \leq k \leq j-1$. \square

Theorem 4.10. *The function ϕ from facets of $AX(c)$ to NCP_c taking a facet F with forward vertices $\tau_{i_1} < \tau_{i_2} < \dots < \tau_{i_k}$ to the product $R(\tau_{i_k})R(\tau_{i_{k-1}}) \dots R(\tau_{i_1})$ is a bijection.*

Proof. Suppose F is a facet with $\phi(F) = v$. By Lemma 4.9 appropriate pairs of factors of

$$c = R(\tau_n)R(\tau_{n-1}) \dots R(\tau_1)$$

can be commuted until the product $R(\tau_{i_k})R(\tau_{i_{k-1}}) \dots R(\tau_{i_1})$ appears on the right. By Lemma 4.4 the forward vertices of F are precisely the vertices of the first facet of $X(v)$. Since $c = (cv^{-1})v$ and $c^{-1}(cv^{-1})c = v^{-1}c$, Lemma 4.8 implies that the backward vertices of F are the images

under c of the vertices of the last facet of $X(v^{-1}c)$. Thus the vertex set of F is completely determined by v and hence ϕ is injective. On the other hand, we know from Theorem 6.4 of [1] that the number of facets of $EX(c)$ is the same as the number of elements of NCP_c . Since $AX(c)$ is the image of $EX(c)$ under the isometry c_+ it follows that ϕ is a bijection. \square

The following result is immediate from Theorem 4.10 and its proof.

Corollary 4.11. *For each $v \in NCP_c$ there is a facet of $AX(c)$ whose vertex set consists of the vertices of the first facet of $X(v)$ and the images under c of the vertices of the last facet of $X(v^{-1}c)$. Moreover, every facet of $AX(c)$ arises in this way.*

5. APPLYING THE μ OPERATOR.

Definition 5.1. *We define the simplicial complex $\mu(AX(c))$ to be the result of applying the operator $\mu = 2(I - c)^{-1}$ to $AX(c)$.*

The vertices and simplices of $\mu(AX(c))$ have a simple characterisation which follows immediately from Proposition 3.2.

Proposition 5.2. *The simplicial complex $\mu(AX(c))$ has vertex set*

$$\{\mu(\rho_1), \dots, \mu(\rho_{nh/2+n})\},$$

and a simplex on $\{\mu(\tau_1), \dots, \mu(\tau_k)\}$ provided

$$\rho_1 \leq \tau_1 < \tau_2 < \dots < \tau_k \leq \rho_{nh/2+n} \quad \text{and} \quad \ell[R(\tau_1) \dots R(\tau_k)c] = n - k.$$

Now we are in a position to show that the cones on the facets of $\mu(AX(c))$ are precisely the cones $F(w)$ defined in the introduction. Recall that

$$F(w) = \{x \in \mathbf{R}^n \mid x \cdot \delta_i \leq 0 \quad \text{and} \quad x \cdot \theta_j \geq 0\},$$

where $\{\delta_1, \dots, \delta_k\}$ is the simple system for the parabolic subgroup determined by w and $\{\theta_1, \dots, \theta_{n-k}\}$ is the simple system for the parabolic subgroup determined by $w' = cw^{-1}$. We note that $F(w)$ is a simplicial cone of dimension n since

$$c = w'w = R(\theta_1) \dots R(\theta_{n-k})R(\delta_1) \dots R(\delta_k)$$

means that $\{\delta_1, \dots, \delta_k, \theta_1, \dots, \theta_{n-k}\}$ is a linearly independent set. We first determine the rays of each $F(w)$.

Proposition 5.3. *Suppose $w \in NCP_c$ and $F(w)$ is the simplicial cone defined above. Then the rays of $F(w)$ are generated by*

$$\{\mu(\epsilon_1), \dots, \mu(\epsilon_{n-k}), \mu[c(\eta_{n-k+1})], \dots, \mu[c(\eta_n)]\},$$

where $\{\eta_{n-k+1}, \dots, \eta_n\}$ is the vertex set of the last facet of $X(w)$ and $\{\epsilon_1, \dots, \epsilon_{n-k}\}$ is the vertex set of the first facet of $X(cw^{-1})$.

Proof. Suppose $\{\tau_1, \dots, \tau_n\}$ is an arbitrary set of positive roots satisfying $c = R(\tau_1) \dots R(\tau_n)$. We are interested in the case

$$\tau_i = \begin{cases} \theta_i & \text{for } 1 \leq i \leq n-k, \\ \delta_{i-n+k} & \text{for } n-k+1 \leq i \leq n, \end{cases}$$

so that the τ_i are positive but may not be in increasing order even though the subsets $\{\delta_1, \dots, \delta_k\}$ and $\{\theta_1, \dots, \theta_{n-k}\}$ are in increasing order. We define

$$\epsilon_i = R(\tau_1) \dots R(\tau_{i-1})\tau_i \quad \text{and} \quad \eta_i = R(\tau_n) \dots R(\tau_{i+1})\tau_i.$$

As in Section 4 we can define the non-crossing partitions

$$a_i = R(\tau_1) \dots R(\tau_i) \quad \text{and} \quad b_i = R(\tau_i) \dots R(\tau_n).$$

Thus $\epsilon_i = -a_i(\tau_i)$ and $\eta_i = -b_i^{-1}(\tau_i)$. Moreover, since $c = a_i R(\tau_i) b_i$, we have $c(\eta_i) = -a_i R(\tau_i)[\tau_i] = a_i[\tau_i] = -\epsilon_i$. We deduce from

$$R(\epsilon_i) = R(\tau_1) \dots R(\tau_{i-1})R(\tau_i)R(\tau_{i-1}) \dots R(\tau_1)$$

that $R(\epsilon_i)c = R(\tau_1) \dots R(\tau_{i-1})R(\tau_{i+1}) \dots R(\tau_n)$ and hence, by Lemma 2.2 of [1] that $\mu(\epsilon_i)$ is orthogonal to τ_j for $j \neq i$. Also, by Lemmas 2.3 and 2.4 of [1],

$$\begin{aligned} \tau_i \cdot \mu(\epsilon_i) &= \tau_i \cdot \mu[-a_i(\tau_i)] \\ &= -\tau_i \cdot a_i(\mu[\tau_i]) \\ &= -\tau_i \cdot (\mu[\tau_i] - 2\tau_i) \\ &= -1 + 2 \\ &= 1. \end{aligned}$$

Thus $\mu(\epsilon_i)$ lies on each of the hyperplanes τ_j^\perp for $j \neq i$ and on the positive side of τ_i^\perp . Since $c(\eta_i) = -\epsilon_i$, it follows that $\mu(c[\eta_i])$ lies on each of the hyperplanes τ_j^\perp for $j \neq i$ but on the negative side of τ_i^\perp .

Now, suppose

$$\tau_i = \begin{cases} \theta_i & \text{for } 1 \leq i \leq n-k, \\ \delta_{i-n+k} & \text{for } n-k+1 \leq i \leq n, \end{cases}$$

corresponding to the factorisation

$$c = (cw^{-1})c = R(\theta_1) \dots R(\theta_{n-k})R(\delta_1) \dots R(\delta_k),$$

where $\{\delta_1, \dots, \delta_k\}$ is the simple system for the parabolic subgroup W_w and $\{\theta_1, \dots, \theta_{n-k}\}$ is the simple system for the parabolic $W_{cw^{-1}}$. The ray of $F(w)$ which is opposite the θ_i^\perp wall and on its positive side is generated by $\mu(\epsilon_i)$, while the ray of $F(w)$ which is opposite the δ_i^\perp wall and on its negative side is generated by $\mu(c(\eta_{n-k+i}))$. We deduce that the rays of $F(w)$ are generated by

$$\{\mu(\epsilon_1), \dots, \mu(\epsilon_{n-k}), \mu[c(\eta_{n-k+1})], \dots, \mu[c(\eta_n)]\}.$$

To conclude, we note that the roots $\epsilon_1, \dots, \epsilon_{n-k}$ are the vertices of the lexicographically first facet of $X(cw^{-1})$ and the roots $\eta_{n-k+1}, \dots, \eta_n$ are the vertices of the lexicographically last facet of $X(w)$, by propositions 3.6 and 3.14 of [1]. \square

Corollary 5.4. *For each $w \in NCP_c$ the rays of $F(w)$ are generated by a subset of the set of vertices of $\mu(AX(c))$.*

Proof. By Proposition 5.3 the rays of $F(w)$ are generated by

$$\{\mu(\epsilon_1), \dots, \mu(\epsilon_{n-k}), \mu[c(\eta_{n-k+1})], \dots, \mu[c(\eta_n)]\},$$

where $\{\eta_{n-k+1}, \dots, \eta_n\}$ is the vertex set of the last facet of $X(w)$ and $\{\epsilon_1, \dots, \epsilon_{n-k}\}$ is the vertex set of the first facet of $X(cw^{-1})$. Since $c\rho_i = \rho_{i+n}$, the rays of $F(w)$ are generated by a subset of the set $\{\mu(\rho_1), \dots, \mu(\rho_{nh/2+n})\}$. \square

Proof of Theorem 1.1. We wish to show that the set of simplicial cones $\{F(w)\}$, where w ranges over the elements of NCP_c is precisely the set of cones on simplices of $\mu(AX(c))$. If $w \in NCP_c$, then by Proposition 5.3, the rays of $F(w)$ are generated by

$$V = \{\mu(\epsilon_1), \dots, \mu(\epsilon_{n-k}), \mu[c(\eta_{n-k+1})], \dots, \mu[c(\eta_n)]\},$$

where $\{\eta_{n-k+1}, \dots, \eta_n\}$ is the vertex set of the last facet of $X(w)$ and $\{\epsilon_1, \dots, \epsilon_{n-k}\}$ is the vertex set of the first facet of $X(cw^{-1})$. On the other hand, by Corollary 4.11 with $v = cw^{-1}$, there is a facet of $AX(c)$ whose vertex set is the union of the vertices of the first facet of $X(cw^{-1})$ and the images under c of the vertices of the last facet of $X(w)$. Since $\mu(AX(c))$ is the image of $AX(c)$ under the action of μ , the complex $\mu(AX(c))$ has a facet with vertex set V . Since every facet of $\mu(AX(c))$ arises in this way by the bijectivity of ϕ and the invertibility of the linear transformation μ , the set of $F(w)$'s coincides with the set of cones on simplices of $\mu(AX(c))$. \square

Theorem 5.5. *The fan determined by the cones $F(w)$ for $w \in NCP_c$ coincides with the c -Cambrian fan.*

Proof. The authors of [5] exhibit a linear isomorphism L from the c -cluster fan of a bipartite Coxeter element c to the c -Cambrian fan. We show that, up to scalar multiple, this map L coincides with $\mu \circ c_+$. Indeed the map L is defined on the basis $\{\alpha_1, \dots, \alpha_n\}$ by

$$\alpha_i \mapsto \begin{cases} -\omega_i & \text{for } i = 1, \dots, s \\ \omega_i & \text{for } i = s + 1, \dots, n. \end{cases}$$

Here $\{\omega_1, \dots, \omega_n\}$ is the dual basis to the basis of coroots $\{\alpha_i^\vee\}$ where

$$\alpha_i^\vee = 2\alpha_i / (\langle \alpha_i, \alpha_i \rangle).$$

Since we have chosen our simple roots to have unit length, the coroot α_i^\vee is simply $2\alpha_i$ and the ‘weight’ ω_i is simply $(1/2)\mu(\rho_i)$ by section 3 of [2]. However, by (1) and (2) of Section 3 above

$$c_+(\alpha_i) = \begin{cases} -\alpha_i & \text{for } i = 1, \dots, s \\ -c(\alpha_i) & \text{for } i = s + 1, \dots, n. \end{cases}$$

Recalling from [2] that

$$\rho_i = \begin{cases} \alpha_i & \text{for } i = 1, \dots, s \\ -c(\alpha_i) & \text{for } i = s + 1, \dots, n. \end{cases}$$

we see that L coincides with $(1/2)(\mu \circ c_+)$. \square

Example 5.6. Let W be the group C_3 (or B_3) of symmetries of the cube in \mathbf{R}^3 . The type C_n generalised associahedron is known as the cyclohedron. We can choose a simple system

$$\alpha_1 = (1, 0, 0), \quad \alpha_2 = (\sqrt{2}/2)(0, 1, -1), \quad \alpha_3 = (\sqrt{2}/2)(-1, 0, 1)$$

so that the dual basis is

$$\mu(\rho_1) = (1, 1, 1), \quad \mu(\rho_2) = \sqrt{2}(0, 1, 0), \quad \mu(\rho_3) = \sqrt{2}(0, 1, 1).$$

Here the Coxeter element is the orthogonal transformation defined by $c(x, y, z) = (-z, x, y)$, so that $h = 6$, $nh/2 = 9$ and $nh/2 + n = 12$. The complex $\mu(\text{AX}(c))$ is shown in Figure 1, where the 2-sphere has been stereographically projected onto the plane from the point $(-1, 0, 1)$. Only the vertices $\mu(\rho_1)$ and $\mu(\rho_{12})$ are labelled in the figure, but the other vertices occur consecutively on the dotted polygonal path between the labelled pair. The reflecting hyperplanes intersect the sphere in circles and segments of these circles form the edges of facets of $\mu(\text{AX}(c))$. The position of a particular hyperplane can be deduced from the fact that ρ_i^\perp passes through $\mu(\rho_{i+1})$ and $\mu(\rho_{i+2})$ since

$$c = R(\rho_{i+2})R(\rho_{i+1})R(\rho_i), \quad \text{for } 1 \leq i \leq 9.$$

The figure also incorporates the map $\phi' = \phi \circ \mu^{-1}$ defined by the bijection ϕ from section 4. Each $\mu(\text{AX}(c))$ region F' is labelled by

a set of integers, the corresponding positive roots forming the simple system for $c[\phi'(F')]^{-1}$. Thus the set of integers labelling a region F' corresponds to a subset of the walls of F' with a wall contributing to the subset if and only if F' lies on the negative side of the wall.

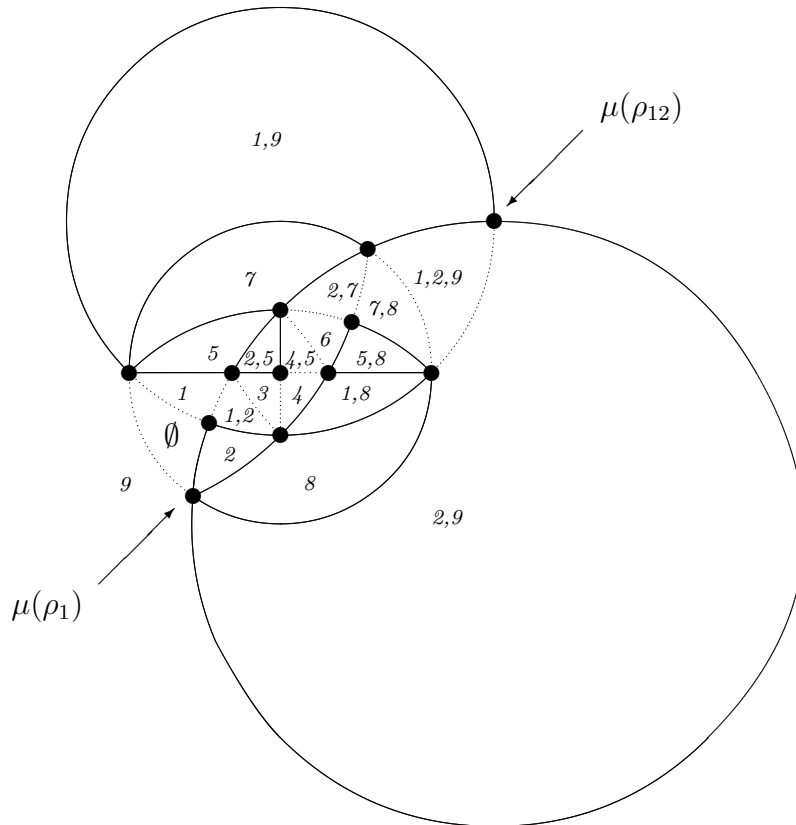


FIGURE 1. The cyclohedron inside the C_3 permutahedron.

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