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Integrable Systems as Fluid Models with Physical Applications

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Dublin Institute of Technology
College of Sciences and Health
School of Mathematical Sciences



Integrable Systems as Fluid Models with Physical Applications

Tony Lyons

Thesis Submitted for the Award of PhD

Supervisor: Dr. Rossen Ivanov

Dublin 2013

Declaration

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Abstract

In this thesis we begin with the development and analysis of hydrodynamical models as they arise in the theory of water waves and in the modelling of blood flow within arteries. Initially we derive three models of hydrodynamical relevance, namely the KdV equation, the two component Camassa-Holm equation and the Kaup-Boussinesq equation. We develop a model of blood flowing within an artery with elastic walls, and from the principles of Newtonian mechanics we derive the two-component Burger's equation as our first integrable model. We investigate the analytic properties of the system briefly, with the aim of demonstrating the phenomenon of wave breaking for the system. In addition we construct a pair of diffeomorphisms which allow us to solve the system explicitly in terms of the initial data. Finally, we show that when we consider the dynamics of the arterial walls themselves, the pressure within the fluid is seen to satisfy the KdV equation.

In the following chapter we investigate the trajectories followed by individual fluid particles in a fluid, as they are subject to the effects of an extreme Stokes wave. In the case of a regular Stokes wave there are no stagnation points or apparent stagnation points, i.e. locations where the fluid velocity and wave velocity are equal, however this condition does not remain true in the context of extreme Stokes waves. The result for the regular Stokes wave then have to be extended to semi-infinite regions with corners, and in doing so we show that the horizontal component of the fluid velocity field is strictly increasing along any stream line, which in turn ensures the non-closure of particle trajectories over the course of a fluid wave.

Next we begin with a review of the inverse scattering transform method of solving the Kortweg-de Vries equation. We construct the one-soliton solution explicitly. We then proceed to examine the Qiao equation, a non-linear partial differential equation with cubic non-linearities. We show that by a suitable change of variables and with a

change of the spectral parameter of its associated spectral problem that we transform it into the spectral problem of the KdV equation. Having already analysed this spectral problem, we then proceed to construct the 1-soliton solution of the Qiao equation with this modified spectral problem. The soliton solutions decay to a non-zero constant value asymptotically. We also investigate the peakon solutions of the Qiao equation, and construct the 1 and 2-peakon profiles, the latter being in the form of travelling M-wave profile.

We then go on to the analysis of a class of equations whose spectral problem are more complicated in the sense that the spectral problem has an energy dependant potential. We develop the inverse scattering transform method for these spectral problems, and construct the one-soliton solution explicitly, which in fact turn out to be a breather type solution. The hydrodynamical relevance of this problem arises from the fact that by an appropriate choice of one of the physical parameters of the system, we obtain the Kaup-Boussinesq equation, a partial differential equation with quadratic and cubic nonlinearities which arises in the theory of water waves in shallow water.

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Introduction

The theory of fluids has provided a rich source of mathematical innovation, in no small part due to the complexities inherent in the nonlinearities present in virtually all models of practical value. The presence of these nonlinearities has required mathematicians to redeploy techniques familiar in other contexts, or to develop entirely new approaches towards analysing the Euler equation and the approximate models which are derived from it.

In addition, the theory of fluids has provided for a rich interaction between several branches of applied and pure mathematics. Among these areas, the nonlinear analysis of partial differential equations and spectral theory applied to integrable systems will arise frequently throughout the thesis. These disciplines, while of purely theoretical interest in their own right, will be applied to several models of physical interest.

In the first chapter we begin with a derivation of the Euler equation from the principles of Newtonian mechanics, which will provide the starting point for all subsequent models to be derived. The nonlinear Euler equation will then be approximated to yield the first of our integrable models, namely the Korteweg-de Vries (KdV) equation. This is perhaps the archetypal integrable system in the theory of fluids, and is included for two reasons, the first reason being the relative simplicity of both its derivation from the Euler equations and the second being its solution via the Inverse Scattering Transform (IST). The method of derivation will follow closely that outlined in [Joh1997]. Secondly, it will be shown the Qiao equation, can in certain circumstance, be solved via

the inverse scattering transform in an identical manner to the IST solution of the KdV equation.

The second model we go on to derive is the two component Camassa-Holm equation, with the derivation following closely the one presented in the work of Ivanov [Iva2009]. The model is included because over the past two decades both the one and two component Camassa-Holm equations have been an important source of new results in the theory of fluids, perhaps one of the most important in relation to this thesis being the demonstration of the phenomenon of wave-breaking. Moreover, the solution of the two component Camassa-Holm equation as presented in the work [HI2011] is the basis of our later treatment of the Kaup-Boussinesq equation via the IST. The Kaup-Boussinesq equation will be solved via the IST in Chapter 6. The chapter follows the work presented in [IL2012c].

The third hydrodynamical model derived in Chapter 1 is the Kaup-Boussinesq equation. The derivation follows closely that in [Iva2009]. It can be seen from the derivation in this thesis that the model is of hydrodynamical relevance in the context of shallow water waves. In addition in Chapter 6 we obtain an explicit solution for the system via the IST. The work in [IL2012c] is a new approach to the inverse scattering transform for the Kaup-Boussinesq equation, which shares several features with the IST of the two component Camassa-Holm equation found in [HI2011]. This chapter will present a comprehensive construction of the Riemann-Hilbert problem for a class of partial differential equations with cubic nonlinearities and the solutions of the inverse problem for this class of spectral problems. In this chapter we also present an explicit breather type solution for the Kaup-Boussinesq equation.

In Chapter 2 we undertake a mathematical investigation of blood flow within arteries. In this chapter we first present an original derivation of the dispersionless two component Burgers equation (this system is also known as the two component Hopf

equation) in the context of arterial blood flow. Next we construct a pair of diffeomorphisms, whereby we can find a solution for the system depending on the initial data for the fluid velocity and arterial cross section. A similar procedure is carried out for the so called wrong sign Burger's equation in the next section. In the following section we present a rigorous demonstration, using the previously constructed diffeomorphisms, of the phenomenon of wave breaking within the system. The work in this chapter is based almost entirely on the results presented in [Lyo2012]. We provide an estimate for the blow up time, which is found to depend on the initial data. Lastly, in this chapter we present a derivation of the KdV equation for the pressure within an artery, which arises from the elastic restoring forces in the arterial walls. This provides further impetus to study the IST solution of the KdV, which we proceed to do in Chapter 4.

In Chapter 3 we investigate the flow of individual fluid elements within a fluid body. Specifically, we investigate to trajectories followed by fluid particles in an extreme Stoke's wave over infinite depth [Lyo2014]. It is shown that both within the fluid domain and on the surface of the fluid, the particles undergo a positive drift over the course of an entire wavelength. The main achievement is in extending the application of maximum principles to a semi infinite domain with continuous boundary with a corner. The results of this chapter rely greatly on the hodograph transform, whereby a free boundary problem is transformed into a nonlinear problem on a domain with fixed boundary. This concludes the first part of the thesis which was concerned with hydrodynamical models in various physical settings.

In the next part of the thesis we investigate the spectral analysis of the models derived earlier via the Inverse Scattering Transform. In Chapter 4 we investigate the KdV equation presenting a comprehensive overview of the associated spectral problem and proving that the discrete spectrum contains a finite number of points, and that the scattering coefficients of the problem are analytic with simple zeros in the upper half of

the complex plane. The Riemann-Hilbert problem for the Jost solutions is constructed, which in turn allows us to solve KdV equation via the inverse problem. We also construct the 1 soliton solution explicitly. This work follows closely the treatment presented in [ZMNP1984].

In Chapter 5 we investigate a PDE with cubic nonlinearities, known as the Qiao equation. It is known that this equation belongs to the bi-Hamiltonian hierarchy of the Camassa-Holm equation. Initially we begin with a construction of the peakon solutions, and demonstrate the existence of the travelling "M"-wave solutions, based on [IL2012a]. We show that under an appropriate change of variables, the spectral problem of the Qiao equation is equivalent to that of the KdV equation, when the potential decays to a non-zero constant in the asymptotic region [IL2012c]. Next we employ the results of Chapter 4 to construct the 1 and 2 soliton solutions of the system.

Finally, Chapter 6 presents an investigation of the spectral problem for another class of PDE with cubic nonlinearities. Among this class of PDE is the Kaup-Boussinesq equation, which in Chapter 1 was shown to be a model of hydrodynamical significance in the shallow water limit. As with the treatment of the KdV equation in Chapter 4 we begin with the construction of the associated spectral problem. In contrast to the case of the KdV equation the spectral problem encountered in this chapter has an energy dependent potential, in that the potential has a term depending on the spectral parameter. This complication requires us to introduce a pair of "conjugate" spectral problems, and investigate the analytic properties of their associated Jost solutions. Again we construct a Riemann-Hilbert problem for the bases of Jost solutions, which in turn allows us to solve the Kaup-Boussinesq equation in terms of the scattering data. Lastly we construct the explicit 1 soliton solution for the system which is found to be a breather type of solution. The work in this chapter relies largely on [IL2012b].

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This thesis is based on the following publications:

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Part I

Fluid Dynamics

Chapter 1

Fluid Models

1.1 The governing equations

1.1.1 The continuity equation

In this section we develop the general theory of fluids embodied in the continuity equation and the Euler equation. This pair of equations follows from the principles of Newtonian mechanics applied to infinitesimal fluid elements and then extended to finite fluid volumes. The work of this section follows closely that presented in [Joh1997]. We consider an arbitrary fluid body in three dimensional space, contained by some region $\Omega \in \mathbb{R}^3$, with a closed boundary $\partial\Omega$. We denote by \mathbf{x} the position of a point in \mathbb{R}^3 and by $\mathbf{u}(\mathbf{x}, t)$ the velocity of the infinitesimal fluid volume contained within d^3x at \mathbf{x} . Written in component form, these quantities are denoted by

$$\mathbf{x} = (x, y, z), \quad \mathbf{u} = (u, v, w), \quad d^3x = dx dy dz,$$

where it is understood $u \equiv u(\mathbf{x}, t)$ and similarly for v and w . Let us now suppose that the fluid density $\rho \equiv \rho(\mathbf{x}, t)$ is dependent on position within the fluid body, and furthermore

the density at some fixed location \mathbf{x} varies with time, in which case we have

$$\int_{\Omega} \rho d^3x = M \quad (1.1.1)$$

where $M \equiv M(t)$ is the mass of fluid contained within Ω . It is clear from the integral above that M depends on time in general via the time dependence of $\rho(\mathbf{x}, t)$. We now introduce $\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}(\mathbf{x}, t)$, which is the unit outward normal to $\partial\Omega$. In this derivation we shall treat Ω as a fixed volume with respect to the coordinate axes (x, y, z) and its boundary is also to be fixed. In this case, the fluid flow in this region is entirely due to the net flow of fluid through the boundary $\partial\Omega$. It follows that one may write the rate of change in M as the net flow of fluid through the boundary $\partial\Omega$,

$$\frac{d}{dt} \int_{\Omega} \rho d^3x = - \int_{\partial\Omega} \rho \hat{\mathbf{n}} \cdot \mathbf{u} d\sigma, \quad (1.1.2)$$

where $d\sigma$ is introduced as the infinitesimal surface element on $\partial\Omega$. We may rewrite the right hand side using Gauss' law, and bringing all terms to one side we obtain

$$\frac{d}{dt} \int_{\Omega} \rho d^3x + \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) d^3x = 0. \quad (1.1.3)$$

Since the region Ω is fixed, we may take the time derivative inside the integral to yield

$$\int_{\Omega} [\rho_t + \nabla \cdot (\rho \mathbf{u})] dx = 0, \quad (1.1.4)$$

where $\rho_t \equiv \partial_t \rho(\mathbf{x}, t)$. Here and throughout variables t, x etc. appearing as subscripts on a function shall denote partial differentiation of that function with respect to those variables. Finally, since the region Ω is chosen arbitrarily, the integral above may be zero in general only if the integrand is zero everywhere,

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.1.5)$$

The above is referred to as the continuity equation, and in the case of incompressible fluid flows, which is equivalent to the statement $\rho = \text{constant}$ it may be equivalently written as

$$\nabla \cdot \mathbf{u} = 0,$$

making \mathbf{u} solenoidal.

1.1.2 Euler's equation

In the previous section we derived an equation to describe the preservation of fluid mass within a fixed region Ω . We must now extend our analysis to include a description of the balance of forces acting on the same fluid volume. In general there are two categories of forces acting upon the fluid body which can be considered internal and external. The external forces are often uniform throughout the body of the fluid, a prime example being the force of gravity which is certainly relevant to the study of fluid waves. In addition there are internal forces, which can be largely categorised as internal pressure, which arises as a result of the action of contiguous fluid element upon each other throughout the fluid body. The pressure acting on the fluid body Ω can be thought of as the net forces exerted on the body through the boundary $\partial\Omega$. We denote by $\mathbf{F}_0 \equiv \mathbf{F}_0(\mathbf{x}, t)$ the external force per unit mass acting on the fluid element at \mathbf{x} at time t . Secondly, $P \equiv P(\mathbf{x}, t)$ shall denote the pressure exerted on a unit element of the boundary $\partial\Omega$. The net forces acting on the fluid body Ω may be expressed as

$$\int_{\Omega} \rho \mathbf{F}_0 d^3x - \int_{\partial\Omega} P \hat{\mathbf{n}} d\sigma = \mathbf{F}, \quad (1.1.6)$$

where $\mathbf{F} \equiv \mathbf{F}(t)$ denotes the resultant force acting on the entire fluid body. Applying Gauss' law to the second integral we may write

$$\int_{\Omega} \rho \mathbf{F}_0 d^3x - \int_{\Omega} \nabla P d^3x = \mathbf{F}. \quad (1.1.7)$$

We now have an expression for the net external force acting on the fluid volume Ω , which induces a change of momentum on the fluid body. An additional contribution to the momentum of this region is due to the net flow of momentum across the boundary $\partial\Omega$, due to fluid elements flowing across the boundary. This may be written as

$$- \int_{\partial\Omega} \rho \mathbf{u} (\hat{\mathbf{n}} \cdot \mathbf{u}) d\sigma = \dot{\mathbf{p}} \quad (1.1.8)$$

where $\dot{\mathbf{p}} \equiv \frac{d\mathbf{p}}{dt}(t)$, denotes the rate of change of momentum of the fluid body due the net flux of momentum across $\partial\Omega$. Again, an application of Gauss' law to both factors of the integrand yields

$$-\int_{\Omega} \mathbf{u} \cdot \nabla(\rho\mathbf{u})d^3x - \int_{\Omega} (\nabla \cdot \mathbf{u})(\rho\mathbf{u})d^3x = \dot{\mathbf{p}} \quad (1.1.9)$$

Finally, the rate of change of momentum of the entire fluid region is simply

$$\frac{d}{dt} \int_{\Omega} \rho\mathbf{u}d^3x = \dot{\mathbf{P}}, \quad (1.1.10)$$

with $\dot{\mathbf{P}}$ denoting the overall rate of change of momentum of the fluid body in Ω . An application of Newton's second law requires that the net external forces acting on the fluid body along with the rate of flow of momentum across the boundary $\partial\Omega$ result in the overall change of momentum of the fluid body, namely

$$\begin{aligned} \int_{\Omega} [\rho\mathbf{F}_0 - \nabla P]d^3x - \int_{\Omega} [\mathbf{u} \cdot \nabla(\rho\mathbf{u}) + (\nabla \cdot \mathbf{u})(\rho\mathbf{u})]d^3x &= \frac{d}{dt} \int_{\Omega} \rho\mathbf{u}d^3x \\ \Leftrightarrow \mathbf{F} + \dot{\mathbf{p}} &= \dot{\mathbf{P}}. \end{aligned} \quad (1.1.11)$$

Rearranging we find, and taking the time derivative through the second integral we have

$$\int_{\Omega} [(\rho\mathbf{u})_t + \mathbf{u} \cdot \nabla(\rho\mathbf{u}) + (\nabla \cdot \mathbf{u})(\rho\mathbf{u})]d^3x = \int_{\Omega} [-\nabla P + \rho\mathbf{F}_0]d^3x. \quad (1.1.12)$$

Expanding the first derivative in the integrand on the left hand side, and applying the continuity equation we find

$$\int_{\Omega} \rho[\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}]d^3x = \int_{\Omega} [-\nabla P + \rho\mathbf{F}_0]d^3x. \quad (1.1.13)$$

Once again, as the above integral equation is true for an arbitrary region Ω , it follows that the integrand appearing in both integrals must be equal everywhere, in which case we find upon dividing both by ρ that

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P + \mathbf{F}_0, \quad (1.1.14)$$

which is the conventional form of Euler's equation. The continuity equation and Euler's equation constitute a coupled pair of nonlinear partial differential equations describing the motion of a fluid element subject to specific external force, for example the gravitational force. It is from these governing equations that all other models considered herein are derived as approximate models in various physical regimes.

1.1.3 Nondimensionalisation

In many circumstances involving fluid waves there exist natural length scales defined by the physical setting such as the mean fluid depth and wavelength. In addition in the case of surface gravity waves, the gravitational acceleration introduces a corresponding velocity scale and therefore a natural time scale. With these scales one may rewrite the governing equations in new non dimensional variables, making it much clearer as to how one may proceed towards approximate models in certain physical regimes. Suppose we consider the case of a surface gravity wave, whose typical wavelength we denote by λ . This wave is assumed to flow over a flat bed located at $z = 0$, with the mean height of the water surface fixed at $z = h$. In addition, the typical amplitude of such a wave is a , where generally $a \ll h$. In general the characteristic speed associated with surface gravity waves is \sqrt{gh} , where g is the gravitational acceleration. In contrast, the velocity scale in the z direction is given by $\frac{h}{\lambda}\sqrt{gh}$, which is necessary so that the continuity equation applies to the non dimensional variables. Finally, we will write the free surface of the fluid as

$$z = h + a\eta(\mathbf{x}_h, t) \quad (1.1.15)$$

where $\mathbf{x}_h = \{\mathbf{x} \in \mathbb{R}^3 : z = h\}$. With the wave amplitude given by a , its clear that the function describing the free surface $\eta \equiv \eta(\mathbf{x}_h, t)$ is dimensionless. To avoid the introduction of an entirely new set of variables for our system of equations we typically denote the rescaled coordinate by the original variable. We write our nondimension-

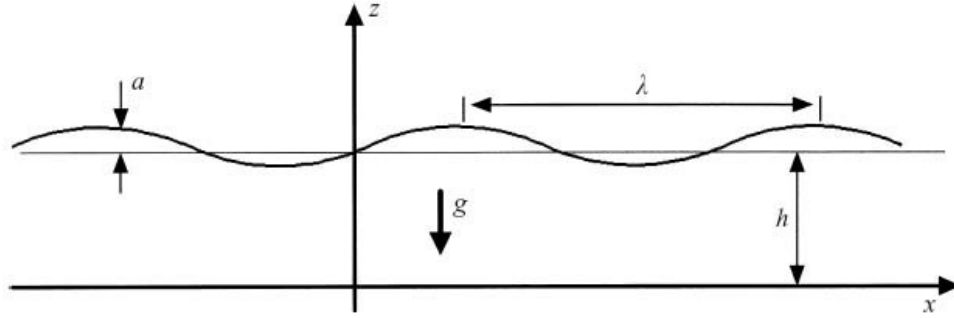


Figure 1.1: The figure shows the propagation of a surface wave of amplitude a and wavelength λ travelling over a flat bed.

alised coordinates as

$$\begin{aligned} x &\rightarrow \lambda x & y &\rightarrow \lambda y & z &\rightarrow hz & t &\rightarrow \frac{\lambda}{\sqrt{gh}}t \\ u &\rightarrow \sqrt{gh}u & v &\rightarrow \sqrt{gh}v & w &\rightarrow \frac{h\sqrt{gh}}{\lambda}w, \end{aligned} \quad (1.1.16)$$

so for example in $x \rightarrow \lambda x$, the left hand x is the original dimensional coordinate while the right hand x is the new scaled dimensionless coordinate. It will also prove convenient to write our pressure as a sum of three contributions,

$$P = P_a + \rho g(h - z) + \rho gh p, \quad (1.1.17)$$

with P_a being the constant atmospheric pressure at the free surface, $\rho g(h - z)$ the hydrostatic pressure, and $\rho gh p \equiv \rho gh p(\mathbf{x}, t)$ the pressure away from hydrostatic equilibrium. When written in nondimensional coordinates in component form, Euler's equation becomes

$$\begin{aligned} u_t + \mathbf{u} \cdot \nabla u &= -p_x + f_0^x \\ v_t + \mathbf{u} \cdot \nabla v &= -p_y + f_0^y \\ \delta^2(w_t + \mathbf{u} \cdot \nabla w) &= 1 - p_z + f_0^z, \quad \delta = \frac{h}{\lambda}, \end{aligned} \quad (1.1.18)$$

where we define the (x, y, z) -components of the dimensionless force per unit mass as follows,

$$f_0^x = \frac{\lambda}{gh} F_0^x \quad f_0^y = \frac{\lambda}{gh} F_0^y \quad f_0^z = \frac{1}{g} F_0^z.$$

The continuity equation retains its form identically on passing from dimensional to dimensionless variables, in which case we simply have

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$

in dimensionless variables. Finally, when written in dimensionless variables, the flat bed $z = 0$ and the free surface $z = h + a\eta$ become

$$z = 0, \quad z = 1 + \varepsilon\eta; \quad \varepsilon = \frac{a}{h}. \quad (1.1.19)$$

The parameters ε and δ are important dimensionless parameters which play an important role in all derivations of approximate physical models. They are especially important when investigating weakly nonlinear models, such as the Korteweg-de Vries (KdV) equation which we shall now derive.

1.2 Shallow water models

1.2.1 The Korteweg-de Vries equation

In this section we derive the prototypical nonlinear partial differential equation of hydrodynamical relevance, the Korteweg-de Vries equation, otherwise known as the KdV equation. The derivation of the KdV presented in this section follows closely that presented in the work [Joh1997]. During the derivation of all three approximate models in this and later sections we consider the case of a two-dimensional wave profile moving in the x -direction with amplitude in the z -direction and whose form is constant in the y -direction. The fluid flow we consider will be incompressible, yielding a constant

fluid density which we normalise to unity i.e. $\rho = 1$. The external body force acting of the fluid body is simply the gravitational force, in which case we have

$$\mathbf{F}_0 = (0, 0, -g). \quad (1.2.1)$$

It is clear that in the free surface $z = 1 + \varepsilon\eta$ the vertical component of velocity of a fluid element is

$$w = \frac{dz}{dt} = \varepsilon(\eta_t + \mathbf{u}_h \cdot \nabla\eta), \quad (1.2.2)$$

where we introduce $\mathbf{u}_h \equiv (u, v, 0)_{\mathbf{x}=\mathbf{x}_h}$. Meanwhile, the deviation away from hydrostatic pressure on the free surface is

$$p = \varepsilon\eta \quad z = 1 + \varepsilon\eta. \quad (1.2.3)$$

In both cases each term is scaled by a factor of ε . It is natural to introduce such a scale factor for all components of \mathbf{u} and the variable p throughout the entire fluid body Ω thereby inducing the change of variables

$$\mathbf{u} \rightarrow \varepsilon\mathbf{u}, \quad p \rightarrow \varepsilon p, \quad (1.2.4)$$

which are the scaled nondimensional variables in which we shall work from now on. When written in these variables, and subject to the conditions described above, the governing equations, in component form become

$$\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x, \\ \delta^2[w_t + \varepsilon(uw_x + ww_z)] &= -p_z, \\ u_x + w_z &= 0, \end{aligned} \quad (1.2.5)$$

with boundary conditions as follows

$$\begin{aligned} p &= \eta, \quad w = \eta_t + \varepsilon u \eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta \\ w &= 0 \quad \text{on} \quad z = 0. \end{aligned} \quad (1.2.6)$$

The asymptotic expansion of the governing equations may be carried out in terms of the nonlinearity parameter ε alone, under the re-scaling

$$x \rightarrow \frac{\delta}{\sqrt{\varepsilon}}x \quad t \rightarrow \frac{\delta}{\sqrt{\varepsilon}}t \quad w \rightarrow \frac{\sqrt{\varepsilon}}{\delta}w, \quad (1.2.7)$$

with all other variables remaining invariant under the re-scaling. The re-scaling introduced yields the following the governing equations

$$\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x, \\ \varepsilon[w_t + \varepsilon(uw_x + ww_z)] &= -p_z, \\ u_x + w_z &= 0, \end{aligned} \quad (1.2.8)$$

while the boundary conditions become

$$\begin{aligned} p = \eta, \quad w = \eta_t + \varepsilon u \eta_x \quad \text{on} \quad z = 1 + \varepsilon \eta \\ w = 0 \quad \text{on} \quad z = 0. \end{aligned} \quad (1.2.9)$$

We consider right moving waves, and in the interest of maintaining uniformity of our asymptotic expansion, we introduce far-field variables,

$$\xi = x - t, \quad \tau = \varepsilon t. \quad (1.2.10)$$

In the far field variables we have

$$\begin{aligned} -u_\xi + \varepsilon(u_\tau + uu_\xi + wu_z) &= -p_\xi, \\ \varepsilon[-w_\xi + \varepsilon(w_\tau + uw_x + ww_z)] &= -p_z, \\ u_\xi + w_z &= 0, \end{aligned} \quad (1.2.11)$$

with boundary conditions

$$\begin{aligned} p = \eta, \quad w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \quad \text{on} \quad z = 1 + \varepsilon \eta \\ w = 0 \quad \text{on} \quad z = 0. \end{aligned} \quad (1.2.12)$$

This system gives rise to an asymptotic expansion in ε for each of u, w, p and η of the form

$$q(\xi, z, \tau; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, z, \tau), \quad \eta(\xi, z, \tau; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi, \tau), \quad (1.2.13)$$

where $q_n \in \{u_n, w_n, p_n\}$, and similarly for the functions without subscripts. The governing equations (1.2.11) along with the boundary conditions (1.2.12) yield

$$u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi}, \quad p_0 = \eta_0, \quad (1.2.14)$$

as the leading order approximation.

At order ε , the boundary conditions yield

$$p_1|_{z=1} = \eta_1, \quad w_1|_{z=1} = -\eta_{1\xi} + (\eta_{0\tau} + 2\eta_0\eta_{0\xi}); \quad w_1|_{z=0} = 0. \quad (1.2.15)$$

The first of the governing equations (1.2.11) give the result,

$$u_{1\xi} = p_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi}, \quad (1.2.16)$$

while the second member gives,

$$p_1 = \eta_1 + \frac{1}{2}(1 - z^2)\eta_{0\xi\xi}. \quad (1.2.17)$$

The continuity equation requires

$$w_{1z} = -u_{1\xi}, \quad (1.2.18)$$

and so upon integrating we find

$$w_1 = \int_{z'=0}^{z'=z} w_{1z'} dz' = -z \left[\eta_{1\xi} + \frac{1}{6}(3 - z^2)\eta_{0\xi\xi\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} \right]. \quad (1.2.19)$$

The boundary conditions for w_1 at $z = 1$ then requires η_0 to satisfy the following

$$2\eta_{0\tau} + 6\eta_0\eta_{0\xi} + \frac{1}{3}\eta_{0\xi\xi\xi} = 0. \quad (1.2.20)$$

At this order there are no conditions imposed on η_1 , rather it is at order ε^2 that we obtain a non-linear partial differential equation for this variable. Re-scaling the far-field variables as follows

$$\xi \rightarrow \frac{1}{\sqrt{3}}\xi \quad \tau \rightarrow \frac{2}{\sqrt{3}}\tau,$$

gives us

$$\eta_{0\tau} + 6\eta_0\eta_{0\xi} + \eta_{0\xi\xi\xi} = 0, \quad (1.2.21)$$

which is the standard form of the KdV equation. In a later chapter we shall discuss in detail the soliton solutions of this equation obtained via the inverse scattering transform (IST).

1.2.2 Irrotational surface waves

We begin the analysis of irrotational surface waves from the scaled governing equations presented in (1.2.5) and (1.2.6) and which we repeat here only for convenience, namely

$$\begin{aligned} u_t + \varepsilon(uu_x + wu_z) &= -p_x, \\ \delta^2[w_t + \varepsilon(uw_x + ww_z)] &= -p_z, \\ u_x + w_z &= 0, \end{aligned}$$

with boundary conditions

$$\left. \begin{aligned} p &= \eta, \\ w &= \eta_t + \varepsilon u \eta_x \end{aligned} \right\} \text{ on } z = 1 + \varepsilon\eta$$

$$w = 0 \quad \text{on } z = 0,$$

which again serve to develop asymptotic series in ε and δ^2 . In addition we require the flow to be irrotational

$$u_z - w_x = 0,$$

which in the scaled coordinates becomes

$$u_z - \delta^2 w_x = 0. \quad (1.2.22)$$

In contrast to the derivation of the KdV equation in the previous section, we develop the asymptotic series above as a two component system in terms of u_0 and η_0 , where to leading order we have

$$p = p_0, \quad u = u_0, \quad w = w_0, \quad \eta \equiv \eta_0. \quad (1.2.23)$$

The first of the boundary conditions yields

$$p_0 = \eta + \mathcal{O}(\varepsilon, \delta^2), \quad (1.2.24)$$

while the second governing equation in (1.2.6) requires $\eta_z = 0 \Rightarrow \eta \equiv \eta(x, t)$. The irrotational condition requires

$$u_{0z} = 0 \quad \Rightarrow \quad u_0 \equiv u_0(x, t). \quad (1.2.25)$$

The continuity equation at leading order requires

$$w_{0z} = -u_{0x} + \mathcal{O}(\varepsilon, \delta^2),$$

and when integrated using the third boundary condition in (1.2.6) we obtain

$$w_0 = -zu_{0x} + \mathcal{O}(\varepsilon, \delta^2), \quad (1.2.26)$$

which is consistent with the second boundary condition at this order. At leading order we also have

$$u_{0t} + \eta_x + \mathcal{O}(\varepsilon, \delta^2) = 0, \quad (1.2.27)$$

which follows from the first governing equation and the previous results. At order $\mathcal{O}(\varepsilon, \delta^2)$ we have

$$p = \eta + p_1 \quad u = u_0 + u_1 \quad w = -zu_0 + w_1, \quad (1.2.28)$$

where all factors of ε , δ^2 have been absorbed into p_1 , u_1 and w_1 . The irrotational condition gives to order $\mathcal{O}(\varepsilon, \delta^2)$

$$u_{1z} = -\delta^2 w_{0x}, \quad (1.2.29)$$

and so upon integrating with respect to z we find

$$u = u_0 - \delta^2 \frac{z^2}{2} u_{0xx} + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2). \quad (1.2.30)$$

The previous result along with the continuity equation gives

$$w_{1z} = -u_{1x} = \delta^2 \frac{z^2}{2} u_{0xx},$$

while the third boundary condition imposes $w_1|_{z=0} = 0$, and so upon integrating with respect to z we find

$$w = -z u_{0x} + \delta^2 \frac{z^3}{6} u_{0xxx} + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2). \quad (1.2.31)$$

Using the expressions (1.2.30) and (1.2.31) for u and w , along with the second boundary condition, we find to order ε, δ^2

$$\eta_t + \left[(1 + \varepsilon \eta) u_0 - \delta^2 \frac{1}{6} u_{0xx} \right]_x + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2) = 0. \quad (1.2.32)$$

The second governing equation gives to first order

$$p_z = \delta^2 z u_{0xt} \quad (1.2.33)$$

while integrating with respect to z using the first and second boundary conditions we find

$$p = \eta - \delta^2 \frac{(1-z)^2}{2} u_{0xt} + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2). \quad (1.2.34)$$

Substituting this expression into the first governing equation we obtain

$$\left(u_0 - \delta^2 \frac{1}{2} u_{0xx} \right)_t + \eta_x + \varepsilon u_0 u_{0x} + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2) = 0. \quad (1.2.35)$$

Letting $\varepsilon, \delta \rightarrow 0$, in both (1.2.32) and (1.2.27), we are left with

$$\begin{aligned}\eta_t + u_{0x} &= 0 \\ u_{0t} + \eta_x &= 0,\end{aligned}\tag{1.2.36}$$

which give us the wave-equation

$$\eta_{tt} - \eta_{xx} = 0,\tag{1.2.37}$$

as a linear approximation. Clearly at leading order we have as approximate solutions left and right travelling waves of the form

$$\eta_{\pm} \equiv \eta(x \pm t).\tag{1.2.38}$$

We may also make the approximation

$$\eta_{\pm} = \pm u_0(x - t) + \mathcal{O}(\varepsilon, \delta^2),\tag{1.2.39}$$

which is valid to leading order. In what follows we shall restrict attention to right moving waves. We introduce a new variable ϱ defined to order $\mathcal{O}(\varepsilon^2, \varepsilon\delta^2)$ according to

$$\varrho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\gamma u_{0xx},\tag{1.2.40}$$

where α, β and γ are as yet undetermined constant coefficients. Using the approximate result obtained in (1.2.39) we find to order $\mathcal{O}(\varepsilon^2, \varepsilon\delta^2)$

$$\varrho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta u_0^2 - \varepsilon\delta^2\gamma u_{0xx} - \mathcal{O}(\varepsilon^2, \varepsilon^2\delta^2),\tag{1.2.41}$$

We may write η in terms of ϱ and u_0 as follows

$$\eta = \frac{\varrho - 1}{\varepsilon\alpha} - \varepsilon\frac{\beta}{\alpha}u_0^2 - \delta^2\frac{\gamma}{\alpha}u_{0xx} + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2),\tag{1.2.42}$$

keeping terms of order $\mathcal{O}(\varepsilon, \varepsilon\delta^2)$. Replacing this expression for η in (1.2.32) and retaining terms of order $\mathcal{O}(\varepsilon, \delta^2)$, we find

$$\frac{\varrho_t}{\varepsilon\alpha} - \varepsilon\frac{\beta}{\alpha}(u_0^2)_t - \delta^2\frac{\gamma}{\alpha}u_{0xxt} + [(1 + \varepsilon\eta)u_0]_x - \delta^2\frac{1}{6}u_{0xxx} + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2) = 0.\tag{1.2.43}$$

Using (1.2.39) we see we may write

$$\varepsilon(u_0^2)_t = (\eta u_0)_t + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2),$$

while using (1.2.36), this may be approximated as

$$\varepsilon(\eta u_0)_t = -\varepsilon(\eta\eta_x + u_0 u_{0x}) + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2),$$

while with one more application of (1.2.39) we may write

$$-\varepsilon(\eta\eta_x + u_0 u_{0x}) = -\varepsilon(u_0\eta_x + \eta u_{0x}) + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2),$$

so overall we have to order $\mathcal{O}(\varepsilon, \delta^2)$,

$$\varepsilon(u_0^2)_t = -\varepsilon(\eta u_0)_x + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2).$$

Meanwhile, using (1.2.36) we have

$$\delta^2 u_{0t} = -\delta^2 \eta_x + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2),$$

while an application of (1.2.39) gives us

$$\delta^2 \eta_x = \delta^2 u_{0x} + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2),$$

from which it follows we may write

$$\delta^2 u_{0xxt} = -\delta^2 u_{0xxx} + \mathcal{O}(\varepsilon\delta^2, \delta^4),$$

to order ε, δ^2 . Using both approximations we now may write,

$$\frac{\rho_t}{\varepsilon\alpha} + \delta^2 \left(\frac{\gamma}{\alpha} - \frac{1}{6} \right) u_{0xxx} + \left[\left(1 + \varepsilon \left(1 + \frac{\beta}{\gamma} \right) \eta \right) u \right]_x + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2) = 0. \quad (1.2.44)$$

As the coefficients α , β and γ were undetermined we may choose

$$\frac{\gamma}{\alpha} - \frac{1}{6} = 0, \quad (1.2.45)$$

in which case the coefficient of u_{0xxx} vanishes and we are left with

$$\frac{\varrho_t}{\varepsilon\alpha} + \left[\left(1 + \varepsilon \left(1 + \frac{\beta}{\gamma} \right) \eta \right) u \right]_x + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2) = 0. \quad (1.2.46)$$

We are also free to choose β such that

$$1 + \frac{\beta}{\gamma} = \alpha, \quad (1.2.47)$$

and so upon neglecting terms of order $\mathcal{O}(\varepsilon^3, \varepsilon^2\delta^2)$, we are left with

$$\varrho_t + \varepsilon\alpha(\varrho u_0)_x = 0, \quad (1.2.48)$$

which later on will be rescaled to become one member of the CH2 equation.

1.2.3 The two component Camassa-Holm equation

To derive the second member of the two component Camassa-Holm equation, we introduce

$$m = u_0 - \delta^2 \left(\frac{1}{2} + \kappa \right) u_{0xx}, \quad (1.2.49)$$

where κ is an as yet undetermined constant. It follows that we may write (1.2.35) as

$$m_t + \delta^2 \kappa u_{0xxt} + \eta_x - \varepsilon u_0 u_{0x} = 0. \quad (1.2.50)$$

We use (1.2.36) to rewrite $\varepsilon u_{0t} = -\varepsilon \eta_x$ and then use (1.2.39) to write $\varepsilon \eta_x = \varepsilon u_{0x}$ in which case we may say $\varepsilon u_{0xxt} = -\varepsilon u_{0xxx} + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2)$. Up to order $\mathcal{O}(\varepsilon^2, \varepsilon\delta^2)$ we have

$$\varrho^2 = 1 + \varepsilon(2\alpha)\eta + \varepsilon^2(\alpha^2 + 2\beta)u_0 u_{0x} + \varepsilon\delta^2\gamma u_{0xx},$$

which allows us rewrite the η_x term above, to obtain

$$m_t - \delta^2 \left(\kappa + \frac{\gamma}{\alpha} \right) u_{0xxx} + \varepsilon \frac{1}{3} \left(1 - \frac{\alpha^2 + 2\beta}{2\alpha} \right) (2m u_{0x} + m_x u_0) + \frac{1}{\alpha\varepsilon} \varrho \varrho_x = 0, \quad (1.2.51)$$

and where we have used,

$$\varepsilon u_0 u_{0x} = \varepsilon \frac{1}{3} (2m u_{0x} + m_x u_0) + \mathcal{O}(\varepsilon^2, \varepsilon\delta^2).$$

Choosing $\kappa = -\frac{\gamma}{\alpha}$ we see the u_{0xxx} term vanishes, and

$$m \rightarrow u_0 - \delta^2 \frac{1}{3} u_{0xx}.$$

Under the change of variables

$$u_0 \rightarrow \frac{1}{\varepsilon\alpha} u, \quad \varrho \rightarrow \rho, \quad x \rightarrow \frac{\delta}{\sqrt{3}} x, \quad t \rightarrow \frac{\delta}{\sqrt{3}} t,$$

and choosing α such that

$$\frac{1}{3\alpha} \left(1 - \frac{(\alpha^2 + 2\beta)}{\alpha} \right) = 1, \quad (1.2.52)$$

the two equations that constitute the CH2 system are

$$\begin{aligned} m_t + 2mu_x + m_x u + \rho\rho_x &= 0; & m &= u - u_{xx}, \\ \rho_t + (\rho u)_x &= 0. \end{aligned} \quad (1.2.53)$$

The system of equations (1.2.45), (1.2.47) and (1.2.52) determine the coefficients α , β and γ exactly and so we find

$$\begin{aligned} \alpha &= \frac{4}{13} \\ \beta &= -\frac{3}{169} \\ \gamma &= \frac{2}{39}, \end{aligned} \quad (1.2.54)$$

while $\kappa = -\frac{\gamma}{\alpha} = -\frac{1}{6}$.

The CH2 system was initially introduced in [OR1996] as a tri-Hamiltonian (integrable) system and was studied further by others, see for example [ELY2007, Hen2009, LZ2005, CLZ2006, Fal2006, Iva2006, CI2008, HT2009, GL2010]. In the Figure (1.2) the system was integrated numerically by J. Percival who considered the evolution of the system from the so called "dam-break" initial conditions. The results shown figures display the development of soliton solutions for (1.2.53) after an initial dam-break. The initial conditions satisfied by the two components of the system are,

$$\begin{aligned} u(x, 0) &= 0, \\ \rho(x, 0) &= 0.1(1 + \tanh(x + 0.5) - \tanh(x - 0.5)), \end{aligned} \quad (1.2.55)$$

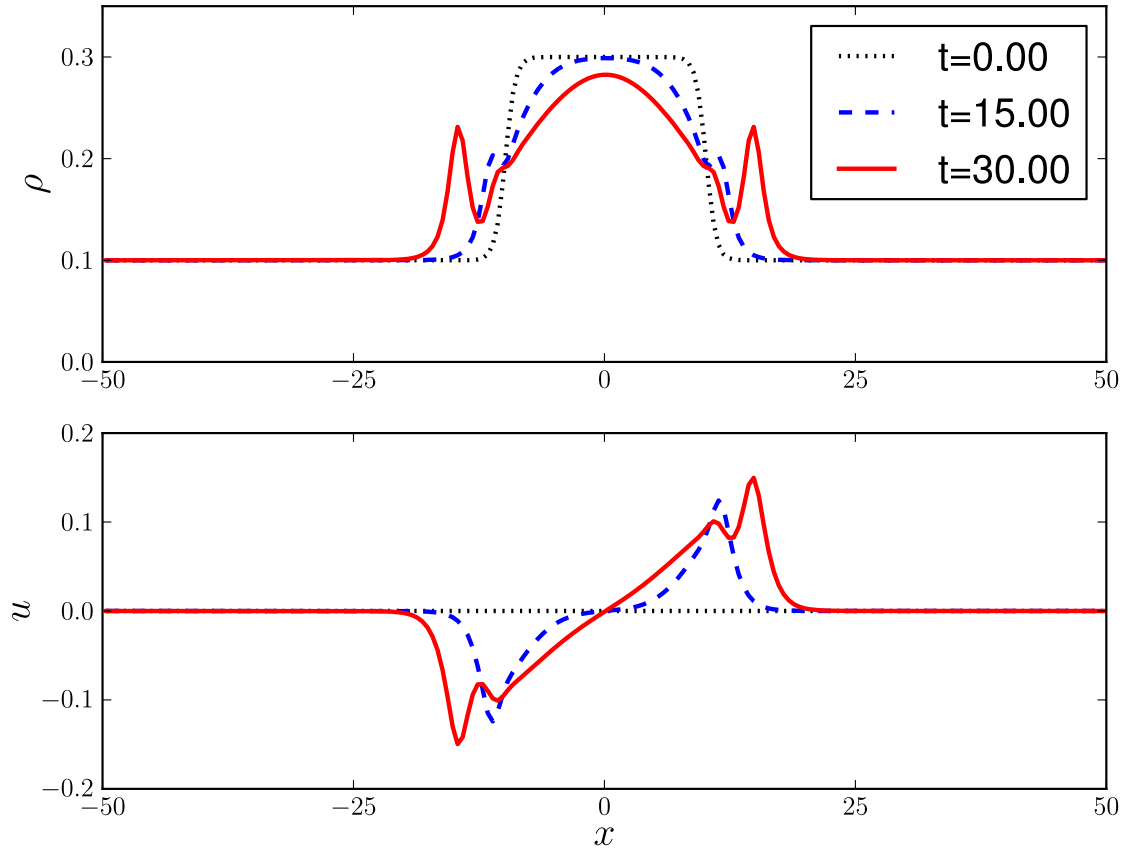


Figure 1.2: Dam-break results for the CH2 system in (1.2.53) arising from initial conditions (1.2.55) in a periodic domain. Figures are courtesy of J. Percival.

The variable m in the CH2 may be interpreted as the momentum associated with the system while the function ρ corresponds to the fluid elevation above the mean surface height. The system is known to be Hamiltonian in the sense that the evolution of the each variable m and ρ may be obtained from a Poisson structure and as such we may write

$$m_t = \{m, H\} \quad \rho_t = \{\rho, H\}, \quad (1.2.56)$$

where H is a Hamiltonian for the system. The Poisson bracket $\{\cdot, \cdot\}$ is linear and skew

symmetric in its entries and also satisfies the Leibnitz condition

$$\{\mathcal{A}, (\mathcal{B}\mathcal{C})\} = \mathcal{B}\{\mathcal{A}, \mathcal{C}\} + \{\mathcal{A}, \mathcal{B}\}\mathcal{C} \quad (1.2.57)$$

and the Jacobi identity

$$\{\mathcal{A}, \{\mathcal{B}, \mathcal{C}\}\} + \{\mathcal{B}, \{\mathcal{C}, \mathcal{A}\}\} + \{\mathcal{C}, \{\mathcal{A}, \mathcal{B}\}\} = 0, \quad (1.2.58)$$

where \mathcal{A} , \mathcal{B} and \mathcal{C} are functionals of the physical variables m and ρ .

Indeed the CH2 system is known to be integrable with a bi-Hamiltonian structure meaning it possess two compatible Poisson brackets. The first Poisson bracket between two functionals \mathcal{F} and \mathcal{G} of the variables m and ρ is given by

$$\{\mathcal{F}, \mathcal{G}\}_1 = - \int \left[\frac{\delta \mathcal{F}}{\delta m} (m\partial + \partial m) \frac{\delta \mathcal{G}}{\delta m} + \frac{\delta \mathcal{F}}{\delta m} \rho \partial \frac{\delta \mathcal{G}}{\delta \rho} + \frac{\delta \mathcal{F}}{\delta \rho} \partial \rho \frac{\delta \mathcal{G}}{\delta m} \right] dx \quad (1.2.59)$$

and has a Hamiltonian

$$H_1 = \frac{1}{2} \int (um + \rho^2) dx. \quad (1.2.60)$$

The second Poisson bracket is given by

$$\{\mathcal{F}, \mathcal{G}\}_2 = - \int \left[\frac{\delta \mathcal{F}}{\delta m} (\partial - \partial^3) \frac{\delta \mathcal{G}}{\delta m} + \frac{\delta \mathcal{F}}{\delta \rho} \partial \frac{\delta \mathcal{G}}{\delta \rho} \right] dx \quad (1.2.61)$$

with associated Hamiltonian

$$H_2 = \frac{1}{2} \int (u\rho^2 + u^3 + uu_x^2) dx. \quad (1.2.62)$$

Meanwhile the compatibility of the Poisson structures $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ means that the sum

$$k_1\{\cdot, \cdot\}_1 + k_2\{\cdot, \cdot\}_2$$

for any constants k_1 and k_2 , has all the properties of a Poisson structure. In general given any function f of the dynamical variables we may write

$$\frac{d}{dt}f = \{f, H_i\}, \quad (1.2.63)$$

in which case given any conserved quantity H_n we may write

$$\{H_n, H_i\} = 0. \quad (1.2.64)$$

As such a sequence of conserved quantities for the two component Camassa-Holm equation may be constructed as in [Iva2006], see also [Mag1978].

In the context of shallow water waves propagating over a flat bottom, u can be interpreted as the horizontal fluid velocity and ρ is the water elevation in the first approximation [CI2008, Iva2009]. In the shallow water regime there are scale characteristics, one of which is $\delta = h/\lambda$. With these scale factors, if $u = \mathcal{O}(1)$, then $m = u - \delta^2 u_{xx}$, see e.g. [CI2008, Iva2009]. In the limit $\lambda \gg h$ or $\delta \rightarrow 0$, we have $m \rightarrow u$ while the CH2 system in (1.2.53) becomes

$$\begin{aligned} u_t + 3uu_x + \rho\rho_x &= 0, \\ \rho_t + (\rho u)_x &= 0, \end{aligned} \quad (1.2.65)$$

which we call the two component Hopf equation. The first Poisson bracket (1.2.61) remains valid in the case $m = u$ with Hamiltonian H_1 . Meanwhile in the limit $\delta \rightarrow 0$ the second Poisson bracket in (1.2.61) becomes

$$\{F, G\}_2 = - \int \left[\frac{\delta F}{\delta u} \partial \frac{\delta G}{\delta u} + \frac{\delta F}{\delta \rho} \partial \frac{\delta G}{\delta \rho} \right] dx, \quad (1.2.66)$$

and the corresponding Hamiltonian is $H_2 = \frac{1}{2} \int (u\rho^2 + u^3) dx$. Dispersionless systems like that above also arise in the context of very long water waves, in particular in the modelling of tsunamis as they approach the shore [CJ2008].

1.2.4 The Kaup-Boussinesq Equation

The final system we shall derive in the context of shallow water models is the Kaup-Boussinesq equation which was derived in [Bou1871] and shown to be completely integrable in [Kau1975]. Like the two component Camassa-Holm equation previously

discussed this is also a two component system. The derivation of this system in the hydrodynamical setting follows that presented in [Iva2009]. It will be shown later that this system like the KdV equation is exactly solvable, in the sense that we may construct explicit soliton solutions for it via the inverse scattering transform method. Unlike the KdV spectral problem however there is an added complication that the spectral problem for Kaup-Boussinesq equation is energy dependent in that the potential has a term in which the spectral parameter appears as a factor. This complication will be discussed in a later chapter but for now we shall concentrate on the hydrodynamical significance of the system. We begin by introducing

$$\mathcal{V} = u_0 - \delta^2 \frac{1}{2} u_{0xx},$$

which we may use to rewrite (1.2.32) as

$$\mathcal{V}_t + \varepsilon \mathcal{V} \mathcal{V}_x + \eta_x + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2) = 0. \quad (1.2.67)$$

Under a shift of the variable η given by

$$\eta \rightarrow \eta - \frac{1}{\varepsilon},$$

equation (1.2.35) transforms to

$$\eta_t + \varepsilon (\eta \mathcal{V})_x - \delta^2 \frac{1}{6} \mathcal{V}_{xxx} + \mathcal{O}(\varepsilon^2, \varepsilon \delta^2) = 0. \quad (1.2.68)$$

Finally we introduce the change of variables

$$x \rightarrow \delta \sqrt{\frac{2}{3}} x, \quad t \rightarrow \delta \sqrt{\frac{2}{3}} t, \quad \mathcal{V} \rightarrow \frac{1}{\varepsilon} \mathcal{V}, \quad \eta \rightarrow \frac{1}{\varepsilon} \eta,$$

under which the system of equations above becomes

$$\begin{aligned} \mathcal{V}_t + \mathcal{V} \mathcal{V}_x + \eta_x &= 0 \\ \eta_t + (\eta \mathcal{V})_x - \frac{1}{4} \mathcal{V}_{xxx} &= 0, \end{aligned} \quad (1.2.69)$$

where we have neglected terms of order $\mathcal{O}(\varepsilon^2, \varepsilon\delta^2)$ and higher. The coupled system in (1.2.69) is the Kaup-Boussinesq equation. This system was investigated by Kaup in [Kau1975], where he investigated its integrability and the inverse scattering method for the system.

In Chapter 6 we are going to investigate the solution of this system via the inverse scattering transform method. In fact we are going to investigate the solution of a general class of partial differential equations with cubic nonlinearities, the Kaup-Boussinesq being a particular member of this class. While the Kaup-Boussinesq equation itself possesses only quadratic nonlinearities, it belongs to a family of equations whose spectral problem depends on a physical parameter κ . Without any restrictions on κ the compatibility condition applied to the Lax pair yields a family of nonlinear PDE with cubic nonlinearities. With a particular value assigned to κ this compatibility condition yields the Kaup-Boussinesq equation and thereby removes the cubic nonlinearities. The particular difficulty of solving this class of equations is in relation to the associated spectral problem, specifically in relation to the energy dependence of the spectral problem which leads to some extra complications when compared to the spectral problems of the KdV equation. The method developed in Chapter 6 allow us to solve the nonlinear system for arbitrary values of κ and so the method may be applied to a family of equations with cubic nonlinearities.

1.3 Conclusion

In this chapter we have introduced the governing equations of fluid dynamics as constituted by the Euler equation and the continuity equation. We presented a brief derivation of these equations as they are presented in many textbooks, in particular we followed [Joh1997] quite extensively. We went on to develop nonlinear models from these governing equations via nondimensionalisation. In particular we presented a derivation of

the KdV equation in line with that presented in [Joh1997] along with a derivation of the CH2 equation and the Kaup-Boussinesq equation both of which followed closely the derivation presented in [Iva2009].

Chapter 2

Blood Flow

2.1 Modeling Newtonian fluids in elastic tubes

The two component Burgers equation, and related systems, are well understood quasi linear systems arising in many physical applications, for example as a model of shock waves in gas dynamics (see e.g. [Str1990], Section 13.2). In this paper we give a derivation of the following system

$$u_t + uu_x = -P_x \tag{2.1.1}$$

$$A_t + (uA)_x = 0, \tag{2.1.2}$$

in the context of a Newtonian fluid within an elastic tube, modelling the flow of blood within arteries. The model is quasi linear in the dynamical variables, u and A the fluid velocity and tubes cross section respectively. The physiological phenomenon which is the focus of the current chapter is the so called *pistol shot pulse* a discussion of which may be found in the work [Ped2003]. This is a popular term for a phenomenon reported by clinicians in which a loud cracking sound is heard by the stethoscope over an artery, caused by a large distension followed by an abrupt collapse of the artery wall. This phenomenon is known to occur in large arteries during aortic regurgitation, a condition

in which some blood leaks back into the left ventricle during systole.

To establish a model of arterial blood flow, we consider a simplified model of blood itself, in the sense that it will be modelled as an incompressible, inviscid and irrotational Newtonian fluid. The derivation we present is an original work but follows closely that presented in [Lyo2012]. The artery is modeled as an axially symmetric elastic tube, with cross section A which depends on the axial coordinate x and time t . Being elastic the artery experiences a restoring force when expanded from it's equilibrium cross section and as such exerts additional pressure on the blood contained within.

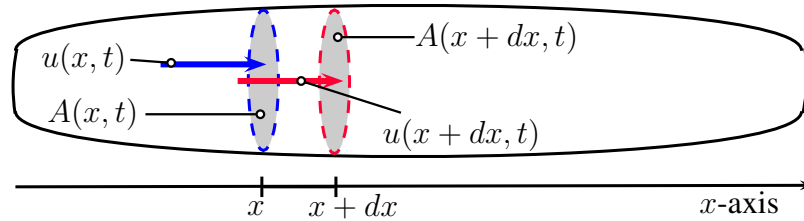


Figure 2.1: The propagation of a pressure wave within an artery. The pressure wave causes a change in the fluid velocity of the blood and an expansion of the artery.

The volume of blood contained in the infinitesimal volume of artery between the axial locations x and $x + dx$ at a fixed instant t is $dV = A(x, t)dx$. In addition the mass content in that same volume of blood will be the material density at the location at that instant $\tilde{\rho}(x, t)$ times the volume itself

$$dM(t) = \tilde{\rho}(x, t)dV = \tilde{\rho}(x, t)A(x, t)dx.$$

We denote the mass by $dM(t)$ to indicate the blood content at a fixed instant of time t . Since we model the blood as an incompressible fluid then the density must be homogenous at all times and so in our model the blood density is a fixed constant $\tilde{\rho}$. Within a finite volume of the artery between the axial locations a and x , with $a < x$ at some

fixed moment t , it follows that the mass content is

$$\int_a^x dM(t) = \tilde{\rho} \int_{V_{ax}} dV = \tilde{\rho} \int_a^x Ad\xi,$$

where V_{ax} denotes the volume between the axial locations a and x .

The velocity of the blood through the artery at a fixed axial coordinate x and time t will be uniform over the cross section at that location and time. During an infinitesimal time dt the blood content at the location x will be displaced by a distance $dX = udt$. In this time interval the total mass of blood to traverse the cross section A will be

$$dM = \tilde{\rho}AdX = \tilde{\rho}Audt.$$

Note that here we do not write dM as $dM(t)$ since we are considering the change in mass content over a time interval dt and so dM does not refer to a mass quantity at a fixed instant. It follows that the change in the blood content in the arterial volume between a and x in the time interval dt is the total blood displacement into the volume across $A(a, t)$ less the total blood displacement out of the volume across $A(x, t)$. Hence we may write,

$$dM|_a^x = \tilde{\rho}u(a, t)A(a, t)dt - \tilde{\rho}u(x, t)A(x, t)dt.$$

We may rewrite the above equation as one involving the rate of change of blood content within a finite arterial volume as follows,

$$\partial_t \left(\int_a^x dM(t) \right) = \partial_t \left(\tilde{\rho} \int_0^x Ad\xi \right) = \tilde{\rho}u(0, t)A(0, t) - \tilde{\rho}u(x, t)A(x, t).$$

Operating on this equation with ∂_x the fundamental theorem of calculus applied to the left hand side gives us

$$\partial_x \partial_t \left(\tilde{\rho} \int_a^x Ad\xi \right) = \tilde{\rho} \partial_t \left(\partial_x \int_a^x Ad\xi \right) = \tilde{\rho} \partial_t A(x, t),$$

while ∂_x applied to the right hand side gives us $-\tilde{\rho} \partial_x (uA)$, and so we arrive at the continuity equation (2.1.2).

The physical interpretation of this equation is as follows. In an infinitesimal time interval dt the infinitesimal increase (decrease) in the blood content dM at x causes an increase (decrease) of the blood volume since the blood itself is incompressible. As the walls of the artery are elastic, this increase (decrease) in blood volume is accommodated by an expansion (contraction) of the artery which is realised as an increase (decrease) in its cross section A .

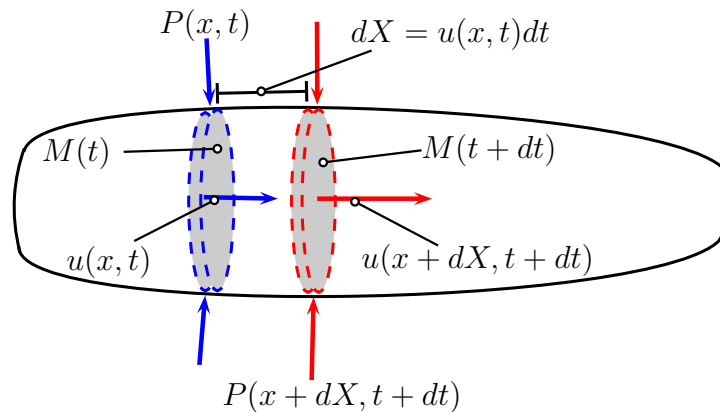


Figure 2.2: The figure illustrates the changing mass, cross section and velocity of an infinitesimal fluid mass as it propagates.

Newton's laws dictate that any forces acting upon the blood will cause a corresponding change of momentum of the blood. In the case of fluids the forces acting upon any individual element may be separated into two categories:

- The external forces acting on the fluid elements, which in this case will be the restoring forces present in the artery walls, thereby exerting an external pressure on the blood at a given location x and some instant t ,
- The internal forces acting upon a fluid element caused by contiguous fluid elements.

Both external and internal forces are found to cause an overall change in the linear

momentum of an individual fluid element, in accordance with Newton's second law.

The linear momentum of an infinitesimal volume of blood at some location x and instant t is $\tilde{\rho}udV_x = \tilde{\rho}uAdx$, that is the infinitesimal mass element times the velocity thereof. The rate of change of this momentum with respect to time is given by $\tilde{\rho}\partial_t(uA)dx$ and will appear as the left hand side of Newton's second law. Extending this to the case of a finite volume of blood we find

$$\tilde{\rho}\partial_t\left(\int_a^x uAd\xi\right),$$

which is the rate of change in momentum of the blood contained within the artery between a and x at the instant t . As we have already seen, an infinitesimal time displacement causes a corresponding spatial displacement of the blood at x by an amount dX . However in moving this distance the velocity of that blood will change by the amount

$$u_x dX = uu_x dt.$$

It follows that the corresponding change in linear momentum of this blood will be its mass times the change in velocity, or

$$\tilde{\rho}uu_x Adx,$$

which for a finite volume becomes

$$\tilde{\rho}\int_a^x uu_\xi Ad\xi.$$

Thus we have established the change in linear momentum of a finite volume of blood due to the spatial displacement thereof during an infinitesimal time increment dt .

Next we must also include the possibility that the linear momentum of the blood will change in an infinitesimal time dt , due to a change in the blood content. Suppose at some instant t the blood flow at x is uA , while the blood flow at $x + dx$ is

$$-uA|_{(x+dx,t)} \simeq -uA - u_x Adx - uA_x dx.$$

It follows that the net of change of the mass will be

$$-\tilde{\rho}(uA)_x dx,$$

and the corresponding change in the linear momentum will be

$$-\tilde{\rho}(uA)_x u dx.$$

In the case of a finite volume of blood the change in linear momentum becomes

$$-\tilde{\rho} \int_a^x u(uA)_\xi d\xi.$$

This accounts for all internal forces acting upon an individual fluid element within the artery.

Next we need to investigate the effects of external forces on a fluid element. In this case the external forces are provided by the restoring forces within the arterial wall which exerts a force on the blood. Between the axial locations x and $x + dx$, the internal surface of the artery has a directed area element $\hat{n}dS$, with \hat{n} being the outward normal and dS being the magnitude of the infinitesimal area element. The force exerted by the distended artery on the blood is $-P\hat{n}dS$, where P denotes the pressure exerted due to the restoring force of the artery acting on the blood at time t and location x . In the case of a finite volume V between a and x , we find the corresponding force exerted by the artery with surface area S to be

$$-\int_S P\hat{n}dS.$$

Applying Gauss' law we may write this as

$$-\int_S P\hat{n}dS = -\int_V \partial_\xi P dV = -\int_a^x P_\xi A d\xi.$$

This accounts for all the forces acting on the blood contained within the artery between axial locations a and x .

Having established the rate of change of momentum along with the various contributing forces we require an equation of motion for the finite fluid element. On applying Newton's second law to the above acceleration and forces we obtain

$$\tilde{\rho} \partial_t \int_a^x u A d\xi = \tilde{\rho} \int_a^x [u u_\xi A - u \partial_\xi (u A)] d\xi - \int_a^x P_\xi A d\xi.$$

Dividing both sides by $\tilde{\rho}$, operating on the resulting equation with ∂_x and applying the fundamental theorem of calculus to both sides we obtain

$$(uA)_t = u u_x A - u (uA)_x - \frac{1}{\tilde{\rho}} P_x A.$$

Expanding the derivatives on each side, applying the continuity equation (2.1.2) and dividing by $A(x, t)$ we find

$$u_t + u u_x + \frac{1}{\tilde{\rho}} P_x = 0. \quad (2.1.3)$$

We see that in our model the blood flow will satisfy Euler's Equation for an incompressible, inviscid and irrotational fluid.

In general it is difficult to obtain from first principles an explicit expression for the transmural pressure $P(x, t)$. However a large body of experimental data exists to suggest a plausible correspondence between the pressure and the cross section of the tube itself. Indeed, a standard example is the so called *Windkessel* model [KS2009], in which the pressure is related linearly to the cross section,

$$P \equiv P(A) = k_W \cdot A.$$

The constant k_W relates the elastic restoring force of the tube when distended to cross section A to the pressure exerted on the blood, and it is determined from clinical data. The Windkessel model may be solved via the method of characteristics and is a well understood model of blood flowing in arteries, [Ped2003]. In this chapter we will adopt a slightly more complicated model, whereby the relationship between pressure and aortic

cross section is quadratic, namely

$$\begin{aligned} P &\equiv P(A) = \kappa A^2 \\ u_t + uu_x + kAA_x &= 0, \quad k = \frac{2\kappa}{\tilde{\rho}} \end{aligned} \tag{2.1.4}$$

In [PL1998] Figure 1 displays a plot of transmural pressure versus cross section for a collapsible tube. It is found that for a certain range of values of A the corresponding transmural pressure is closely approximated by a quadratic relationship. Furthermore in [Ped2003] the author presents a table data obtained from clinical experiments on dogs which corroborates this quadratic relationship for a range of values of arterial blood pressure. The equations (2.1.2) and (2.1.4) when taken together constitute a two component dispersionless Burgers equation.

2.2 Solutions when $k > 0$

In this section we consider in more detail the behaviour of the system (2.1.2) and (2.1.4). To simplify matters we also introduce the scaling

$$A \rightarrow \frac{1}{\sqrt{|k|}} A,$$

whereby the Euler equation becomes

$$\begin{aligned} u_t + uu_x + AA_x &= 0 \\ A_t + (uA)_x &= 0, \end{aligned} \tag{2.2.1}$$

and where we have chosen $k > 0$. As we already mentioned one of the main motivations for investigating this system is that it may be solved directly by the method of characteristics. To illustrate this we shall define a pair of diffeomorphisms $\psi_{\pm} \equiv \psi_{\pm}(x, t)$ by the following criteria:

$$\begin{aligned} \partial_t \psi_{\pm} &= u(\psi_{\pm}, t) \pm A(\psi_{\pm}, t), \\ \psi_{\pm}(x, 0) &= x. \end{aligned} \tag{2.2.2}$$

In this analysis we are considering the propagation of pressure wave in an artery over the half line $x \geq 0$ and so strictly speaking the mappings ψ_{\pm} are diffeomorphisms only when $x > 0$. Applying ∂_t once more to (2.2.2) and using (2.2.1) we find

$$\partial_t^2 \psi_{\pm} = 0, \quad (2.2.3)$$

and as such it follows that $\partial_t \psi_{\pm}$ depends on x only.

Using this and the definition supplied in (2.2.2), we see that the physical variables evaluated along the flow of $\psi_{\pm}(x, t)$ are conserved, namely

$$u(\psi_{\pm}, t) \pm A(\psi_{\pm}, t) = u_0 \pm A_0, \quad (2.2.4)$$

where $u_0 \equiv u(x, 0)$ and $A_0 \equiv A(x, 0)$ are obviously time independent. Moreover (2.2.3) requires ψ_{\pm} be linear in t while the second equation in (2.2.2) imply ψ_{\pm} satisfy

$$\psi_{\pm} = x + t\gamma_0^{(\pm)},$$

where it is understood $\gamma_0^{(\pm)}$ depends only on x . Applying ∂_t to ψ_{\pm} as it is given in this expression and comparing to (2.2.2) and (2.2.4), we see that the functions $\gamma_0^{(\pm)}$ may be written in terms of the initial data as follows

$$\gamma_0^{(\pm)} = u_0 \pm A_0.$$

So we see that we may solve (2.2.2) to find

$$\psi_{\pm} = x + t(u_0 \pm A_0). \quad (2.2.5)$$

allowing us to express ψ_{\pm} in terms of the given initial data u_0 and A_0 .

Since 2.2.2 are diffeomorphisms, at least for appropriately chosen initial data, it is in principle possible to invert (2.2.5), and upon doing so one may obtain explicit solutions for the physical variables u and A in terms of the initial data $(u_0, A_0) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$\begin{aligned} u(x, t) &= \frac{1}{2} [u_0(\psi_+^{-1}) + u_0(\psi_-^{-1})] + \frac{1}{2} [A_0(\psi_+^{-1}) - A_0(\psi_-^{-1})], \\ A(x, t) &= \frac{1}{2} [u_0(\psi_+^{-1}) - u_0(\psi_-^{-1})] + \frac{1}{2} [A_0(\psi_+^{-1}) + A_0(\psi_-^{-1})] \end{aligned} \quad (2.2.6)$$

where it is understood that $\psi_{\pm}^{-1} \equiv \psi_{\pm}^{-1}$.

Example: We consider a solution in which our initial data is of the form $u_0 \sim A_0 \sim x^{\frac{1}{3}}$.

Specifically, the initial data is defined by

$$u_0 \pm A_0 = a_{\pm} x^{\frac{1}{3}}, \quad (2.2.7)$$

where a_{\pm} are constants. It follows from (2.2.5) that the diffeomorphisms ψ_{\pm} may be written as,

$$\psi_{\pm} = x + a_{\pm} t x^{\frac{1}{3}}. \quad (2.2.8)$$

Making the invertible substitution $x \rightarrow w^3$ and with the corresponding change of variables $\psi_{\pm}(x, t) \rightarrow y_{\pm}(w, t)$ we may rewrite the diffeomorphisms in (2.2.8)

$$y_{\pm}(w, t) = w^3 + a_{\pm} t w.$$

We now have a pair of monic polynomials with argument $w \in \mathbb{R}$, namely

$$w^3 + a_{\pm} t w - y_{\pm}^3(w, t) = 0, \quad (2.2.9)$$

the discriminants of which are given by

$$D = -4(a_{\pm} t)^3 - 27. \quad (2.2.10)$$

The values of the discriminant determine the quantity and nature of solutions for w . In particular we are interested in real solutions $w(y_{\pm}, t)$.

Equivalently this solution allows us to solve for x in terms of ψ_{\pm} and t , that is it offers us an expression for ψ_{\pm}^{-1} . Depending on the discriminant D we may have several real roots all of which may be distinct or some of which may be equal. In the case $D = 0$ we have two equal real roots for $a_{\pm} \in \mathbb{R}$ and real ψ_{\pm} . Moreover the discriminant allows us to explicitly calculate the time at which the wave breaking occurs and corresponds to the value of t for which the discriminant has no real roots, namely

$$T_{\pm} = -\frac{3}{\sqrt[3]{4a_{\pm}}}. \quad (2.2.11)$$

In this case the functions ψ_{\pm} do not have unique inverses and so no longer behave as diffeomorphisms. Furthermore the corresponding solutions u and A as defined by (2.2.6) will no longer satisfy $|u_x| < \infty$ and $|A_x| < \infty$, since ψ_{\pm} are no longer strictly monotone increasing functions of x for all $t > 0$.

Remark The functions ψ_{\pm} are diffeomorphisms only in the case where wave breaking does not occur. We assume $(u_0, A_0) \in C^1 \times C^1$ and also u_0 and A_0 are bounded, that is

$$\sup_{x \in \mathbb{R}} (|u_0| + |A_0|) < \infty.$$

Therefore our solutions blow up only if $u'_0 \pm A'_0 < 0$ at some point and $t > 0$ otherwise the solutions are global.

2.3 Solutions when $k < 0$

A related system is the so called "wrong" sign Burgers equation, given by,

$$\begin{aligned} u_t + uu_x - (A^2)_x &= 0 \\ A_t + (uA)_x &= 0, \end{aligned} \tag{2.3.1}$$

which differs from (2.1.1, 2.1.2) by the sign of the $(A^2)_x$ term. In analogy to the case of solution via characteristics we may construct a pair of complex conjugate mappings χ_{\pm} which are formally defined by,

$$\partial_t \chi_{\pm}(x, t) = u(\chi_{\pm}, t) \pm iA(\chi_{\pm}, t). \tag{2.3.2}$$

As in the case of (2.2.4) we find the analogous relations

$$u(\chi_{\pm}, t) \pm iA(\chi_{\pm}, t) = u_0 \pm iA_0, \tag{2.3.3}$$

where $u_0 \equiv u(x, 0)$ and similarly for A_0 . The construction of such a pair of solutions was given in [KK2002] wherein the authors obtained solutions by the method of Rie-

mann invariants with the mappings χ_{\pm} given by

$$\begin{aligned}\chi_{\pm} &= x - t(1 \pm i\sqrt{3}) \left(\frac{x}{\lambda}\right)^{\frac{1}{3}} \\ &= x - 2te^{\pm \frac{i\pi}{3}} \left(\frac{x}{\lambda}\right)^{\frac{1}{3}},\end{aligned}\tag{2.3.4}$$

It follows that the initial data corresponding to each of these mappings is given by

$$u_0(x) = -\left(\frac{x}{\lambda}\right)^{\frac{1}{3}} \quad A_0(x) = \mp \left(\frac{x}{\lambda}\right)^{\frac{1}{3}},\tag{2.3.5}$$

with the parameter λ being freely adjustable.

The corresponding solutions are found to be

$$A(\chi_{\pm}, t) = \sqrt{\frac{8t}{\lambda} + 3u^2(\chi_{\pm}, t)},\tag{2.3.6}$$

where the solution $u(\cdot, t)$ is the real root of the cubic polynomial

$$\lambda u^3(\xi, t) + 2tu(\xi, t) + \xi = 0,\tag{2.3.7}$$

for arbitrary $\xi \in \mathbb{R}$.

Having an explicit expression for $A(\chi_{\pm}, t)$ in term of $u(\chi_{\pm}, t)$ we now require an explicit expression for $u(\chi_{\pm}, t)$ in terms of x and t . To proceed, we notice from (2.3.3) that our solution $u(\chi_{\pm}(x, t), t)$ must satisfy

$$u(\chi_{\pm}) = u_0(x) \pm iA_0(x) \mp iA(\chi_{\pm}(x, t), t).\tag{2.3.8}$$

It follows from (2.3.5) and (2.3.6) that the solutions $u(\chi_{\pm}, t)$ may be written as,

$$u(\chi_{\pm}, t) = -2e^{\pm \frac{i\pi}{3}} \left(\frac{x}{\lambda}\right)^{\frac{1}{3}} \mp \sqrt{\frac{8t}{\lambda} + 3u^2(\chi_{\pm}, t)}.\tag{2.3.9}$$

Substituting this expression into the cubic polynomial in (2.3.7), we find,

$$u(\chi_{\pm}, t) = -\frac{e^{\pm \frac{2i\pi}{3}}}{2} \left(\frac{x}{\lambda}\right)^{\frac{1}{3}} \pm \sqrt{-\frac{3e^{\pm \frac{i\pi}{3}}}{8} \left(\frac{x}{\lambda}\right)^{\frac{2}{3}} - \frac{t}{\lambda}},\tag{2.3.10}$$

and so we also have explicit solutions for $A(\chi_{\pm}, t)$ in terms of x and t , as follows from (2.3.6).

2.4 Bounded solutions and wave breaking

The phenomenon of wave breaking was one of the most interesting and exciting aspects of the Camassa-Holm equation to be investigated following its discovery in the work [CH1993]. In this article, although we have focussed on somewhat more simplified system to model the flow of blood in arteries, and we would still like to investigate the possibility of the wave breaking phenomenon arising in the context of this model. A clinical interpretation of this is in relation to the phenomenon of the *pistol shot pulse*. Such behaviour arises during systole of patients with *aortic insufficiency*, when blood ejected into the artery is regurgitated back through the aortic valve, into the ventricles. To maintain systole pressure, the heart will contract more during ventricular systole, thereby ejecting blood into the artery with greater pressure. After systole, the ejected blood will induce a pressure gradient along the radial artery, however, owing to the initial aortic regurgitation, there will be insufficient blood mass to maintain the pressure throughout the length of the artery. This in turn will cause a sudden increase followed by a rapid decrease in the aortic cross section, which is observed as the femoral pistol shot pulse.

In this section we aim to establish the conditions under which a sudden expansion in the aortic cross section is followed by a collapse thereof, in rapid succession. We aim to show that the phenomena of wave breaking arising in the system (2.1.1)-(2.1.2), which we are using as a simplified model of aortic blood flow, is sufficient to account for this clinical phenomenon. Mathematically speaking, wave breaking occurs when our solutions $u(x, t)$ and $A(x, t)$ remain bounded for all $(x, t) \in \mathbb{R} \times [0, T)$, while the magnitude of their gradients become singular in finite time [CE1998, Whi1980],

$$|u_x| + |A_x| \rightarrow \infty, \quad t \rightarrow T.$$

We begin by constraining our initial data, such that,

$$(u_0, A_0) \in H^1 \times H^1 \subset C^1 \times C^1.$$

We will show that continuous solutions u and A do indeed remain finite, if their slopes remain finite. To do so, we first multiply each member of the system in (2.2.1) by u and A respectively. Then integrating over \mathbb{R} and imposing the boundary conditions

$$\lim_{x \rightarrow \pm\infty} u = 0, \quad \lim_{x \rightarrow \pm\infty} A = 0, \quad (2.4.1)$$

we find the following conditions

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int u^2 dx &= \frac{1}{2} \int u_x A^2 dx \\ \frac{1}{2} \frac{d}{dt} \int A^2 dx &= -\frac{1}{2} \int u_x A^2 dx. \end{aligned} \quad (2.4.2)$$

Indeed we see from this result that the quantity

$$H = \frac{1}{2} \int (u^2 + A^2) dx,$$

is an integral of motion and acts as a Hamiltonian for our system. Analogous to the case of the CH2 equation the Hamiltonian above may be used to find the equations of motion for u and A using the Poisson structure

$$\{\mathcal{A}, \mathcal{B}\} = \int \left[\frac{\delta \mathcal{A}}{\delta u} \frac{\delta \mathcal{B}}{\delta A} - \frac{\delta \mathcal{A}}{\delta A} \frac{\delta \mathcal{B}}{\delta u} \right] dx,$$

where \mathcal{A} and \mathcal{B} are functionals of u and A .

We apply the operator ∂_x to each member of (2) and multiply by u_x and A_x respectively. Upon Imposing the boundary conditions (2.4.1) and integrating over \mathbb{R} we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int u_x^2 dx &= -\frac{1}{2} \int u_x^3 dx - \int u_x A_x^2 dx - \int u_x A A_{xx} dx \\ \frac{1}{2} \frac{d}{dt} \int A_x^2 dx &= \int u_x A A_{xx} dx - \frac{1}{2} \int u_x A_x^2 dx. \end{aligned} \quad (2.4.3)$$

Using (2.4.2) and (2.4.3) we find that

$$\frac{1}{2} \frac{d}{dt} \int (u^2 + u_x^2 + A^2 + A_x^2) dx = -\frac{1}{2} \int u_x (u_x^2 + 3A_x^2) dx. \quad (2.4.4)$$

It follows from (2.4.4) that

$$\frac{d}{dt} \int (u^2 + u_x^2 + A^2 + A_x^2) dx \leq 3M_1 \int (u^2 + u_x^2 + A^2 + A_x^2) dx, \quad (2.4.5)$$

where $\sup_{x \in \mathbb{R}} |u_x| = M_1$.

Gronwall's inequality [BN1969] states that for a pair of functions $f(t)$ and $g(t)$ continuous on some interval $\alpha \leq t < \beta$ and with f differentiable on (α, β) and such that

$$\begin{aligned} f' &\leq fg, \quad t \in (\alpha, \beta) \Rightarrow \\ f(t) &\leq f(\alpha) \exp \left(\int_{\alpha}^t g(\xi) d\xi \right). \end{aligned} \quad (2.4.6)$$

Upon applying Gronwall's inequality to (2.4.5) we find that

$$\int (u^2 + u_x^2 + A^2 + A_x^2) dx \leq K_1 e^{3M_1 t} < \infty, \quad t \in [0, T]. \quad (2.4.7)$$

Here we introduce $K_1 = \int_{\mathbb{R}} (u_0^2 + u_0'^2 + A_0^2 + A_0'^2) dx$, with $u_0(x) \equiv u(x, 0)$ and $A_0(x) \equiv A(x, 0)$. Moreover for any function $u \in H^1$ we have

$$|u|^2 \leq \|u\|_1^2 = \int_{\mathbb{R}} (u^2 + u_x^2) d\xi.$$

Since we assumed $(u, A) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ it follows from this inequality along with the result in (2.4.7) that

$$|u|^2 + |A|^2 \leq \|u\|_{H^1}^2 + \|A\|_{H^1}^2 \leq K_1 e^{3M_1 t}, \quad (x, t) \in \mathbb{R} \times [0, T], \quad (2.4.8)$$

It follows that the solutions $|u|$ and $|A|$ remain bounded for all $(x, t) \in \mathbb{R} \times [0, T]$ if $|u_x|$ and $|A_x|$ remain bounded for the same values of x and t .

Next we would like to establish under what conditions the functions $u_x(x, t)$ and $A_x(x, t)$ actually become singular while $u(x, t)$ and $A(x, t)$ remain bounded. We return

to the diffeomorphisms introduced in Section 3 in particular (2.2.4). Differentiating this once with respect to x we find

$$[u_x(\psi_{\pm}, t) \pm A_x(\psi_{\pm}, t)] \cdot \psi_{\pm, x} = u'_0 \pm A'_0. \quad (2.4.9)$$

Substituting (2.2.5) into this we find

$$u_x(\psi_{\pm}, t) \pm A_x(\psi_{\pm}, t) = \frac{u'_0 \pm A'_0}{1 + t \cdot (u'_0 \pm A'_0)}. \quad (2.4.10)$$

Under the condition $(u_0, A_0) \in H^1 \times H^1$ and with $\inf_{x \in \mathbb{R}} [u'_0 \pm A'_0] < 0$, it follows that

$$u_x(\psi_{\pm}, t) \pm A_x(\psi_{\pm}, t) \rightarrow -\infty, \quad t \rightarrow \inf_{x \in \mathbb{R}} \frac{1}{|u'_0 \pm A'_0|}. \quad (2.4.11)$$

It follows that with these initial conditions the system (2.1.1)-(2.1.2) develops wave breaking. Conversely for $u_0 \pm A_0 > 0$, $x \in \mathbb{R}$, we see that the denominator is nonzero for all $t > 0$. In this case the functions ψ_{\pm} are invertible for all $(x, t) \in \mathbb{R} \times [0, \infty)$. In this case the functions in (2.4.10) remain bounded so that

$$|u_x(\psi_{\pm}, t) \pm A_x(\psi_{\pm}, t)| < \infty, \quad t > 0. \quad (2.4.12)$$

The solutions u and A are global if our initial data satisfies $u'_0 \pm A'_0 > 0$, $x \in \mathbb{R}$.

We include here a graph illustrating the development of breaking waves for the dam break scenario applied to the two component Burger's equation, cf. Figure 1.2 where the dam break scenario is illustrated for the two component Camassa-Holm equation. The solutions illustrated above arise from the initial data given by

$$\begin{aligned} u(x, 0) &= 0, \\ \rho(x, 0) &= 0.1 (1 + \tanh(x + 5) - \tanh(x - 5)), \end{aligned} \quad (2.4.13)$$

which is repeated periodically in x with periodicity 100. As can be seen the solution has steepening wavefront whose slope eventually becomes unbounded in a finite time cf. equation (2.4.10). In Figure (1.2) relating to the two component Camassa-Holm equation under the same initial conditions it was seen that a train of solitons develop.

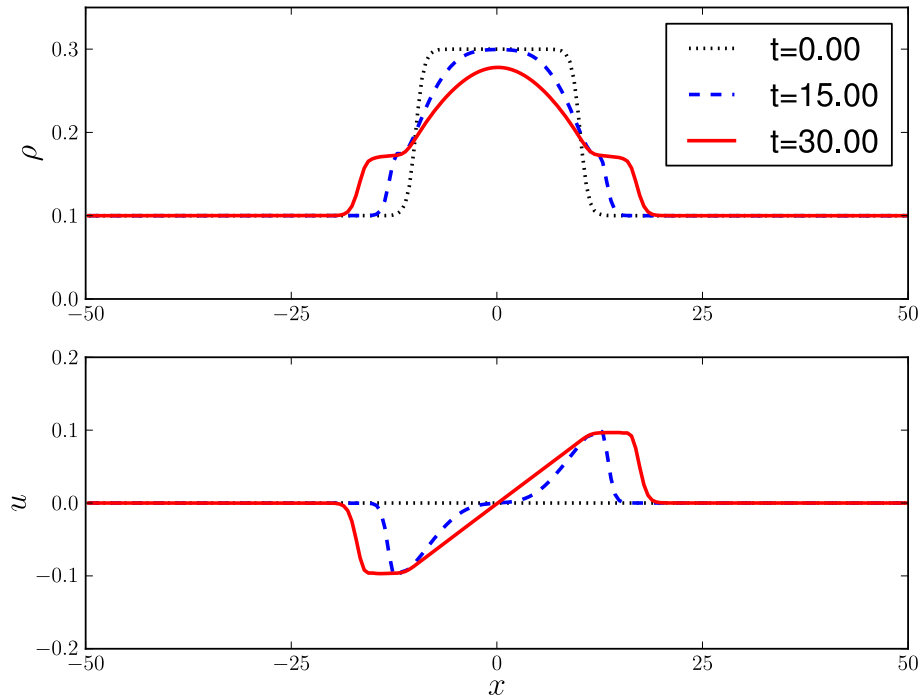


Figure 2.3: Dam break results for the two component Burger's system in (2.1.1-2.1.2) arising from initial conditions (2.4.13) in a periodic domain. Figures are courtesy of J. Percival.

2.5 KdV from arterial blood flow

The preceding discussion concerned the two component Hopf equation as it arises in the theory of arterial blood flow. The system is quasi linear in that there are no dispersion terms present. This restricts the possible solutions to the system and as we saw under certain initial conditions there are no global solutions. That is to say the nonlinearities in the system are not counter balanced by the effects of dispersion and so lead to the phenomenon of wave breaking. We have already presented a derivation of the KdV equation as it arises in the theory of shallow water wave. The KdV is perhaps the archetypal and best understood example of a nonlinear system with dispersion. The

presence of dispersion and nonlinearities together allow for a completely new class of solutions that are not possible in either linear or quasi linear systems, namely soliton solutions.

In this regard we shall now present a derivation of the KdV equation as it arises in the theory of arterial blood flow, thereby allowing for the possibility of solutions which are global and therefore more realistic in the context. Moreover, the derivation relies on the inclusion of effects so far neglected in our previous derivation of the Hopf equation, namely the inclusion of the elastic restraining forces acting on the arterial wall itself. The figure which follows is a simple diagram illustrating the effects of arterial distension on the arterial wall due to its own internal restoring forces. The material presented in this section is included for the sake of continuity in that we see the emergence of a nonlinear system with dispersion arising in the physical setting of arterial blood flow. In this regard there are no original results presented and the material follows closely the derivation presented in [DP2006].

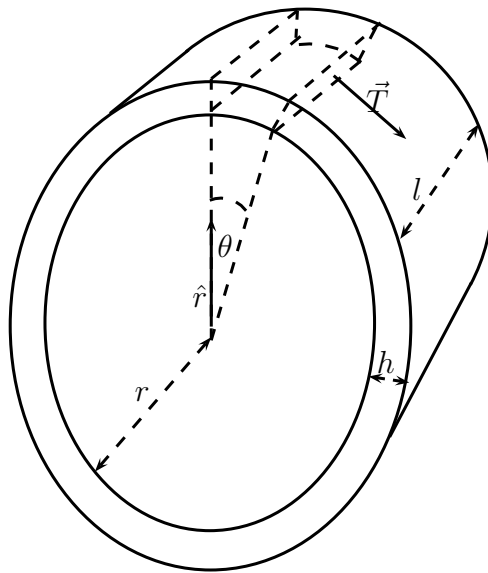


Figure 2.4: The elastic restoring forces in an artery.

The blood contained within the artery exerts a force along the line \hat{r} on the region

contained between the opening of angular width θ . The magnitude of force exerted on this area is simply the pressure time the area itself, in which

$$\vec{F}(P) = Prl\theta\hat{r}. \quad (2.5.1)$$

The pressure P is the difference within the artery between systolic and diastolic pressure. Letting the ring have a an equilibrium radius of r_0 during diastole, the expansion of the artery to radius r during diastole induces elastic restoring forces within the arterial wall. The arc lengths subtending the angle θ during diastole and systole are

$$\mathcal{A}(r_0) = r_0\theta \quad \& \quad \mathcal{A}(r) = r\theta \quad (2.5.2)$$

respectively. Consequently there is a relative stretching of this section of the arterial wall given by

$$\frac{r\theta - r_0\theta}{r_0\theta} = \frac{r - r_0}{r_0}$$

The elastic restoring forces within the wall ensure the remainder of the arterial section exert a force \vec{T} on the section illustrated. If we denote by Y_0 the Young modulus of the arterial tissue, then Y_0 and the associated tensile force T are related by

$$Y_0 \frac{r - r_0}{r_0} = \frac{T}{lh}. \quad (2.5.3)$$

The tensile force has a radial component

$$\vec{F}_T = -T \sin \theta \hat{r} \sim -T\theta\hat{r} = -Y_0 \frac{r - r_0}{r_0} lh\theta\hat{r},$$

which acts to restore the illustrated section to its equilibrium radius.

We suppose the arterial tissue has a uniform density ρ_0 , in which case the mass of the arterial section is given by

$$m(\theta) = \rho_0 r_0 lh\theta. \quad (2.5.4)$$

The mass of the section times its radial acceleration must be equal to the net force exerted on the section, and so after simplifying we find

$$\rho_0 r_0 h \frac{d^2 r}{dt^2} = Pr - Y_0 h \frac{r - r_0}{r_0}, \quad (2.5.5)$$

which is a statement of Newton's second law.

The arterial cross section is simply

$$A(r) = \pi r^2,$$

and the second time derivative of A is

$$A_{tt}(r) = \frac{d^2 A}{dt^2}(r) = 2\pi \left(r \frac{d^2 r}{dt^2} + \left(\frac{dr}{dt} \right)^2 \right).$$

As we are looking for a mathematical description of regular blood flow we are interested in relatively small radial velocities of the arterial wall in which case the second term on in the right hand side above may be neglected. Thus we make the approximation

$$A_{tt}(r) \sim 2\pi r \frac{d^2 r}{dt^2}. \quad (2.5.6)$$

Meanwhile since the variation in radius is small, it follows that the variation in cross section may be written as

$$A(r) - A(r_0) \sim 2\pi r_0 (r - r_0), \quad (2.5.7)$$

and so we may approximately rewrite (2.5.5) as

$$\rho_0 h A_{tt}(r) = 2\pi r_0 P + \frac{Y_0 h}{r_0^2} [A(r) - A(r_0)]. \quad (2.5.8)$$

The forces exerted net forces acting on the wall must be considered in conjunction with the forces actin on the blood within the artery to provide a complete description of the dynamics.

Like the derivation of the Hopf equation earlier we will suppose that significant variation in blood pressure and blood velocity take place along the axial direction only.

In the axial direction along the x -axis, the Euler equation is

$$u_t + uu_x = -P_x,$$

while the equation of mass conservation requires

$$A_t + (uA)_x = 0,$$

as before. We assume that blood has a uniform density $\rho = 1$ in the units we are working in.

Under the change of variables

$$A \rightarrow \pi r_0^2 A, \quad P \rightarrow \frac{Y_0 h}{2r_0} P, \quad u \rightarrow \lambda \omega u, \quad x \rightarrow \lambda x, \quad t \rightarrow \frac{1}{\omega} t \quad (2.5.9)$$

where we introduce

$$\lambda = \sqrt{\frac{\rho_0 r_0 h}{2}}, \quad \omega = \sqrt{\frac{Y_0}{\rho_0 r_0^2}},$$

equation (2.5.8) along with the two component Hopf equation become

$$\begin{aligned} A_{tt} + A &= p + 1, \\ u_t + uu_x &= -p_x, \\ A_t + (Au)_x &= 0, \end{aligned} \quad (2.5.10)$$

which is the system in non-dimensional variables.

The equilibrium values of the variables are

$$A_0 = 1, \quad P_0 = 0, \quad u_0 = 0,$$

and so we expand each of the variables according to

$$A = 1 + a \quad P = p \quad u = \tilde{u}, \quad (2.5.11)$$

where A, p and \tilde{u} etc are small perturbation about the equilibrium values. Neglecting nonlinear terms the system (2.5.9) becomes

$$\begin{aligned} a_{tt} + a &= p, \\ \tilde{u}_t &= -p_x, \\ a_t + \tilde{u}_x &= 0. \end{aligned} \tag{2.5.12}$$

when expanded about the equilibrium values.

The system in (2.5.12) has plane wave solutions

$$a = a_0 e^{i(\nu t - kx)}, \quad p = p_0 e^{i(\nu t - kx)}, \quad \tilde{u} = \tilde{u}_0 e^{i(\nu t - kx)}, \tag{2.5.13}$$

provided we have the dispersion relation

$$\nu^2 = \frac{k^2}{1 + k^2}. \tag{2.5.14}$$

It follows that the phase velocity is

$$v_\phi = \frac{1}{\sqrt{1 + k^2}}, \tag{2.5.15}$$

which depends on k , making the system dispersive.

As with the derivation of KdV in the shallow water regime we consider weakly nonlinear solutions

$$a = \varepsilon a_1 + \varepsilon^2 a_2, \quad \tilde{u} = \varepsilon u_1 + \varepsilon^2 u_2, \quad p = \varepsilon p_1 + \varepsilon^2 p_2. \tag{2.5.16}$$

We require terms of order ε^2 since solution to order ε correspond to the linear solutions already discussed. In addition we consider the system in the moving frame whose coordinates are given in terms of (x, t) -coordinates by

$$\xi = \varepsilon^{1/2}(x - t), \quad \tau = \varepsilon^{3/2}t.$$

The system (2.5.10) yields to order ε^2 in the moving frame

$$p_{1,\tau} + \frac{3}{2}p_1 p_{1,\xi} + \frac{1}{2}p_{1,xxx} = 0,$$

and with $\tau \rightarrow 2\tau$ we obtain

$$p_{1,\tau} + 3p_1p_{1,\xi} + p_{1,xxx} = 0, \quad (2.5.17)$$

which is the KdV equation for p_1 in the (ξ, τ) -coordinate system.

2.6 Conclusion

In this chapter we have examined our first nonlinear model, namely the two-component Hopf equation, as it arises in the modelling of arterial blood flow. We have seen that it is possible to solve the system in terms of the initial data via the method of characteristics and from there demonstrated the phenomenon of wave breaking. We also demonstrated the necessary conditions under which the solutions remain bounded. The system while nonlinear possessed no dispersion terms and so for this reason we consider it a quasi-linear system, lacking some of the structure necessary to allow for global solutions. In the final section we have shown that if one includes the nonlinearities induced by the elastic restoring forces in the wall then we arrive at the KdV equation as a model, which will be studied in much greater detail in Chapter 4.

Chapter 3

Particle Trajectories

3.1 Introduction

In the current chapter we analyse qualitative properties of the underlying motion for the Stokes wave of greatest height, over a fluid of infinite depth. A Stokes wave is a symmetric wave profile over an irrotational flow, which rises and falls exactly once per period between crest and trough. The wave of greatest height or extreme wave shares many features of a regular Stokes wave, but crucially from a mathematical perspective it displays certain irregularities at the surface. Specifically at the wave crest a cusp develops, whereby the profile is continuous but it is no longer differentiable. Physically this is due to the presence of a so-called stagnation point where the horizontal velocity of the particle equals the wave speed.

The existence of the extreme wave was first conjectured by Stokes [Sto1880], and has been the subject of an extensive body of research over for a century and was rigorously established in [AFT1982] (cf. [Tol2006] for an overview of Stokes waves in general). Subsequently a number of further interesting features of the extreme wave have been established, for example [Con2012, PT2004], and from both a mathematical and physical point of view the extreme wave remains a subject of great interest.

The aim of this chapter is to provide a clear qualitative picture of the particle trajectories throughout the fluid domain for an infinitely deep extreme Stokes wave. It had been assumed for many years that the trajectories followed by fluid particles in regular waves should be closed, either in the form of ellipses for water of finite depth, or approaching circular paths at infinite depth [Joh1997]. However in contrast it has been shown that in a periodic surface gravity wave there are no closed particle trajectories in the linear approximation, see [CEV2008, CV2008]. In recent years there have been several papers which dealt with various aspects of the flow beneath a regular travelling water wave, in irrotational flow or in a flow with vorticity, see discussions in [Con2001, CE2004, CE2007, CE2011, CS2010, CV2011, Ehr2008, Hen2008, Var2007]. The analysis employed in these papers is not transferable to the case of the extreme wave, since the presence of a stagnation point at the crest generates a number of insurmountable mathematical difficulties.

Rather, a different approach must be employed, and in [Con2012] an analysis of the particle trajectories for the finite depth extreme Stokes wave was undertaken. In this chapter we extend the analysis of particle trajectories for extreme Stokes waves to the setting of infinitely deep fluid, thereby completing the work which was initiated in [Hen2006, Hen2008]. Despite the cusp at the wave crest, the velocity field is shown to be continuous throughout the closure of the fluid domain. We then use techniques from conformal mapping theory, together with the approach developed for regular Stokes waves, to prove that the particle trajectories in deep-water extreme Stokes are not closed, but rather undergo a positive drift in the direction of wave propagation.

3.2 Preliminaries

3.2.1 The governing equations

We consider a two-dimensional flow, periodic in the horizontal direction, which moves with a constant speed $c > 0$. The flow is assumed to be irrotational and propagates in an infinitely deep body of water. The fluid body is given by $\Omega = \{(X, Y) \in \mathbb{R} \times (-\infty, \eta(X, t))\}$, where $\eta(X, t)$ is the surface profile of the fluid. In addition, the fluid is considered to be inviscid, incompressible and of constant density $\rho = 1$, with the flow being subject to a gravitational acceleration g . The Euler equations are

$$\begin{aligned} u_t + uu_X + vv_Y &= -P_X, \\ v_t + uv_X + vv_Y &= -P_Y - g \quad \text{for } (X, Y) \in \Omega. \end{aligned} \quad (3.2.1)$$

The incompressibility of the fluid flow is expressed by

$$u_X + v_Y = 0, \quad \text{for } (X, Y) \in \Omega, \quad (3.2.2)$$

while the irrotational character of the flow is described by

$$u_Y - v_X = 0, \quad \text{for } (X, Y) \in \Omega. \quad (3.2.3)$$

In the absence of surface tension, the boundary conditions for the flow are given by

$$\begin{aligned} v &= \eta_t + u\eta_X, \\ P &= P_0 \quad \text{on } Y = \eta(X, t), \end{aligned} \quad (3.2.4)$$

$$(u, v) \rightarrow (0, 0) \quad \text{uniformly in } X \text{ as } Y \rightarrow -\infty. \quad (3.2.5)$$

In the boundary condition (3.2.4) above, P_0 is the constant pressure exerted by the atmosphere on the free surface. Each of the unknown functions u , v , P and η is required to be periodic in the variable X , with a period $L > 0$. Moreover each function assumes the form of a travelling wave profile and so depends on the (X, t) -variables via the

combination $X - ct$. Without loss of generality we assume the solutions have period $L = 2\pi$. The decay of the velocity field deep down as in (3.2.5), is interpreted as the statement that at great depth there is very little fluid motion.

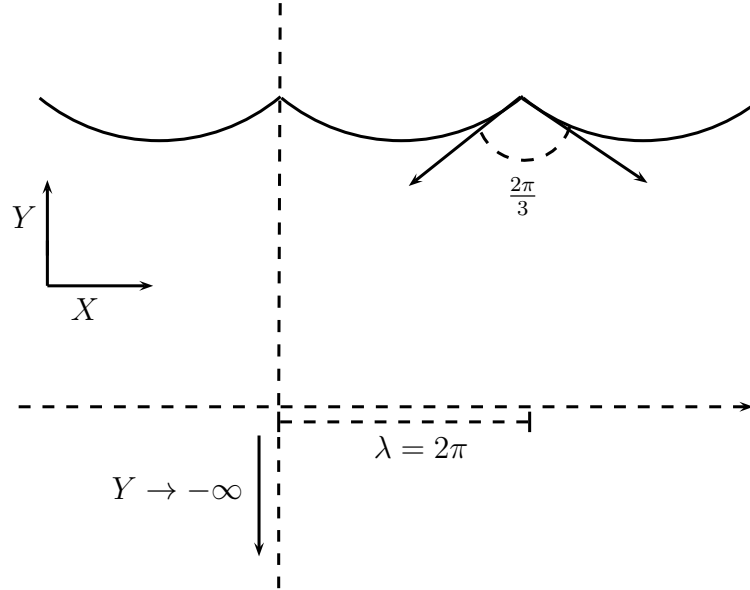


Figure 3.1: An extreme Stokes wave of wavelength 2π over an infinite depth as seen in the laboratory frame.

3.2.2 The moving frame

It will be convenient for us to analyse the system in the moving frame, namely the frame of reference in which the free surface assumes a stationary wave form. The transformation to this frame of reference is induced by the following change of coordinates

$$x = X - ct, \quad y = Y, \quad t = t. \quad (3.2.6)$$

Under this change of coordinates the Euler equations in (3.2.1) transform as

$$\begin{aligned} (u - c)u_x + vu_y &= -P_x, \\ (u - c)v_x + vv_y &= -P_y - g \quad \text{for } (x, y) \in \mathbb{R} \times (-\infty, \eta(x)), \end{aligned} \quad (3.2.7)$$

while the incompressibility and irrotationality conditions (3.2.2)-(3.2.3) become

$$\begin{aligned} u_x + v_y &= 0, \\ v_x - u_y &= 0 \quad \text{for } (x, y) \in \mathbb{R} \times (-\infty, \eta(x)). \end{aligned} \tag{3.2.8}$$

The boundary conditions (3.2.4)-(3.2.5) in the moving frame are

$$\begin{aligned} v &= (u - c) \cdot \eta', \\ P &= P_0 \quad \text{on } y = \eta(x), \end{aligned} \tag{3.2.9}$$

$$(u - c, v) \rightarrow (-c, 0) \quad \text{uniformly in } x \text{ as } y \rightarrow -\infty. \tag{3.2.10}$$

The system of equations (3.2.7)-(3.2.8) along with the boundary conditions (3.2.9)-(3.2.10) serve to define the free boundary problem in the moving frame. In addition to the boundary conditions (3.2.9), there are several symmetries of the functions that apply in the moving frame: η is symmetric with respect to the crest line $x = 0$, while u is even and v is odd in the x -variable. The advantage of transferring the free boundary problem to the moving frame is that the system of equations now has no explicit time dependence.

In the case of a smooth Stokes wave we always have the condition

$$u(x, y) < c, \tag{3.2.11}$$

for all points in the fluid domain and its boundary. In the case of an extreme Stokes wave we have

$$u(x, y) \leq c, \tag{3.2.12}$$

where equality is achieved at the wave crest. Equations (3.2.7) and (3.2.8) ensure that u , v , η , and P are analytic in the interior of the fluid domain, while (3.2.9) and (3.2.12) ensure that these functions are merely continuous on the free surface. In the moving frame, we may say that the wave crest is located at $x = 0$, $y = \eta(0)$, while the tangent

lines to the profile at the cusp create an opening of 120° . For a discussion of these facts, see [BT2003].

The irrotationality condition in (3.2.8) implies

$$\int_{-\pi}^{\pi} \int_{y_0}^{y_1} [u_y - v_x] dy dx = 0,$$

for any fixed depths $y = y_0$ and $y = y_1$ below the trough level. Integrating we find

$$\int_{-\pi}^{\pi} u(x, y_1) dx - \int_{-\pi}^{\pi} u(x, y_0) dx = 0,$$

where we have reversed the order of integration in the second integral and used $v = 0$ along the trough lines $((\pm\pi, -\infty), (\pm\pi, \eta(\pm\pi)))$. The relation

$$\kappa = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x, y_0) dx < c \quad (3.2.13)$$

follows at once, with the situation being similar to that encountered in the case of a flat bed cf. [Con2013]. It is clear that κ may be interpreted as the average horizontal current of the fluid body. In this chapter it will be assumed that $\kappa = 0$.

3.3 The hodograph transform

It follows from (3.2.8) that we may introduce a stream function $\psi(x, y)$ defined by

$$\psi_y = u - c, \quad (3.3.1)$$

$$\psi_x = -v. \quad (3.3.2)$$

Differentiating (3.3.1)-(3.3.2) with respect to y and x respectively and applying (3.2.8), we deduce $\psi(x, y)$ is harmonic throughout the fluid domain. Integrating (3.3.1)-(3.3.2) we obtain

$$\psi(x, y) = \psi(0, y_0) + \int_{y_0}^y [u(x, \zeta) - c] d\zeta - \int_0^x v(\xi, y_0) d\xi \quad y \in (-\infty, \eta(x)], \quad (3.3.3)$$

where y_0 is some fixed depth below the trough level. The 2π -periodicity in x of $u(x, y)$, along with the fact $v(x, y)$ is odd in the x -variable together ensure that $\psi(x, y)$ also has a period of 2π in the x -variable at any fixed depth below the free surface.

The first boundary condition in (3.2.9) gives us

$$\frac{d}{dx}\psi(x, \eta(x)) = -v(x, \eta(x)) + (u(x, \eta(x)) - c) \cdot \eta' = 0, \quad (3.3.4)$$

and so the stream function is constant along the free surface. Since $\psi(x, y)$ is harmonic in the fluid domain the stream function must attain its maximum and minimum on the boundary. Furthermore by (3.2.11) - (3.2.12) we have $\psi_y \leq 0$ along the free surface $y = \eta(x)$, in which case the minimum of $\psi(x, y)$ must occur on the surface, since $\psi_y \rightarrow -c$ as $y \rightarrow -\infty$.

In the fluid domain the Euler equations are alternatively stated using Bernoulli's law

$$\frac{(u - c)^2 + v^2}{2} + gy + P = Q, \quad (3.3.5)$$

where Q is constant. Using Bernoulli's law we may reconstruct Euler's equations in (3.2.7) along with the boundary conditions (3.2.9), in terms of the stream function $\psi(x, y)$ and height function $\eta(x)$, both of which are periodic and even in the x -variable. The free boundary problem (3.2.7) and (3.2.9), when expressed in terms of $\eta(x)$ and $\psi(x, y)$ using (3.3.5) may be written as

$$\begin{aligned} \Delta\psi &= 0 \quad \text{in } -\infty < y < \eta(x), \\ \frac{1}{2}|\nabla\psi|^2 + gy + P_0 &= Q \quad \text{on } y = \eta(x), \\ \psi &= 0 \quad \text{on } y = \eta(x), \\ \nabla\psi &\rightarrow (0, -c) \quad \text{uniformly in } x \text{ as } y \rightarrow -\infty. \end{aligned} \quad (3.3.6)$$

The level sets of the steam function provide a foliation of the closure of the fluid domain $\bar{\Omega}$, with $\psi = 0$ on the surface.

We divide the fluid domain Ω into distinct regions as follows

$$\begin{aligned}\Omega_+ &= \{(x, y) \in \mathbb{R}^2 : x \in (0, \pi), -\infty < y < \eta(x)\}, \\ \Omega_- &= \{(x, y) \in \mathbb{R}^2 : x \in (-\pi, 0), -\infty < y < \eta(x)\},\end{aligned}\tag{3.3.7}$$

which are two halves of the fluid domain in one period. In addition we have the two halves of the surface over one period given by

$$\begin{aligned}S_+ &= \{x \in (0, \pi), y = \eta(x)\}, \\ S_- &= \{x \in (-\pi, 0), y = \eta(x)\}.\end{aligned}\tag{3.3.8}$$

The regions Ω_{\pm} are separated by the crest line $((0, -\infty), (0, \eta(0)))$ and are bounded laterally by the trough lines $((\pm\pi, -\infty), (\pm\pi, \eta(\pm\pi)))$. The height function η is even in x , in which case $\eta(-\pi) = \eta(\pi)$, while $\eta' \leq 0$ on S_+ and $\eta' \geq 0$ on S_- .

The irrotationality condition in (3.2.8) allows us to introduce a velocity potential $\phi(x, y)$ such that

$$\phi_x = u - c, \quad \phi_y = v,\tag{3.3.9}$$

with $\phi = 0$ along the crest line. Integrating (3.3.9) we find

$$\phi(x, y) = \int_0^x [u(\xi, y_0) - c] d\xi + \int_{y_0}^y v(x, \zeta) d\zeta,\tag{3.3.10}$$

where y_0 is a fixed depth below the trough level. It is clear from this integral representation that $\phi(x, y) + cx$ is 2π -periodic in x . In particular we have

$$\phi(2\pi, y) = \int_0^{2\pi} [u(\xi, y) - c] d\xi = -2c\pi,$$

for all y below the trough level.

We can use the stream function and the potential function to perform a conformal hodograph transform induced by the change of variables

$$\begin{aligned}q &= -\phi(x, y), \\ p &= -\psi(x, y).\end{aligned}\tag{3.3.11}$$

The image of the semi infinite strip Ω_{\pm} is

$$\begin{aligned}\hat{\Omega}_+ &= \{(p, q) \in \mathbb{R}^2 : q \in (0, c\pi), p \in (-\infty, 0)\}, \\ \hat{\Omega}_- &= \{(p, q) \in \mathbb{R}^2 : q \in (-c\pi, 0), p \in (-\infty, 0)\},\end{aligned}\tag{3.3.12}$$

while the free surface region S_{\pm} transforms as

$$\begin{aligned}\hat{S}_+ &= \{q \in (0, c\pi), p = 0\}, \\ \hat{S}_- &= \{q \in (-c\pi, 0), p = 0\},\end{aligned}\tag{3.3.13}$$

under the conformal mapping in (3.3.11).

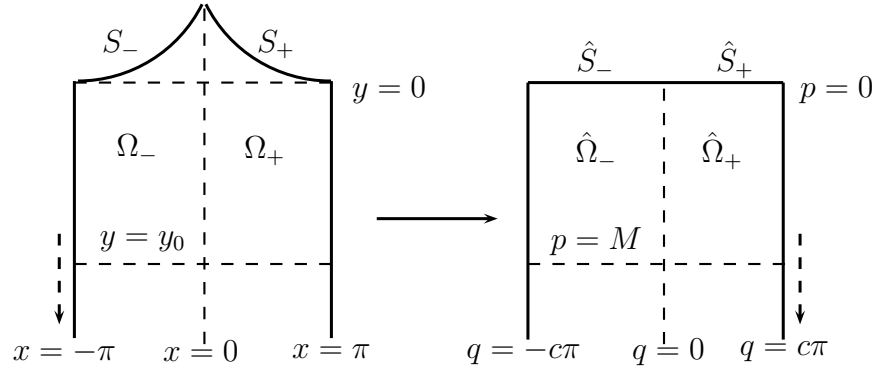


Figure 3.2: The transformation of the fluid domain under the hodograph transform to a domain with fixed boundary.

We introduce a new harmonic function $h(q, p)$, written in terms of the new variables such that

$$h(q, p) = y.\tag{3.3.14}$$

In terms these new variables the free boundary problem may be alternatively written as a new nonlinear fixed boundary problem

$$\begin{aligned}\Delta_{q,p}h &= 0, \quad \text{for } q \in (-c\pi, c\pi), p \in (-\infty, 0], \\ 2(E_0 - gh)(h_p^2 + h_q^2) &= 1, \quad p = 0, \\ \nabla\psi &\rightarrow \left(0, \frac{1}{c}\right), \quad \text{uniformly in } q \in (-c\pi, c\pi) \text{ as } p \rightarrow -\infty.\end{aligned}\tag{3.3.15}$$

The function h is even and $2\pi c$ -periodic in the q -variable.

We introduce the relations

$$\begin{aligned} \partial_q &= h_p \partial_x + h_q \partial_y, & \partial_x &= (c - u) \partial_q + v \partial_p, \\ \partial_p &= -h_q \partial_x + h_p \partial_y, & \partial_y &= -v \partial_q + (c - u) \partial_p, \end{aligned} \quad (3.3.16)$$

from which it follows

$$\begin{aligned} h_q &= -\frac{v}{(c - u)^2 + v^2} = -\frac{\partial x}{\partial p} = \frac{\partial y}{\partial q}, \\ h_p &= \frac{c - u}{(c - u)^2 + v^2} = \frac{\partial x}{\partial q} = \frac{\partial y}{\partial p}. \end{aligned} \quad (3.3.17)$$

While in the above we may use the same symbol to denote h , u etc. in either the moving frame coordinates (x, y) or its transformation under (3.3.11) with coordinates (p, q) , it should be understood the $u(p, q)$ does not have the same functional form as $u(x, y)$ and like wise for v , h etc. Rather it should be understood that $u(p, q)$ depends on the (x, y) coordinates via $u(\phi(x, y), \psi(x, y))$. Similar considerations apply to functions such as v and h etc.

3.4 The velocity field of an extreme Stokes wave

We introduce the function $f : \hat{\Omega}_+ \rightarrow \Omega_+$ defined by $f : \xi \mapsto x + iy$, with $\xi = q + ip$, where f is analytic in $\hat{\Omega}_+$ and continuous on the closure $\hat{\Omega}_+ \cup \partial\hat{\Omega}_+$. Furthermore the function has analytic continuation to any point on $\partial\hat{\Omega}_+$ except the point $(0,0)$. However due to the cusp in the wave crest the behaviour of f' is singular at the point $\xi = 0$.

Nevertheless the fact that the fluid domain Ω has a corner with two Hölder continuous curves issuing from $f(0)$, the tangents of which form an angle of $\frac{2\pi}{3}$ at the cusp, ensures that the function $\xi \mapsto \xi^{1/3} f'(\xi)$ is continuous on the closure of $\hat{\Omega}_+$. Specifically we may say that

$$\lim_{\xi \rightarrow 0} \xi^{1/3} f'(\xi) = \xi_0 \neq 0, \quad (3.4.1)$$

where ξ_0 is constant, cf. [Pom1992]. Using the results of (3.3.17) along with the definition of $f(\xi)$ we see that

$$\frac{1}{f'(\xi)} = \frac{1}{x_q + iy_q} = (c - u) + iv. \quad (3.4.2)$$

Meanwhile (3.4.1) ensures $\frac{1}{f'(\xi)}$ is continuous in the closure of $\hat{\Omega}_+$, which together with (3.4.2) imply $[(c - u) + iv]$ is also continuous in the closure of $\hat{\Omega}_+$.

In fact we may specifically write

$$\frac{(c - u) + iv}{\xi^{1/3}} \rightarrow \frac{1}{\xi_0} \quad \text{as } \xi \rightarrow 0, \quad (3.4.3)$$

and so it follows that

$$\lim_{\xi \rightarrow 0} \frac{[(u - c) + iv]\xi^{2/3} - 0}{\xi} = \lim_{\xi \rightarrow 0} [(u - c) + iv]\xi^{-1/3} = \frac{1}{\xi_0}. \quad (3.4.4)$$

The function $[(u - c) + iv]\xi^{2/3}$ is therefore differentiable at $\xi = 0$.

Evaluating the derivative of $\frac{\xi^{2/3}}{f'(\xi)}$ along $\xi = ip$ with $p \uparrow 0$, i.e. where $v = 0$, we see that $\text{Re}[\xi_0] \neq 0$ and $\text{Im}[\xi_0] \neq 0$. It follows from this and evaluating the same derivative along $\xi = q$, that the limit of $[c - u(q, 0)]q^{-1/3}$ as $q \downarrow 0$ is also non-zero. As such the map $q \mapsto u(q, 0)$ is continuous and periodic in \mathbb{R} but does not belong to the Sobolev space $W^{1,k}(0, 1)$ for $k \geq 3/2$. However if this were so then $u(0, 0) - c = 0$ would yield

$$\lim_{q \rightarrow 0} ([c - u(q, 0)]q^{-1/3}) = 0,$$

which we see from,

$$0 \leq c - u(q, 0) = \int_0^q |u_q(s, 0)| ds \leq q^{(k-1)/k} \left(\int_0^q |u_q(s, 0)|^k ds \right)^{1/k},$$

$$\text{as } \lim_{q \downarrow 0} \int_0^q |u_q(s, 0)|^k ds = 0.$$

3.4.1 The vertical velocity

In the case of almost extreme water waves we have that $v > 0$ in $\hat{\Omega}_+$ cf. [CS2010], while for the extreme Stokes wave we have $v \geq 0$ in $\hat{\Omega}_+$ with equality achieved on

the boundary see [Con2012]. In both cases $\hat{\Omega}_+$ is a bounded domain since the fluid domain is of finite depth. In our present case we would like to extend these results to the case when $\hat{\Omega}_+$ is a semi infinite domain, corresponding to the case of an infinitely deep Stokes wave.

To begin we assume there exists a point $(q_0, p_0) \in \hat{\Omega}_+$ such that $v(q_0, p_0) = 0$ for the extreme Stokes wave. We may choose $\varepsilon \in (0, \sqrt{q_0^2 + p_0^2})$ and consider the modified region $\hat{\Omega}_+^\varepsilon$, which is the region $\hat{\Omega}_+$ with a quarter disc of radius ε and centre $(0, 0)$ removed. Clearly (q_0, p_0) is an interior point of $\hat{\Omega}_+^\varepsilon$ in which case the harmonic function v attains its minimum at an interior point $\hat{\Omega}_+^\varepsilon$. Maximum principles then require $v \equiv 0$ throughout this region cf. [GT2001].

We also see from (3.2.7) that

$$\nabla^2 P = -2(u_x^2 + u_y^2) \leq 0 \quad \text{in } \Omega, \quad (3.4.5)$$

in which case P is superharmonic. The weak maximum principle and the periodicity of P then ensure that the minimum must be attained on the free surface $y = \eta(x)$ cf. [GT2001]. Furthermore since $P = P_0$ all along the free surface it follows that P attains its minimum all along the free surface.

Supposing there exists a point $(q_0, 0)$ with $q_0 \in (0, \pi)$ where the harmonic function $v = 0$, then this corresponds to a point $(x_0, \eta(x_0)) \in \Omega_+$ where $v = 0$. The mapping $x \mapsto \eta(x)$ is strictly decreasing for all $x \in (0, \pi)$ in which case $\eta'(x_0) < 0$. Since v is harmonic in the region Ω_+^ε , where Ω_+^ε is the pre-image of $\hat{\Omega}_+^\varepsilon$ under the conformal transformation (3.3.11), it follows that $(x_0, \eta(x_0))$ must be a minimum point of v in Ω_+^ε . Hopf's maximum principle then requires that $v_y(x_0, \eta(x_0)) < 0$ cf. [Fra2000].

However the first member of (3.2.8) now requires $u_x > 0$ at $(x_0, \eta(x_0))$ which together with (3.2.12) gives $(u - c)u_x \leq 0$ at $(x_0, \eta(x_0))$. It follows that

$$P_x = (c - u)u_x \geq 0 \quad \text{at } (x_0, \eta(x_0)). \quad (3.4.6)$$

Since $P = P_0$ on the free surface and combined with the maximum principle requires

$$\begin{aligned} P_x + P_y \eta' &= 0, \\ P_y &< 0 \quad \text{on } y = \eta(x), \end{aligned} \tag{3.4.7}$$

and so it follows from $\eta' < 0$ on $x \in (0, \pi)$ that $P_x < 0$ at $(x_0, \eta(x_0))$, thus (3.4.6) is a contradiction. It follows that the velocity component v is strictly positive in Ω_+ except at the wave crest and wave trough.

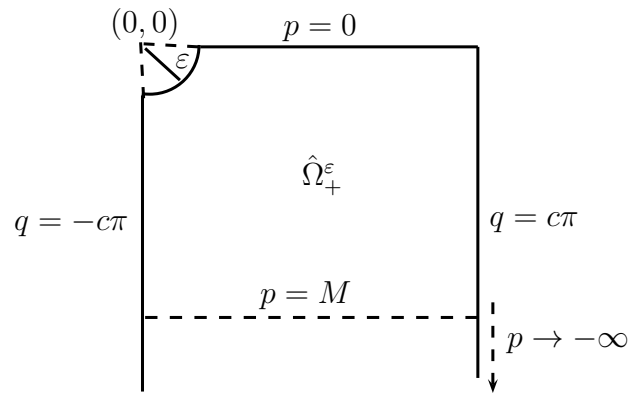


Figure 3.3: The region $\hat{\Omega}_+$ with the quarter disc of radius ε and centre $(0, 0)$ removed.

3.4.2 The horizontal velocity

Along the lateral sides of the excised region Ω_+^ε we have $v = 0$, therefore v is also zero along the lateral edges of its image $\hat{\Omega}_+^\varepsilon$. Furthermore we now know $v > 0$ in the interior of $\hat{\Omega}_+$ which together with Hopf's maximum principle requires

$$v_q(0, p) > 0 \quad v_q(c\pi, p) < 0,$$

for $p \in (-\infty, 0]$. The incompressibility and irrotationality of the flow in (3.2.8) along with the relations in (3.3.16) give

$$u_q + v_p = 0, \quad u_p - v_q = 0. \tag{3.4.8}$$

We see that $u_p(0, p) = v_q(0, p) > 0$ and $u_p(c\pi, p) = v_q(c\pi, p) < 0$, and so u decreases as we descend along the crest line while u increases as we descend the trough line of $\hat{\Omega}_+^\varepsilon$ cf. [Con2012]. Moreover the relations (3.3.16)-(3.3.17) give us

$$u_q = \frac{(c-u)u_x - vu_y}{(c-u)^2 + v^2}, \quad (3.4.9)$$

which is well defined for all points of the fluid domain and its boundary except at the wave crest. Therefore along the lateral edges of $\hat{\Omega}_+^\varepsilon$ where $v = 0$, it follows that

$$u_q = \frac{u_x}{c-u} = -\frac{v_y}{c-u} = 0, \quad (3.4.10)$$

and so u_q is also zero along the lateral edges of $\hat{\Omega}_+^\varepsilon$. In addition since the transformation (3.3.11) is a conformal mapping, while u is harmonic in the fluid domain Ω_+^ε , it follows that u is also harmonic in the domain $\hat{\Omega}_+^\varepsilon$.

The first of the Euler equations gives us

$$P_x = (c-u)u_x - vu_y,$$

which together with the relations (3.3.16)-(3.3.17) give

$$u_q = \frac{P_x}{h_q^2 + h_p^2}, \quad (3.4.11)$$

which holds at all points in the fluid domain and along the free boundary except at the wave crest. Along the free surface it was previously found that

$$P_x < 0 \quad \text{on } y = \eta(x),$$

except at the wave crest and wave trough where $P_x = 0$. It follows that

$$u_q(q, 0) < 0 \quad \text{for } q \in (0, c\pi), \quad (3.4.12)$$

in which case u is strictly decreasing along the free surface, except at the wave crest and wave trough cf. [Con2012].

Since u is harmonic in $\hat{\Omega}_+^\varepsilon$, then u_q is also harmonic in the same region and so attains its maximum and minimum values along the boundary $\partial\hat{\Omega}_+^\varepsilon$. If we suppose that $u_q(q_0, p_0) = m > 0$, where (q_0, p_0) is an interior point of $\hat{\Omega}_+^\varepsilon$, then we have a contradiction of the maximum principle. To see this we note that since $(u, v) \rightarrow (0, 0)$ as $p \rightarrow -\infty$, we may choose $p_1 < p_0$ such that $u_q(q, p_1) < m$. On the bounded region consisting of $\hat{\Omega}_+^\varepsilon$ truncated by the line $p = p_1$ u_q is larger at an interior point than on the boundary, which is a contradiction and so $u_q \leq 0$ in the interior of $\hat{\Omega}_+^\varepsilon$. The strong maximum principle also requires that if $u_q = 0$ at some interior point, then $u_q \equiv 0$ throughout $\hat{\Omega}_+^\varepsilon$ contradicting (3.4.12). Thus we conclude $u_q < 0$ in the interior of $\hat{\Omega}_+^\varepsilon$ cf. [Hen2008, Hen2011].

Along the streamline defined via $\psi(x, y(x)) = \psi_0 < 0$ by definition we have $\psi_x(x, y(x)) + \psi_y(x, y(x))y'(x) = 0$ and so using (3.3.17) we find $y'(x) = -\frac{v}{c-u} = \frac{h_q}{h_p}$. Differentiating u along the streamline and using (3.3.16) we find

$$\frac{d}{dx}u(x, y(x)) = \frac{u_q}{h_p} < 0, \quad (3.4.13)$$

since $h_p > 0$ and $u_q < 0$ in the interior of Ω_+^ε . As such u is strictly decreasing along any streamline in the interior of Ω_+^ε . By a limiting process we therefore deduce that $u_q \leq 0$ in the interior of $\hat{\Omega}_+$ for the wave of greatest height cf. [Con2012]. Maximum principles now ensure a strict inequality in the interior since u_q is harmonic. Consequently, along any streamline u is strictly decreasing between the crest line and a successive trough line.

3.5 Particle trajectories

Along a stream line $(x(t), y(t))$ we have $\psi(x(t), y(t)) = \psi_0$, in which case we also have $\partial_t\psi(x(t), y(t)) = 0$. Along the image of the stream line p is constant and so we may write $p(t) = p(0) = p_0$. Furthermore since we always have the strict inequality

$u - c < 0$ in the fluid domain except at the wave crest, it follows that $x(t)$ goes from $+\infty$ to $-\infty$ as t goes from $-\infty$ to $+\infty$. It follows that there exist a time t_0 such that $x(t_0) = 0$ if we are beneath the free surface. Moreover there also exists a time say $t = 0$ when $x(0) = \pi$, and a later time $t = \theta$ such that $x(\theta) = -\pi$. Since $x(0) = \pi$ and $x(\theta) = -\pi$ are the endpoints of one period, it follows that θ is the time required for a particle to traverse one period in the moving frame. That is to say θ is the elapsed time per period along a stream line beneath the surface.

It follows that in the region $\hat{\Omega}_+$ with $p < 0$ we have

$$\frac{dq}{dt} = \dot{q} = -\phi_x \dot{x} - \phi_y \dot{y} = -(c - u)^2 - v^2 < 0, \quad (3.5.1)$$

while at the endpoints we have,

$$q(0) = c\pi > 0, \quad q(\theta) = -c\pi < 0. \quad (3.5.2)$$

From (3.5.1) we see that $dt = -\frac{1}{(u-c)^2+v^2}dq$. Along the particle path the vertical displacement over time θ is given by

$$y(\theta) - y(0) = \int_0^\theta v(x(t), y(t))dt = \int_{-c\pi}^{c\pi} \frac{v}{(u-c)^2+v^2}dq = 0, \quad (3.5.3)$$

since v is odd in q while the denominator in the integrand is even in q . In addition the period along a streamline p may be written as

$$\theta(p) = \int_0^\theta \frac{\dot{x}}{u(x(t), y(t)) - c} dt = \int_{-\pi}^{\pi} \frac{1}{c - u(x, y(x))} dx, \quad (3.5.4)$$

which we now use to demonstrate $\theta(p) > \frac{2\pi}{c}$ when $p \in (-\infty, 0)$.

To begin we note that the Cauchy-Schwarz inequality gives

$$\left[\int_{-\pi}^{\pi} \frac{1}{c - u(x, y(x))} dx \right] \left[\int_{-\pi}^{\pi} (c - u(x, y(x))) dx \right] \geq \left[\int_{-\pi}^{\pi} dx \right]^2 = 4\pi^2. \quad (3.5.5)$$

We now consider the region $D \subset \Omega_+$, which is bounded above by the stream line $\{(x, y(x)) : x \in (-\pi, \pi), \psi(x, y(x)) = \psi_0 < 0\}$ and below by the line segment

$\{(x, y_0) : x \in (-\pi, \pi)\}$, while it is laterally bounded by the line segments $\{(-\pi, y) : y \in (y_0, y(-\pi))\}$ and $\{(\pi, y) : y \in (y_0, y(\pi))\}$. The divergence theorem applied to the vector field $(v, u - c)$ in the region D gives

$$2\pi c = \int_{-\pi}^{\pi} [c - u(x, y(x))][1 + y'(x)^2] dx > \int_{-\pi}^{\pi} [c - u(y(x))] dx. \quad (3.5.6)$$

Taken together the relations (3.5.4), (3.5.5) and (3.5.6) imply

$$\theta(p) \geq \frac{4\pi^2}{\int_{-\pi}^{\pi} [c - u(x, y(x))] dx} > \frac{2\pi}{c}, \quad (3.5.7)$$

the inequality being strict since $[c - u(x, y(x))]$ only achieves a constant value as $y \rightarrow -\infty$.

The horizontal drift of a particle is defined as the net horizontal distance moved by the particle between two consecutive trough lines. That is to say,

$$X(\theta) - X(0) = c\theta - 2\pi = X(t + \theta) - X(t), \quad t \in \mathbb{R}, \quad (3.5.8)$$

which corresponds to the motion of a particle over one period in the stationary frame. In the (X, Y) -frame the particle trajectory is governed by the system

$$\begin{aligned} \dot{X}(t) &= u(X(t), Y(t)), \\ \dot{Y}(t) &= v(X(t), Y(t)), \end{aligned} \quad (3.5.9)$$

in which case a solution of (3.5.9) of period $\theta = \frac{2\pi}{c}$ corresponds to a closed trajectory in the physical frame (X, Y) .

3.5.1 Particle trajectories beneath the free surface

It is clear from (3.5.7) and (3.5.9) that the particle drift in the stationary frame is in the positive X -direction. The level set $\{u = 0\}$, consists of a continuous curve \mathcal{C}_+ in Ω_+ that intersects each streamline $\psi = p$ exactly once where $p \in (-\infty, 0]$. The

corresponding level set \mathcal{C}_- in Ω_- where $\{u = 0\}$ is the reflection of \mathcal{C}_+ about the line $x = 0$. As there is a unique point along each streamline in Ω_+ at which $u = 0$, it follows that $u < 0$ between \mathcal{C}_+ and $x = \pi$. Similarly between $x = -\pi$ and \mathcal{C}_- we have $u < 0$, while between the two level sets \mathcal{C}_- and \mathcal{C}_+ we have $u > 0$.

We consider a particle initially located at $a = (\pi, y_0)$ when $t = 0$ which moves to the left and intersects the level set \mathcal{C}_+ at point b , and later intersects the crest line at point c before moving on to intersect \mathcal{C}_- at point d , and finally intersecting the trough line at $e = (-\pi, y_0)$ at time $t = \theta(y_0) = \theta$.

Between the trough lines and the level sets we have $u < 0$, in which case the motion of the particle between the points a and b and the points d and e must be in the negative X -direction when viewed from the (X, Y) -frame. While moving from b to d in the moving frame we have $u > 0$. Thus between b and d the particle trajectory in the stationary frame is in the positive X -direction.

In Section 3.4.1 it was demonstrated that $v > 0$ in Ω_+ , and when combined with the antisymmetry of v about the crest line it follows that $v > 0$ between a and c , while $v < 0$ between c and e . Therefore we deduce that the particle experiences no net vertical drift over the course of one period. The initial particle position is $(X(0), Y(0)) = (\pi, Y_0)$ while the final location is $(X(\theta), Y(\theta)) = (-\pi + c\theta, Y_0)$, which together with (3.5.7) implies $X(\theta) > \pi$. Consequently $X(\theta) > X(0)$, thereby indicating a drift to the right experienced by the particle after one complete period. Thus beneath the free surface the particle trajectory along any streamline is not a closed loop, but rather drifts to the right over the course of one period when observed in the (X, Y) -frame.

3.5.2 Particle trajectories on the free surface

In order to demonstrate the existence of non-trivial solutions on the free surface we want to show that if a particle is initially located at $(x_0, \eta(x_0))$ when $t = 0$ and $x_0 \in (0, \pi)$

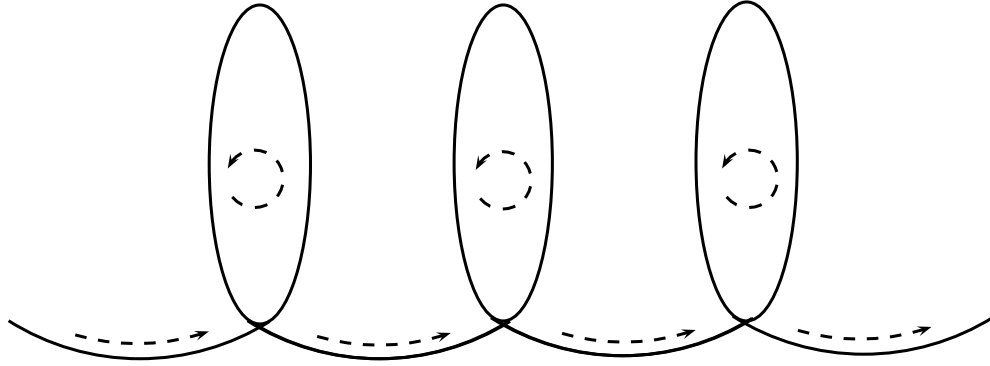


Figure 3.4: The trajectory followed by a fluid particle in an extreme Stokes wave in the fluid body

then it will always reach the wave crest $(0, \eta(0))$ in finite time. If there exists a time τ such that for all $t \in (0, \tau)$ we have $x(t) \neq 0$, then we also have

$$\dot{x}(t) = u(x(t), \eta(x(t))) - c < 0, \quad t \in (0, \tau), \quad (3.5.10)$$

as $u \leq c$ except at the wave crest $(0, \eta(0))$. Integrating (3.5.10) we obtain,

$$\tau = \int_{x(\tau)}^{x_0} \frac{1}{c - u(\xi, \eta(\xi))} d\xi. \quad (3.5.11)$$

At the wave crest $(0, \eta(0))$ we have a symmetric cusp whose tangents form an angle of $\frac{2\pi}{3}$ and so the magnitude of opening between the horizontal and each tangent is $\frac{\pi}{6}$, in which case

$$\lim_{x \rightarrow 0} [\eta'(x)]^2 = \lim_{x \rightarrow 0} \frac{v^2(x, \eta(x))}{[u(x, \eta(x)) - c]^2} = \tan^2\left(\frac{\pi}{6}\right) = \frac{1}{3}. \quad (3.5.12)$$

In addition the free boundary problem (3.3.6) implemented at the wave crest requires

$$g\eta(0) + P_0 = Q. \quad (3.5.13)$$

Applying Bernoulli's law in (3.3.5) then gives us

$$[u(x, \eta(x)) - c]^2 + v^2(x, \eta(x)) = 2[Q - P_0 - g\eta(x)] = 2g[\eta(0) - \eta(x)], \quad (3.5.14)$$

which implies

$$\lim_{x \rightarrow 0} \frac{[u(x, \eta(x)) - c]^2 + v^2(x, \eta(x))}{|x|} = 2g \lim_{x \rightarrow 0} \frac{\eta(0) - \eta(x)}{|x|} = \frac{2g}{\sqrt{3}}. \quad (3.5.15)$$

The velocity components of a fluid particle approaching the cusp are related through (3.5.12) which together with (3.5.15) gives

$$\lim_{x \rightarrow 0} \frac{[c - u(x, \eta(x))]^2}{|x|} = \frac{g\sqrt{3}}{2}. \quad (3.5.16)$$

From relation (3.5.16) we conclude

$$\int_0^\pi \frac{1}{c - u(s, \eta(s))} ds < \infty, \quad (3.5.17)$$

thus the time it takes the particle to travel from the initial location $(x_0, \eta(x_0))$ to the wave crest $(0, \eta(0))$ is finite. Physical considerations require that the particle can only occupy the crest point $(0, \eta(0))$ for an instant before it is replaced by a new particle, since particles resting there for a finite time would accumulate, something which is not observed. It is clear then that the crest is an apparent stagnation point.

With these issues resolved arguments analogous to those presented in Section 3.5.1 allow us to define the elapsed time $\theta(0)$ along with the horizontal drift $c\theta(0) - 2\pi$ over one period for a particle travelling on the free surface. Evaluating the relations (3.5.4) and (3.5.8) in the limit $p \uparrow 0$, we find

$$\theta(0) \geq \frac{2\pi}{c}, \quad \int_{-\pi}^\pi [c - u(\xi, \eta(\xi))] d\xi \leq 2\pi. \quad (3.5.18)$$

On the other hand the Cauchy-Schwarz inequality (3.5.5) imposes

$$\left[\int_{-\pi}^\pi \frac{1}{c - u(x, \eta(x))} dx \right] \left[\int_{-\pi}^\pi (c - u(x, \eta(x))) dx \right] \geq 4\pi^2, \quad (3.5.19)$$

with equality possible only if $c - u(x, \eta(x))$ is constant over $x \in [-\pi, \pi]$. In contrast to particle trajectories beneath the free surface there are no horizontal tangents at the wave crest, but rather a pair of tangents which create an opening of 120° . The same observation remains true upon passing from the moving frame to the (X, Y) -frame cf. [Con2012].

3.6 Conclusion

The main result of this chapter has been the demonstration of non-closed particle trajectories in an extreme Stokes wave over infinite depth. This result followed from the application of maximum principles to the components of the velocity field within the fluid body which ensured that the particle velocity was strictly increasing along any streamline in the fluid domain. In addition the vertical velocity was found to be anti-symmetric in the x variable and strictly positive within the fluid body except along the crest and trough lines. These results allowed us to obtain strict inequalities for the period of a particle trajectory during its motion through one entire wavelength, thereby leading to the non-closure of particle paths, when observed in the (X, Y) -frame. These results followed an extension of results for the case of almost extreme Stokes wave, of which the extreme Stokes wave may be considered a limiting example. It remains an open question however as to whether every extreme wave can be obtained from such a limiting procedure.

Part II

Spectral Analysis

Chapter 4

Spectral Theory of the KdV Equation

The inverse scattering transform (IST) is a method of solving nonlinear Cauchy problems. In the case of rapidly decreasing functions as one approaches the asymptotic region of the spatial coordinate, the inverse scattering method is the analogue of Fourier's method for solving linear partial differential equations. In the case of linear partial differential equations, the Fourier transform of the system converts the system into a system of linear ordinary differential equations. Furthermore, when the coefficients of the original linear PDE are constant, the Fourier transform consists of a linearly independent set of ODE for the Fourier harmonics which are readily integrated.

In the inverse scattering method, there is an analogous process at work, whereby a linear differential operator whose coefficients depend a priori on space and time are transformed into a set of scattering data. In the case of current interest, the KdV equation is obtained via a consistency condition applied to a pair of spectral operators, one of which is simply the Schrödinger operator familiar from quantum mechanics,

$$-\frac{d^2\psi}{dx^2} + u\psi = k^2\psi,$$

The potential $u(x, t) \equiv u$ is transformed into a reflection coefficient $r(k, t) \equiv r$ via the IST. The KdV equation itself then allows us to obtain the time evolution of the reflection

coefficient, which will later be show to be

$$r(k, t) = r(k, 0)e^{-8ik^3t}.$$

Solving the KdV then becomes a case of reconstructing the potential $u(x)$ from the time dependant scattering coefficient $r(k, t)$, which is the so called inverse problem.

The potential u depends on the variables x and t while the function r depends on the spectral parameter k ant time t . In addition the spectral functions $\psi \equiv \psi(x, t; k)$ depend on all three variables. In general the variables on which each function depends will be suppressed until needed. In what follows from Sections (4.1)-(4.5) all considerations will only be concerned with the x dependence of the potential u , the k dependence of the scattering coefficient r and the (x, k) dependence of the spectral functions ψ . In Section (4.6) the explicit time dependence of the scattering coefficient r will be investigated thereby introducing the time dependence of the spectral functions. Upon solving the Riemann-Hilbert problem we will then obtain the explicit time dependence of the potential u . Thus until Section (4.6) there shall be no consideration of time dependence but it should be kept in mind that time dependence is implicit for all functions considered.

The material presented in this chapter is well known and is included to provide a comprehensive background. There are many textbooks in which the IST for the KdV is treated, it being the archetypal example of a nonlinear system solvable by this method. The treatment presented in this chapter closely follows that found in [ZMNP1984].

4.1 The spectral problem

The solutions u of the KdV which we seek are smooth in $x \in \mathbb{R}$ and such that u and its derivatives to all orders vanish more rapidly than any finite power of x . We consider

the eigenvalue problem

$$-\frac{d^2\psi}{dx^2} + u\psi = k^2\psi, \quad (4.1.1)$$

where we have made the replacement $\lambda \rightarrow k^2$ from the usual version of the Schrödinger equation. The special problem consists of two distinct regions, the continuous spectrum which consists of all $k \in \mathbb{R}$ and the discrete spectrum. The discrete spectrum in the case of KdV at least consists of a finite number of imaginary eigenvalues $k = i\kappa_n$, with $n \in \{1, \dots, N\}$ and where $\kappa_n > 0$.

The basis of Jost solutions are defined asymptotically according to

$$\left. \begin{array}{l} \psi_1(x; k) \rightarrow e^{-ikx} \\ \psi_2(x; k) \rightarrow e^{ikx} \end{array} \right\} \text{ as } x \rightarrow \infty, \quad \left. \begin{array}{l} \phi_1(x; k) \rightarrow e^{-ikx} \\ \phi_2(x; k) \rightarrow e^{ikx} \end{array} \right\} \text{ as } x \rightarrow -\infty, \quad (4.1.2)$$

for all $k \in \mathbb{R}$. We see from (4.1.1) that in either basis of solutions in (4.1.2) one member must be the complex conjugate of the other. It follows that

$$\psi_1(x; k) = \bar{\psi}_2(x; k), \quad \phi_1(x; k) = \bar{\phi}_2(x; k). \quad (4.1.3)$$

Given $k \in \mathbb{R}$ it is also apparent that

$$\psi_1(x; k) = \psi_2(x; -k), \quad \phi_1(x; k) = \phi_2(x; -k). \quad (4.1.4)$$

Moreover since $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ both form independent bases for any particular $k \in \mathbb{R}$, we may write

$$\phi_a(x; k) = \sum_{b=1}^2 T_{ab}(k)\psi_b(x; k), \quad a \in \{1, 2\}, \quad (4.1.5)$$

where $T_{ab}(k)$ as defined is the scattering matrix. Given the relationship (4.1.3) it is clear that $T_{ab}(k)$ must be of the form

$$T_{ab}(k) = \begin{pmatrix} a(k) & b(k) \\ \bar{b}(k) & \bar{a}(k) \end{pmatrix}, \quad (4.1.6)$$

and with this in mind it will be more convenient henceforth to drop the labels 1, 2 on $\{\phi_1, \phi_2\}$ and replace them with $\{\phi, \bar{\phi}\}$, and similarly with the ψ -basis. The scattering relation between bases may now be written as

$$\phi(x; k) = a(k)\psi(x; k) + b(k)\bar{\psi}(x; k), \quad (4.1.7)$$

with the scattering relation for $\bar{\phi}$ obtained by complex conjugation.

Given two solutions to (4.1.1) which we denote by $g_1(x; k)$ and $g_2(x; k)$ for any $k \in \mathbb{C}$, we define the Wronskian

$$W[g_1, g_2] = g_1 \frac{dg_2}{dx} - g_2 \frac{dg_1}{dx}. \quad (4.1.8)$$

Differentiating once with respect to x we find

$$\frac{d}{dx} W[g_1, g_2] = g_1 \frac{d^2 g_2}{dx^2} - g_2 \frac{d^2 g_1}{dx^2}, \quad (4.1.9)$$

whereupon using (4.1.1) we see that the right hand side becomes identically zero. Clearly the Wronskian $W[\cdot, \cdot]$ of two solutions of the spectral problem is independent of position. In particular we find using the asymptotic behaviour of the bases $\{\psi, \bar{\psi}\}$ and $\{\phi, \bar{\phi}\}$ that

$$\lim_{x \rightarrow \infty} W[\psi, \bar{\psi}] = 2ik, \quad \lim_{x \rightarrow -\infty} W[\phi, \bar{\phi}] = 2ik, \quad (4.1.10)$$

and so we see from our previous results that,

$$W[\phi, \bar{\phi}] = 2ik, \quad W[\psi, \bar{\psi}] = 2ik, \quad (4.1.11)$$

since the Wronskian is independent of location x .

Using (4.1.7) in (4.1.11) we obtain

$$2ik = W[\phi, \bar{\phi}] = (|a(k)|^2 - |b(k)|^2)2ik, \quad (4.1.12)$$

from which it is immediately clear

$$\det[T_{ab}(k)] = |a(k)|^2 - |b(k)|^2 = 1. \quad (4.1.13)$$

making T unimodal.

Considering the scattering equation (4.1.7) where we let both sides approach the asymptotic region $x \rightarrow \infty$, in which case we find

$$\lim_{x \rightarrow -\infty} \frac{1}{a(k)} \phi(x; k) = e^{-ikx} + \frac{b(k)}{a(x)} e^{ikx} + \mathcal{O}(1). \quad (4.1.14)$$

Recall as $x \rightarrow -\infty$ we have $\phi(x; k) \rightarrow e^{-ikx}$, in which case we have

$$\lim_{x \rightarrow -\infty} \frac{1}{a(k)} \phi(x; k) = \frac{1}{a(k)} e^{-ikx} + \mathcal{O}(1). \quad (4.1.15)$$

The coefficients $t(k) = \frac{1}{a(k)}$ and $r(k) = \frac{b(k)}{a(k)}$ are naturally interpreted as the transmission and reflection coefficients respectively for a right-moving plane wave e^{-ikx} , originating in the asymptotic region $x \rightarrow -\infty$ and scattering off the potential $u(x)$.

Dividing both sides of (4.1.13) by $|a(k)|^2$ we find

$$|t(x)|^2 + |r(k)|^2 = 1. \quad (4.1.16)$$

The properties of the continuous spectrum of the original Schrödinger problem are contained entirely within the scattering matrix $T(k)$, which in turn is essentially described in full by the reflection coefficient $r(k)$. The relations (4.1.3) and (4.1.4) also ensure that we only need worry about the half axis $k > 0$. Recall $\bar{\phi}(x; k) = \phi(x; -k)$ when $k \in \mathbb{R}$, while the analogue is true for $\psi(x; k)$ so we must have,

$$0 \equiv \phi(x; k) - \bar{\phi}(x; -k) = [a(k) - \bar{a}(-k)]\bar{\psi}(x; k) + [b(k) - \bar{b}(-k)]\bar{\psi}(x; k).$$

Given the linear independence of $\psi(x; k)$ and $\bar{\psi}(x; k)$ it follows that

$$\bar{a}(-k) = a(k) \quad \& \quad \bar{b}(-k) = b(k) \quad \Rightarrow \quad \bar{r}(-k) = r(k), \quad (4.1.17)$$

for $k \in \mathbb{R}$

4.2 Analytic properties of the Jost solutions

We introduce the modified Jost solutions defined by

$$\xi^+(x; k) = e^{ikx}\phi(x; k), \quad \xi^-(x; k) = e^{ikx}\psi(x; k), \quad (4.2.1)$$

whose asymptotic behavior clearly satisfies

$$\xi^\pm(x; k) \rightarrow 1 \quad \text{as} \quad x \rightarrow \mp\infty$$

respectively. Differentiating once with respect to x we find

$$e^{ikx}\phi_x(x; k) = \xi_x^+(x; k) - ik\xi^+(x; k). \quad (4.2.2)$$

Differentiating once more and using (4.1.1) we find

$$\xi_{xx}^+(x; k) - 2ik\xi_x^+(x; k) - u(x)\xi^+(x; k) = 0. \quad (4.2.3)$$

In addition the asymptotic behaviour of $\xi^+(x; k)$ along with the spectral problem (4.1.1) allows us to write

$$\xi^+(x; k) = 1 + \int_{-\infty}^x \frac{e^{2ik(x-y)} - 1}{2ik} P(y; k) \xi^+(y; k) dy, \quad (4.2.4)$$

so differentiating once with respect to x and applying the fundamental of calculus we find

$$\xi_x^+(x; k) = \int_{-\infty}^x e^{2ik(x-y)} P_+(y; k) \xi^+(y; k) dy. \quad (4.2.5)$$

Differentiating once more we find

$$\xi_{xx}^+(x; k) = P_+(y; k) \xi^+(x; k) + 2ik\xi_x^+(x; k). \quad (4.2.6)$$

In comparison to (4.2.3) we see immediately that

$$P_+(x; k) = u(x). \quad (4.2.7)$$

Thus we may write

$$\xi^+(x; k) = 1 + \int_{-\infty}^x \frac{e^{2ik(x-y)} - 1}{2ik} u(y) \xi(y; k) dy, \quad (4.2.8)$$

in which case it is obvious that the integral on the right hand side above is bounded if $k \geq 0$. Consequently $\xi^+(x; k)$ is analytic if $k \in \mathbb{C}_+$.

Similarly one may write

$$\xi^-(x; k) = 1 + \int_x^{\infty} \frac{e^{2ik(x-y)} - 1}{2ik} P_-(y; k) \xi^-(y; k) dy, \quad (4.2.9)$$

and find

$$P_-(x; k) = -u(x). \quad (4.2.10)$$

The same argument now implies that $\xi_-(x; k)$ is analytic for all $k \in \mathbb{C}_-$.

Returning to the definitions (4.2.1) we see that

$$\phi(x; k) = e^{-ikx} \xi^+(x; k), \quad (4.2.11)$$

in which case $\phi(x; k)$ must be analytic throughout \mathbb{C}_+ , since e^{-ikx} & $\xi^+(x; k)$ are both analytic in the upper half-plane. Moreover we see that as $|k| \rightarrow \infty$ with $k \in \mathbb{C}_+$ that $\xi^+(x; k) \rightarrow 1$ and $\phi(x; k) \rightarrow 0$. Similarly we find $\xi^-(x; k)$ is analytic throughout \mathbb{C}_- and $\xi^-(x; k) \rightarrow 1$ & $\psi(x; k) \rightarrow 0$ as $|k| \rightarrow \infty$ with $k \in \mathbb{C}_-$.

4.3 Analytic properties of $a(k)$

The relations obtained in (4.1.11) apply only when $k \in \mathbb{R}$. However returning to the spectral problem (4.1.1), it is clear the $\psi(x; k)$ and $\bar{\psi}(x; \bar{k})$ are solutions to the same spectral problem for arbitrary $k \in \mathbb{C}$. With this observation we may extend (4.1.11) for all $k \in \mathbb{C}$. In the previous section it was found that $\psi(x; k)$ could be analytically continued into the region \mathbb{C}_- , in which case see $\bar{\psi}(x; \bar{k})$ has analytic continuation

throughout \mathbb{C}_+ . Moreover, it was found the $\phi(x; k)$ had analytic continuation throughout \mathbb{C}_+ , in which case we see

$$a(k) = \frac{1}{2ik} \mathbf{W}[\phi(x; k), \bar{\psi}(x; \bar{k})] \quad (4.3.1)$$

is analytic throughout $k \in \mathbb{C}_+$. Thus the scattering coefficient $a(k)$ is analytic in \mathbb{C}_+ .

In the case where $k_0 = \mu_0 + i\kappa_0 \in \mathbb{C}_+$ such that $a(k_0) = 0$, the Wronskian (4.3.1) becomes

$$\mathbf{W}[\phi(x; k_0), \bar{\psi}(x; \bar{k}_0)] = \phi(x; k_0)\bar{\psi}_x(x; \bar{k}_0) - \phi_x(x; k_0)\bar{\psi}(x; \bar{k}_0) = 0. \quad (4.3.2)$$

Consequently, at $k = k_0 \in \mathbb{C}_+$ we must have

$$\phi(x; k_0) = b_0 \bar{\psi}(x; \bar{k}_0), \quad (4.3.3)$$

with b_0 a nonzero constant. The definition of $\phi(x; k)$ in (4.1.2) implies that

$$\lim_{x \rightarrow -\infty} \phi(x; k_0) = \lim_{x \rightarrow -\infty} e^{-ik_0 x} = 0. \quad (4.3.4)$$

Meanwhile, as $x \rightarrow +\infty$ we find

$$\lim_{x \rightarrow \infty} \bar{\psi}(x; \bar{k}_0) = \lim_{x \rightarrow \infty} e^{ik_0 x} = 0. \quad (4.3.5)$$

We see that the function $\phi(x; k_0)$ is exponentially decreasing as $|x| \rightarrow \infty$.

In general if $\phi(x; k)$ a solution of (4.1.1) then we have

$$\bar{\phi}_{xx}(x; k) = [-\bar{k}^2 + u(x)]\bar{\phi}(x; k), \quad (4.3.6)$$

and so we have

$$\frac{d}{dx} \mathbf{W}[\phi(x; k), \bar{\phi}(x; k)] = (k^2 - \bar{k}^2)|\phi(x; k)|. \quad (4.3.7)$$

In the specific case of $k = k_0$, in which case we know $\phi(x; k_0) \rightarrow 0$ as $x \rightarrow \pm\infty$, we see that

$$0 = \mathbf{W}[\phi(x; k_0), \bar{\phi}(x; k_0)] \Big|_{x=-\infty}^{x=\infty} = (k^2 - \bar{k}^2) \int_{-\infty}^{\infty} |\phi(x; k)| dx. \quad (4.3.8)$$

The L_2 -norm satisfies

$$\|\phi(\cdot; k)\|_2^2 = \int_{-\infty}^{\infty} |\phi(x; k)|^2 dx > 0,$$

for any solution of (4.1.1) and as such it must be the case

$$k^2 - \bar{k}^2 = 0. \quad (4.3.9)$$

Thus we may write

$$\mu_0 \kappa_0 = 0,$$

and since we have $k_0 \in \mathbb{C}_+$, it follows that $\mu_0 = 0$. This must be true for all zeros of $a(k)$, and so

$$k_n = i\kappa_n \quad \kappa_n \in \mathbb{R}_+ \quad n \in \{1, \dots, N\}, \quad (4.3.10)$$

i.e. the discrete spectrum of (4.1.1) must be purely imaginary.

Finally we must show that the zeros of $a(k)$ are simple. To do so, we differentiate the spectral problem (4.1.1) once with respect to the spectral parameter k , to obtain

$$\phi_{xxk} = -2k\phi + [-k^2 + u(x)]\phi_k. \quad (4.3.11)$$

Multiplying (4.3.11) by ϕ and (4.1.1) by ϕ_k and subtracting the latter from the former we obtain

$$\begin{aligned} [\phi\phi_{xxk} - \phi_{xx}\phi_k]_x &= -2k\phi^2 \\ \text{or } \frac{d}{dx} \mathbf{W}[\phi, \phi_k] &= -2k\phi^2, \end{aligned} \quad (4.3.12)$$

and so upon integrating we find

$$\mathbf{W}[\phi, \phi_k] \Big|_{x=-\infty}^{x=\infty} = -2k \int_{-\infty}^{\infty} \phi^2 dx. \quad (4.3.13)$$

Meanwhile, we also have

$$\mathbf{W}[\phi, \psi] = 2ika(k),$$

and so differentiating with respect to the spectral parameter we find

$$\mathbf{W}[\phi_k, \psi] + \mathbf{W}[\phi, \psi_k] = 2ia(k) + 2ik\dot{a}(k), \quad (4.3.14)$$

where we introduce $\dot{a}(k) = \frac{d}{dk}a(k)$. Choosing $k = i\kappa_0$ such that $a(i\kappa_0) = 0$, where we have $\phi(x, i\kappa_0) = b_0\psi(x, i\kappa_0)$, then equation (4.3.14) becomes

$$b_0^2\mathbf{W}[\psi(x; i\kappa), \psi_k(x; i\kappa)] - \mathbf{W}[\phi(x; i\kappa_0), \phi_k(x; i\kappa_0)] = -2b_0\kappa_0\dot{a}(i\kappa_0), \quad (4.3.15)$$

where we have multiplied both sides by b_0 . In addition, (4.3.13) evaluated at $k = i\kappa_0$ yields

$$\mathbf{W}[\phi(\infty; i\kappa_0), \phi_k(\infty; i\kappa_0)] - \mathbf{W}[\phi(-\infty; i\kappa_0), \phi_k(-\infty; i\kappa_0)] = -2i\kappa_0 \int_{-\infty}^{\infty} \phi(x; i\kappa_0)^2 dx. \quad (4.3.16)$$

which when added to (4.3.15) evaluated as $x \rightarrow \infty$ yields

$$b_0^2\mathbf{W}[\psi(x; i\kappa), \psi_k(x; i\kappa)] - \mathbf{W}[\phi(-\infty; i\kappa_0), \phi_k(-\infty; i\kappa_0)] = -2b_0\kappa_0\dot{a}(i\kappa_0) - 2i\kappa_0 \int_{-\infty}^{\infty} \phi(x; i\kappa_0)^2 dx. \quad (4.3.17)$$

It has already been observed that $\psi(x; i\kappa_0)$ and $\phi(x; \kappa_0)$ both decay exponentially as $x \rightarrow \pm -\infty$, respectively, in which case we find

$$\dot{a}(i\kappa_0) = -\kappa_0 \int_{-\infty}^{\infty} \phi(x; i\kappa_0)^2 dx. \quad (4.3.18)$$

Replacing $k = i\kappa_0$ in (4.2.9) ensures $\xi^-(x; i\kappa_0)$ is real which combined with (4.2.1) ensures $\phi(x; i\kappa_0)$ is also real. It follows from these observations and the previous result that

$$\dot{a}(i\kappa_0) < 0, \quad (4.3.19)$$

in which case the zeros of $a(k)$ must be simple.

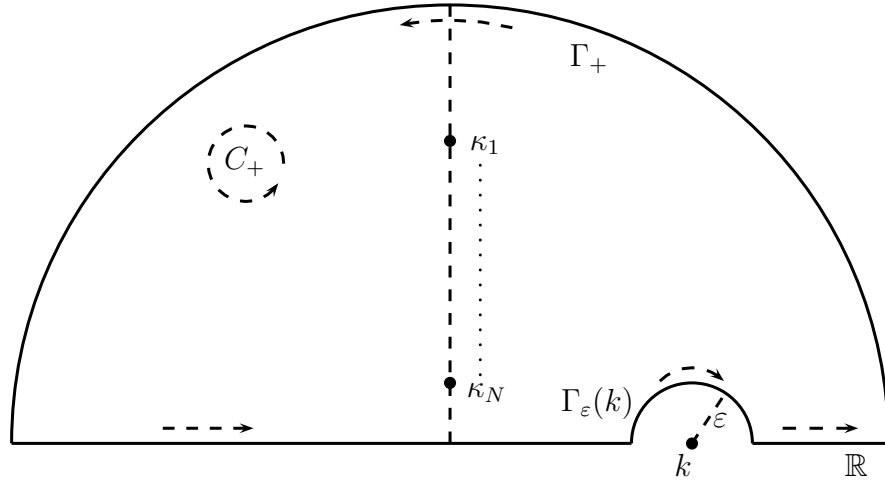


Figure 4.1: The integration contour in \mathbb{C}_+ for the analytic continuation of $a(k)$

4.4 Analytic continuation of $a(k)$

We now know that the spectral function $a(k)$ is analytic in \mathbb{C}_+ with a finite number of simple zeros along the imaginary axis. Given the Wronskian in (4.3.1) we see from the asymptotic properties of ϕ and ψ that

$$\lim_{|k| \rightarrow \infty} a(k) = 1, \quad (4.4.1)$$

where $k \in \Gamma_+$ as shown in the contour diagram below.

We introduce the auxiliary function

$$A(k) = a(k) \prod_{n=1}^N \frac{k + i\kappa_n}{k - i\kappa_n}, \quad (4.4.2)$$

which has no zeros or poles in \mathbb{C}_+ , given the analytic properties of $a(k)$ discussed in the previous section. Furthermore we see that as $|k| \rightarrow \infty$ with $k \in \Gamma_+$ then $A(k) \rightarrow 1$. Given that $\ln A(k)$ is both analytic and without zeros in \mathbb{C}_+ , and such that $\ln A(k) = 0$ when $k \in \Gamma_+$, we may write

$$0 = \oint_{C_+} \frac{\ln A(k)}{k - k'} dk' = \int_{-\infty}^{\infty} \frac{\ln A(k')}{k - k'} dk' + \int_{\Gamma_\varepsilon(k)} \frac{\ln A(k')}{k - k'} dk', \quad (4.4.3)$$

An application of the residue theorem yields

$$\int_{\Gamma_\varepsilon(k)} \frac{\ln A(k')}{k - k'} dk' = -i\pi \ln A(k), \quad (4.4.4)$$

and so we see at once

$$\ln A(k) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\ln A(k')}{k' - k} dk'. \quad (4.4.5)$$

It is clear from the definition of $A(k)$ in (4.4.2) that $|A(k)| = |a(k)|$, and so we may write

$$\ln A(k) = \ln |a(k)| + i \arg A(k),$$

and so with this replacement made in (4.4.5), we find upon comparing both sides that

$$\ln |a(k)| = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\arg A(k')}{k' - k} dk' \quad \arg A(k) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k} dk', \quad (4.4.6)$$

which are the Kramers-Kronig relations. We may write an integral representation for $\ln A(k)$ in the form

$$\ln A(k) = \ln |a(k)| - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k} dk' = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k - i0^+} dk', \quad (4.4.7)$$

while (4.4.2) also implies

$$\ln A(k) = \ln a(k) + \sum_{n=1}^N \ln \left(\frac{k + i\kappa_n}{k - i\kappa_n} \right) \quad (4.4.8)$$

Comparing both expressions we find

$$\ln a(k) = \sum_{n=1}^N \ln \left(\frac{k - i\kappa_n}{k + i\kappa_n} \right) - \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k - i0^+} dk'. \quad (4.4.9)$$

It is clear that upon analytic continuation into \mathbb{C}_+ this integral representation become

$$\ln a(k) = \ln |a(k)| + \sum_{n=1}^N \ln \left(\frac{k - i\kappa_n}{k + i\kappa_n} \right) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k} dk', \quad (4.4.10)$$

and so $a(k)$ is completely determined by $|a(k)|$ which in turn is determined by $r(k)$. The scattering coefficient $b(k)$ may be simply written as $b(k) = a(k)r(k)$, and so as was claimed earlier, the scattering data is essentially contained in $r(k)$.

4.5 The Riemann-Hilbert problem

We now know the analytic properties of the Jost solutions away from the real axis, and moreover we are also able to determine the scattering coefficients $a(k)$ and $b(k)$ given $r(k)$. Rewriting the scattering equation (4.1.7) in terms of the modified solutions $\xi^\pm(x; k)$ and dividing by $a(k)$ we find

$$\frac{\xi^+(x; k)}{a(k)} = \xi^-(x; k) + r(k)\bar{\xi}^- e^{2ikx}, \quad (4.5.1)$$

where as always $k \in \mathbb{R}$. As we now know, the left hand side is analytic in \mathbb{C}_+ except at the zeros of $a(k)$ as shown in Figure 4.5, while the right hand side is analytic in \mathbb{C}_- .

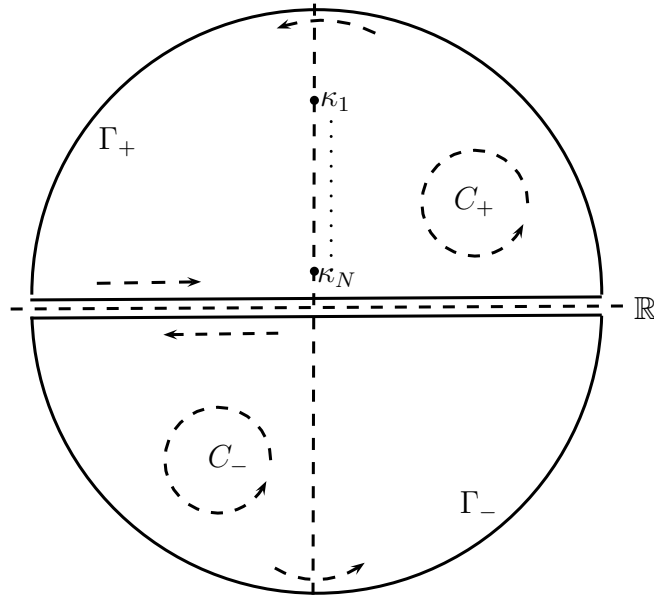


Figure 4.2: The contours C_+ & C_- for the Riemann-Hilbert problem of $\Phi(x; k)$.

We may integrate the left hand term about the contour C_+ , shown in Figure 4.5, since $\xi^+(x; k)$ and $a(k)$ have analytic continuation away from \mathbb{R} in \mathbb{C}_+ . The residue theorem applied to this integral yields

$$\frac{1}{2\pi i} \oint_{C_+} \frac{\xi^+(x; k')}{a(k')(k' - k)} dk' = \sum_{n=1}^N \frac{\xi_n^+(x)}{a(i\kappa_n)(i\kappa_n - k)}, \quad k \in \mathbb{C}_-, \quad (4.5.2)$$

where we define $\xi_n(x) = \xi(x; i\kappa_n)$. Meanwhile, expanding the integral into two distinct integrals over Γ_+ and \mathbb{R} we find

$$\frac{1}{2\pi i} \int_{C_+} \frac{\xi^+(x; k')}{a(k')(k' - k)} dk' = \frac{1}{2\pi i} \int_{\Gamma_+} \frac{1}{(k' - k)} dk' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\xi^+(x; k')}{a(k')(k' - k)} dk',$$

where in the first integral on the right hand side we have used $\xi^+(x; k) \rightarrow 1$ and $a(k) \rightarrow 1$ when $k \rightarrow \Gamma_+$. Using the scattering relation (4.5.1) we may rewrite the integral over \mathbb{R} as follows:

$$\int_{-\infty}^{\infty} \frac{\xi^+(x; k')}{a(k')(k' - k)} dk' = \int_{-\infty}^{\infty} \frac{\xi^-(x; k')}{(k' - k)} dk' + \int_{-\infty}^{\infty} \frac{r(k')\bar{\xi}^-(x; k')}{k' - k} dk'. \quad (4.5.3)$$

In addition we may use the analyticity of $\xi^-(x; k)$ throughout \mathbb{C}_- and its asymptotic behaviour $\xi^-(x; k) \rightarrow 1$ as $k \rightarrow \Gamma_-$, to write

$$\xi^-(x; k) = \frac{1}{2\pi i} \int_{C_-} \frac{\xi^-(x; k')}{k' - k} dk' = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\xi^-(x; k')}{k' - k} dk' + \frac{1}{2\pi i} \int_{\Gamma_-} \frac{1}{k' - k} dk'. \quad (4.5.4)$$

Combining the results of (4.5.3), (4.5.3) and (4.5.4) we find

$$\xi^-(x; k) = 1 + \sum_{n=1}^N \frac{\xi_n^+(x)}{i\kappa_n(i\kappa_n - k)} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(k')\bar{\xi}^-(x; k')}{k' - k} dk'. \quad (4.5.5)$$

We previously saw that along the discrete spectrum $\phi(x; i\kappa_n)$ and $\psi(x; i\kappa_n)$ are related by $\phi(x; i\kappa_n) = b_n\psi(x - i\kappa_n)$, and so it follows

$$\xi^+(x; i\kappa_n) = \phi(x; i\kappa_n)e^{-\kappa_n x} = b_n\psi(x; -i\kappa_n)e^{-\kappa_n x} = b_n\xi_n^-(x)e^{-2\kappa_n x}.$$

With this relationship in mind we rewrite (4.5.5) as

$$\xi^-(x; k) = 1 - i \sum_{n=1}^N \frac{R_n \xi_n^-(x) e^{-2\kappa_n x}}{k - i\kappa_n} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(k')\bar{\xi}^-(x; k')}{k' - k} dk', \quad (4.5.6)$$

where we define $R_n = \frac{b_n}{i\bar{a}(i\kappa_n)}$.

4.6 Time dependence of the scattering data

In order to integrate the KdV equation for $u(x)$ using the IST method, we need to obtain the time dependence of the scattering data $\{R_n, r(k) : n = 1, \dots, N, k \in \mathbb{R}\}$. This is achieved using the KdV equation itself. Specifically, we introduce a second time dependant spectral problem

$$\phi_t(x; k) = -4\phi_{xxx}(x; k) - 3u_x(x)\phi(x; k) + 6u(x)\phi_x(x; k) + \gamma\phi(x; k), \quad (4.6.1)$$

with gamma an arbitrary constant. The consistency condition $\psi_{xxt}(x; k) = \psi_{txx}(x; k)$ is possible only if $u(x)$ satisfies the KdV equation

$$u_t + 3uu_x + u_{xxx} = 0.$$

The pair of equations (4.1.1) and (4.6.1) are said to form the Lax pair for the KdV equation. Here, and throughout, we have suppressed the variable t in the potential u , the spectral function ψ and the scattering data $r(k)$ & R_n .

While $\phi(x; k)$ is defined by its asymptotic behaviour as $x \rightarrow -\infty$ we may obtain its asymptotic behaviour as $x \rightarrow +\infty$ via the behaviour of $\psi(x; k)$ in the same region. The scattering relation (4.1.7) yields

$$\lim_{x \rightarrow \infty} \phi(x; k) = a_t(k)e^{-ikx} + b_t(k)e^{ikx}, \quad k \in \mathbb{R}, \quad (4.6.2)$$

where it is understood the asymptotic behaviour of $\psi(x; k)$ and is independent of time as $x \rightarrow +\infty$. In addition we substitute (4.1.7) in (4.6.1) and let $x \rightarrow +\infty$ to obtain

$$\lim_{x \rightarrow \infty} \phi_t(x; k) = (-4ik^3 + \gamma)a(k)e^{-ikx} + (4ik^3 + \gamma)b(k)e^{ikx}, \quad k \in \mathbb{R}, \quad (4.6.3)$$

where we have also used the limiting behaviour $\lim_{|x| \rightarrow \infty} u(x) = 0$ & $\lim_{|x| \rightarrow \infty} u_x(x) = 0$ at all times.

The scattering coefficient $a(x)$ is required to be time independent hence $a_t(k) = 0$ which in turn yields

$$0 = a_t(k) = (-4ik + \gamma)a(k),$$

in which case $\gamma = 4ik^3$. This in turn yields time evolution equation for the scattering coefficient $b(k)$, namely

$$b_t(k) = 8ik^3b(k), \quad k \in \mathbb{R}$$

which is readily solved to yield

$$b(k) = b_0(k)e^{8ik^3t}, \quad b_0(k) = b(k)|_{t=0}, \quad (4.6.4)$$

thus giving the time dependence of the scattering coefficient $b(k)$. In addition, time dependence of the reflection coefficient $r(k)$ is readily obtained from that of $b(k)$, namely

$$r(k) = r_0(k)e^{8ik^3t}, \quad r_0(k) = r(k)|_{t=0} = \frac{b_0(k)}{a(k)}. \quad (4.6.5)$$

The time dependence of $r(k)$ identical to that of $b(k)$ since $a(k)$ is independent of time. Thus we have obtained to time dependence of the scattering data along the continuous spectrum $k \in \mathbb{R}$.

Recall that along the discrete spectrum we have $\phi_n(x) = \phi(x; i\kappa_n) = b_n\psi(x - i\kappa_n) = b_n\psi_n(x)$. Strictly speaking $\phi_n(x)$ only defined in the asymptotic region $x \rightarrow -\infty$, however we may describe its behaviour as $x \rightarrow \infty$ through the asymptotic behaviour of $\psi_n(x)$. It follows

$$\lim_{x \rightarrow \infty} \phi_n(x) = b_n \lim_{x \rightarrow \infty} \psi_n(x) = b_n e^{-\kappa_n x}, \quad (4.6.6)$$

which we substitute into (4.6.1) to obtain

$$\lim_{x \rightarrow \infty} \phi_n(x) = b_{n,t} e^{-\kappa_n x}, \quad (4.6.7)$$

since the asymptotic behaviour of $\psi_n(x)$ as $x \rightarrow \infty$ is independent of time. Along the discrete spectrum (4.6.1) simplifies to become

$$\phi_{n,t}(x) = (-4\kappa_n^3 b_n - \gamma_n b_n) e^{-\kappa_n x},$$

where we define $\gamma_n = \gamma|_{k=i\kappa_n} = 4i(i\kappa_n)^3 = 4\kappa_n^3$.

Comparing the two expressions for $\phi_{n,t}(x)$ in the asymptotic region $x \rightarrow \infty$, we find the time dependence of b_n is governed by

$$b_{n,t} = 8\kappa_n^3 b_n, \quad (4.6.8)$$

whose solution

$$b_n = b_n^0 e^{8\kappa_n^3 t}, \quad b_n^0 = b_n|_{t=0}, \quad (4.6.9)$$

gives the time dependence of b_n explicitly. In addition, since $a(k)$ is time independent for all $k \in \mathbb{C}$, it follows

$$0 = \partial_t a(i\kappa_n) = i \frac{d\kappa_n}{dt} \dot{a}(i\kappa_n) \Rightarrow \frac{d\kappa_n}{dt} = 0,$$

as the zeros of $a(k)$ were already shown to be simple zeros. It follows that $\dot{a}(i\kappa_n)$ is also independent of time, in which case the R_n depend on time via b_n only. Thus

$$R_n = R_n^0 e^{8\kappa_n t}, \quad R_n^0 = R_n|_{t=0} = \frac{b_n^0}{i\dot{a}(i\kappa_n)}, \quad (4.6.10)$$

gives the time dependence of R_n explicitly.

4.7 Solving KdV via IST

Recall equations (4.2.9) and (4.2.10), which now allow us to write the asymptotic expansion for $\xi^-(x; k)$, namely

$$\xi^-(x; k) = 1 - \int_x^\infty \frac{e^{2ik(x-y)} - 1}{2ik} u(y) \xi^-(y; k) dy. \quad (4.7.1)$$

In the asymptotic region $k \rightarrow \Gamma_-$, we know from the asymptotic behaviour of $\xi^-(x; k)$ this the integral behaves as

$$\xi^-(x; k) \simeq 1 + \frac{1}{2ik} \int_x^\infty u(y) dy + \mathcal{O}\left(\frac{1}{k^2}\right), \quad k \rightarrow \Gamma_-. \quad (4.7.2)$$

Equation (4.5.5) provides us with a second expression for $\xi^-(x; k)$, which when evaluated in the asymptotic region $k \rightarrow \Gamma_-$ becomes,

$$\xi(x; k) \simeq 1 + \frac{1}{2ik} \left[2 \sum_{n=1}^N R_n \xi_n^-(x) e^{-2\kappa_n x} + \frac{1}{\pi} \int_{-\infty}^{\infty} r(k') \xi^-(x; k') dk' \right] + \mathcal{O}\left(\frac{1}{k^2}\right), \quad k \rightarrow \Gamma_-.$$
(4.7.3)

Clearly then we may write the potential $u(x)$ in terms of the modified spectral functions $\xi^-(x; k)$ and $\xi_n(x)$ with

$$u(x) = -\frac{d}{dx} \int_x^{\infty} u(y) dy = -\frac{d}{dx} \left[2 \sum_{n=1}^N R_n \xi_n^-(x) e^{-2\kappa_n x} + \frac{1}{\pi} \int_{-\infty}^{\infty} r(k') \xi^-(x; k') dk' \right],$$
(4.7.4)

being the expression we seek.

In general the system (4.5.6) contains closed form solutions for a large class of reflection coefficients $r(k)$. However, in the case of reflectionless potentials, when $r(k) = 0$, the system (4.5.6) becomes purely algebraic, thus allowing us to solve for $\xi_n(x)$ exactly. This in turn allows us to solve for the potential $u(x)$ exactly, leading to a class of solutions referred to as N -soliton solutions. In this case, the system reduces to become

$$\xi^-(x; k) = 1 - i \sum_{n=1}^N \frac{R_n \xi_n(x) e^{-2\kappa_n x}}{k - i\kappa_n}, \quad k \in \mathbb{C}_-.$$
(4.7.5)

With $k = -i\kappa_m$ we have

$$\xi^-(x; -i\kappa_m) = \xi_m(x) = 1 + \sum_{n=1}^N \frac{R_n \xi_n^-(x) e^{-2\kappa_n x}}{\kappa_m + \kappa_n},$$
(4.7.6)

which obviously allows us to solve for $\xi_m^-(x)$ exactly in terms of $\{R_n, \kappa_n : n = 1, \dots, N\}$.

In the case to the one-soliton solutions, we obviously have $N = 1$ and $\kappa_1 \equiv \kappa$ for some $\kappa > 0$. The system (4.7.6) is readily solved to yield

$$\xi_1(x) = \frac{1}{1 - e^{-2\kappa(x-x_0-4\kappa^2 t)}} \tag{4.7.7}$$

where we define x_0 according to

$$\frac{R_1^0}{2\kappa} = e^{2\kappa x_0},$$

and which corresponds to the soliton centre of mass location at time $t = 0$. It follows that

$$u(x, t) = -\frac{d}{dx} \left[\frac{4\kappa e^{-2\kappa(x-x_0-4\kappa^2 t)}}{1 - e^{-2\kappa(x-x_0-4\kappa^2 t)}} \right] = 2\kappa^2 \operatorname{sech}^2 \kappa(x - x_0 - 4\kappa^2 t), \quad (4.7.8)$$

which is the one soliton solution for the KdV equation. This solution is realised in the context of water waves and was first observed by Russell in 1834 and reproduced by him thereafter in multiple experiments [Rus1844].

4.8 Conclusion

The KdV equation has served as the simplest motivating example to demonstrate the effectiveness of the IST method of solving a nonlinear PDE. The IST is in a sense the nonlinear analogue of the Fourier transform method for solving linear PDE, in that we uncover the time evolution of the solutions via the time evolution of the scattering coefficients $r(k, t)$ and $R_n(t)$. The IST applied to the KdV is well understood problem and appears in many textbooks as a first introduction. Nevertheless, we have included the treatment of this problem as an introduction to the methods of the IST for two reasons. Firstly, the it will be seen in the following chapter that the spectral problem associated with the Qiao equation may be identified with the spectral problem of the KdV equation when we require the solution of the former to have constant boundary values. The second reason we have studied this problem in such detail is that the methods introduced may be extended to allow us to solve a more general class of PDE with cubic nonlinearities, which develop in greater detail in Chapter 6. Indeed, given the applicability of the KdV to many nonlinear phenomena of physical interest, the study of the IST for

the KdV is of great interest in its own right, and as such its inclusion is justified by the occurrence of the KdV as a model equation in previous chapters.

Chapter 5

The Qiao Equation

The interest inspired by the Camassa-Holm (CH) equation and its singular peakon solutions [CH1993] prompted the search for other integrable equations with similar properties. An integrable peakon equation with cubic nonlinearities was first discovered by Qiao [Qia2006] and studied further in e.g. [Qia2007, QL2009]. Another equation with cubic nonlinearities was introduced by V. Novikov in [Nov2009]. Actually the Qiao equation

$$m_t + (m(u^2 - u_x^2))_x = 0, \quad m = u - u_{xx} \quad (5.0.1)$$

appears in the class of integrable equations discussed in [Fok1995].

It is known that the Qiao equation has a distinctive W/M -shape travelling wave solution [Qia2006, Qia2007]. The peakons of Novikov's equation were studied in [HLS2009] while $2 + 1$ dimensional generalizations of Qiao's hierarchy are studied in [Est2011]. Single and, muting-peakon dynamics, weak kink, kink-peakon, and stability analysis of the Qiao equation were studied in [QXL2012] and [GLOQ2012], while other types of solitons are studies in [IL2012c]. Equation (5.0.1) may also be written as

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0. \quad (5.0.2)$$

Qiao introduced a 2×2 Lax pair for this equation given by the linear system $\Psi_x = \mathbf{U}\Psi$

and $\Psi_t = \mathbf{V}\Psi$ where

$$\begin{aligned} \mathbf{U} &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}m\lambda \\ -\frac{1}{2}m\lambda & \frac{1}{2} \end{pmatrix}, \\ \mathbf{V} &= \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}m\lambda(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}m\lambda(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}. \end{aligned} \quad (5.0.3)$$

Another equation from the same hierarchy is

$$m_t + \left(\frac{1}{m^2}\right)_x - \left(\frac{1}{m^2}\right)_{xxx} = 0. \quad (5.0.4)$$

The (white) soliton solutions of (5.0.1) and (5.0.4) were previously found in [Sak2011, Zha2013]. These results rely on the fact that the spectral problem for (5.0.1) is gauge-equivalent to the one for the mKdV equation. In this chapter we first discuss the peakon solutions of (5.0.1). Then we present soliton solutions which approach a nonzero, constant value as $|x| \rightarrow \infty$ (dark solitons). To this end we are going to reformulate the spectral problem in the form of a Schrödinger operator, which is also the spectral problem for the KdV equation.

5.1 Peakon solutions

In [HW2008] there is a remark on the peakons of Qiao's equation, stating that their computation is problematic since one encounters a square of a delta-function. This difficulty can be avoided by the following transformation of (5.0.1). Assuming peakon solutions which vanish as $x \rightarrow \pm\infty$ and writing

$$m(x, t) = \sum_{k=1}^N p_k(t) \delta(x - x_k(t))$$

one can integrate (5.0.1) to find

$$\partial_t \int_{-\infty}^x m(y, t) dy + (u^2 - u_x^2)m(x, t) = 0,$$

giving

$$\partial_t \left(\sum_{k=1}^N p_k(t) \theta(x - x_k(t)) \right) + (u^2 - u_x^2) m(x, t) = 0,$$

It follows that

$$\sum_{k=1}^N \dot{p}_k(t) \theta(x - x_k(t)) - \sum_{k=1}^N p_k(t) \dot{x}_k(t) \delta(x - x_k(t)) + (u^2 - u_x^2) \sum_{k=1}^N p_k(t) \delta(x - x_k(t)) = 0,$$

which is only possible if

$$\dot{p}_k(t) = 0, \quad (5.1.1)$$

$$\dot{x}_k(t) = (u^2 - u_x^2)_{x=x_k(t)}. \quad (5.1.2)$$

The $N = 1$ peakon solution is easily obtained from the above system, $u(x, t) = \pm \sqrt{c} e^{-|x-ct|}$ where $p_1 = 2\sqrt{c} = \text{constant}$. This solution was reported in [HW2008].

To compute the two-peakon solution we notice that

$$H = \frac{1}{2} \int m u \, dx = \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 + p_1 p_2 e^{-|x_1 - x_2|}$$

is a conserved quantity. Therefore $\Delta = x_1 - x_2$ is time independent, i.e. the distance between the two peakons is constant and they move together. This explains the M-shape travelling wave solution mentioned earlier, see for example Fig.5.1. The solution is

$$u(x, t) = \frac{1}{2} p_1 e^{-|x-ct|} + \frac{1}{2} p_2 e^{-|x-ct-\Delta|}, \quad c = \frac{1}{2} p_1 p_2 e^{-|\Delta|}.$$

A rigorous discussion of the weak solutions of the Qiao equation may be found in the publication [GLOQ2013].

5.2 Soliton solutions

5.2.1 Reformulation of the spectral problem

Let us consider solutions such that

$$m(x, t) > 0, \quad \lim_{x \rightarrow \pm\infty} m(x, t) = M_0, \quad (5.2.1)$$

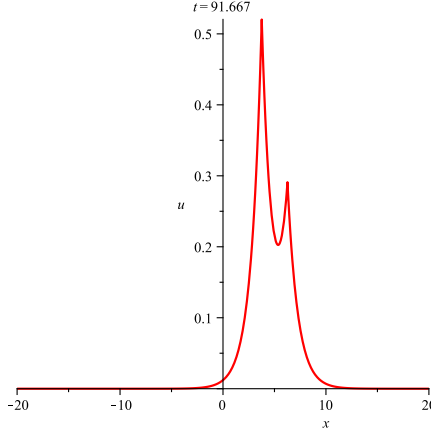


Figure 5.1: Two peakon profile, $p_1 = 1$, $p_2 = 0.5$, $\Delta = 2.5$.

where M_0 is a positive constant. Let us also assume that $m(x, \cdot) - M_0 \in \mathcal{S}(\mathbb{R})$ for any value of t . One can reformulate the spectral problem into a scalar one as follows. Introducing $\Psi = (\psi, \phi)^T$ the matrix Lax pair written in components becomes

$$2\psi_x = -\psi + m\lambda\phi$$

$$2\phi_x = -m\lambda\psi + \phi.$$

We introduce a change of coordinates given by

$$\partial_y = \frac{2}{M}\partial_x, \quad \psi = \frac{1}{\lambda} \left[\frac{\phi}{M} - \phi_y \right] \quad (5.2.2)$$

where we define $M(y, t) \equiv m(x(y, t), t)$ and $\varphi(y, t) = \phi(x(y, t), t)$. With this change of variables we obtain the following scalar spectral problem for $\phi(y, \lambda)$ (we omit the argument t which acts as an external parameter for the spectral problem being considered)

$$-\varphi_{yy} + \left[\left(\frac{1}{M} \right)_y + \frac{1}{M^2} \right] \varphi = \lambda^2 \varphi. \quad (5.2.3)$$

Note that this is a Schrödinger's operator with a potential

$$U(y, t) = \left(\frac{1}{M} \right)_y + \frac{1}{M^2}. \quad (5.2.4)$$

It is well known how to recover $U(y, t)$ from the scattering data of (5.2.3), however the solution is $m(y, t)$ and its recovery from $U(y, t)$ necessitates solving a nonlinear (Riccati) equation. We can express $m(y, t)$ in terms of the eigenfunctions of the Schrödinger's operator. We introduce $\rho(y, \lambda) = \frac{\phi_y}{\phi}$ from which we immediately obtain

$$\rho_y + \rho^2 = \frac{\varphi_{yy}}{\varphi} = U(y) - \lambda^2.$$

If we define $\rho_0(y) = \rho(y, 0)$ then we have

$$U(y) = \rho_{0,y} + \rho_0^2.$$

However, comparing this with (5.2.4) we find a solution $\frac{1}{M} = \rho_0$ or

$$M(y, t) = \frac{1}{\rho_0(y, t)} = \frac{\varphi(y, t, \lambda)}{\varphi_y(y, t, \lambda)} \Big|_{\lambda=0} \quad (5.2.5)$$

So far we have worked with y as our variable instead of x . However we can treat y as a parameter, and then (5.2.5) represents the solution in parametric form, while the original variable x follows from (5.2.2), (5.2.5) by:

$$x(y, t) = 2 \ln \varphi(y, t, 0) + \text{const.} \quad (5.2.6)$$

Assuming that $\varphi(y, t, 0)$ is positive everywhere, we have a solution in parametric form (5.2.5) (5.2.6) given entirely in terms of the eigenfunctions $\varphi(y, t, 0)$. We can formally write this solution as

$$m(x, t) = 2 \int_{-\infty}^{\infty} M(y, t) \delta(x - 2 \ln \varphi(y, t, 0)) \, dy. \quad (5.2.7)$$

where we neglect the constant appearing in (5.2.6).

5.2.2 Inverse scattering and Soliton solutions

It follows from (5.2.1) and (5.2.4) that $U(y)$ does not decay to 0 as $y \rightarrow \pm\infty$. As such we need to introduce the modified potential

$$\tilde{U}(y) = U(y) - \frac{1}{M_0^2}, \quad (5.2.8)$$

which clearly satisfies $\lim_{|y| \rightarrow \infty} \tilde{U}(y) = 0$. So we have

$$-\varphi_{yy} + \left[U(y) - \frac{1}{M_0^2} \right] \phi = \left(\lambda^2 - \frac{1}{M_0^2} \right) \phi,$$

or, introducing a new spectral parameter

$$k^2 = \lambda^2 - \frac{1}{M_0^2} \quad (5.2.9)$$

we have a standard spectral problem

$$-\varphi_{yy}(k, y) + \tilde{U}(y)\varphi(k, y) = k^2\varphi(k, y), \quad \tilde{U}(y) \in \mathcal{S}(\mathbb{R}). \quad (5.2.10)$$

However when $\lambda = 0$ we find $k = \pm \frac{i}{M_0}$. This means that if we take an eigenfunction $\phi(k, y)$ of (5.2.10) which is analytic in the upper (lower) complex k -plane, we should evaluate it at $k = \frac{i}{M_0}$ ($k = -\frac{i}{M_0}$):

$$M(y, t) = \left. \frac{\varphi(y, t, k)}{\varphi_y(y, t, k)} \right|_{k=\pm \frac{i}{M_0}} \quad (5.2.11)$$

$$x(y, t) = 2 \ln \varphi \left(y, t, \pm \frac{i}{M_0} \right). \quad (5.2.12)$$

5.3 Time Dependence of the scattering data and Soliton Solutions

The spectral theory for the problem (5.2.10) is well developed, e.g. [ZMNP1984]. We are going to use these results to construct the soliton solutions of (5.0.1), (5.0.4). One can introduce scattering data as usual. For the time-dependence of the scattering data one needs the time-evolution of the eigenfunction $\phi(k, x)$. The Lax-pair in x and t variables for (5.0.1) has the form

$$\phi_{xx} = \frac{m_x}{m} \phi_x + \left(\frac{1}{4} - \frac{m_x}{2m} - \frac{m^2}{4} \lambda^2 \right) \phi, \quad (5.3.1)$$

$$\phi_t = \frac{1}{\lambda^2} \left[\frac{u_x + u_{xx}}{m} \right] \phi - \left[\frac{u + u_x}{\lambda^2 m} + \frac{u^2 - u_x^2}{2} \right] \phi_x + \gamma \phi, \quad (5.3.2)$$

where γ is an arbitrary constant. Asymptotically as $x \rightarrow \pm\infty$ equation (5.3.2) becomes

$$\phi_t \rightarrow - \left[\frac{1}{\lambda^2} + \frac{M_0^2}{2} \right] \phi_x + \gamma \phi.$$

In terms of the (y, k) -variables letting $y \rightarrow \pm\infty$ we find,

$$\varphi_t \rightarrow - \frac{M_0^3}{2} \left[\frac{k^2 M_0^2 + 3}{k^2 M_0^2 + 1} \right] \varphi_y + \gamma \varphi, \quad (5.3.3)$$

since $\lim_{|y| \rightarrow \infty} m = \lim_{|y| \rightarrow \infty} u = M_0$. Defining Jost solutions by

$$\lim_{y \rightarrow \pm\infty} \xi_{\pm}(y, k) e^{iky} = 1, \quad (5.3.4)$$

such that

$$\xi_-(y, k) = a(k) \xi_+(y, k) + b(k) \bar{\xi}_+(y, k), \quad k \in \mathbb{R} \quad (5.3.5)$$

and noting that $\varphi_- \rightarrow a e^{-iky} + b e^{iky}$ when $y \rightarrow \infty$ it follows from (5.3.3)

$$a_t = \frac{M_0^3}{4} \left[\frac{k^2 M_0^2 + 3}{k^2 M_0^2 + 1} \right] (ika) + \gamma a, \quad b_t = - \frac{M_0^3}{4} \left[\frac{k^2 M_0^2 + 3}{k^2 M_0^2 + 1} \right] (ikb) + \gamma b.$$

Requiring $a_t = 0$, we find

$$b_t = -ik \frac{M_0^3}{2} \left(\frac{k^2 M_0^2 + 3}{k^2 M_0^2 + 1} \right) b(k, t)$$

and thus for the scattering coefficient $r \equiv b/a$ we have

$$r(k, t) = r(k, 0) \exp \left[-ik \frac{M_0^3}{2} \left(\frac{k^2 M_0^2 + 3}{k^2 M_0^2 + 1} \right) t \right], \quad (5.3.6)$$

while the analogue on the discrete spectrum $k = i\kappa_n$, is given by

$$R_n(t) \equiv \frac{b(i\kappa_n)}{ia'(i\kappa_n)} = R_n(0) \exp \left[\frac{\kappa_n M_0^3 (3 - \kappa_n^2 M_0^2)}{2(1 - \kappa_n^2 M_0^2)} t \right]. \quad (5.3.7)$$

It is convenient to define a dispersion law $f(\kappa) = \frac{\kappa M_0^3 (3 - \kappa^2 M_0^2)}{2(1 - \kappa^2 M_0^2)}$. Then we can write

$$R_n(t) = R_n(0) \exp(f(\kappa_n)t). \quad (5.3.8)$$

Furthermore for convenience we introduce

$$\chi_n \equiv y - \frac{f(\kappa_n)}{2\kappa_n} t - \frac{1}{2\kappa_n} \ln \frac{R_n(0)}{2\kappa_n}.$$

5.4 Soliton Solutions

The eigenfunctions of the spectral problem (5.2.10) are well known, see e.g. [ZMNP1984].

In the purely N -soliton case the eigenfunction analytic in the lower complex k -plane is the Jost solution $\varphi_+(y, k)$ defined in (5.3.4) which has the form

$$\xi_+(y, t, k) = e^{iky} \left(1 + \sum_{n=1}^N \frac{\Gamma_n(y, t)}{k - i\kappa_n} \right) \quad (5.4.1)$$

with the residues $\Gamma_n(y, t)$ satisfying a linear system

$$\Gamma_n(y, t) = iR_n(t)e^{-2\kappa_n y} \left(1 + i \sum_{m=1}^N \frac{\Gamma_m(y, t)}{\kappa_n + \kappa_m} \right).$$

The time-dependence of the scattering data is given by (5.3.8). The N -soliton solution then is given in parametric form by (5.2.11) and (5.2.12) for the eigenfunction (5.4.1).

The condition $0 < \kappa_n < M_0^{-1}$ is sufficient to ensure smoothness of the solitons.

5.4.1 Example: One-Soliton Solution

The one-soliton solution corresponds to one discrete eigenvalue $k_1 = i\kappa_1$, where κ_1 is real, positive and $\kappa_1 < M_0^{-1}$. The eigenfunction in this case is (5.4.1)

$$\xi_+(y, t, k) = e^{iky} \left(1 + \frac{1}{k - i\kappa_1} \cdot \frac{iR_1(t)e^{-2\kappa_1 y}}{1 + \frac{R_1(t)}{2\kappa_1}e^{-2\kappa_1 y}} \right). \quad (5.4.2)$$

Evaluated at $k = \frac{-i}{M_0}$ we find

$$\xi_+(y, t, \frac{-i}{M_0}) = e^{\frac{y}{M_0}} \left(1 - \frac{1}{\frac{1}{M_0} + \kappa_1} \cdot \frac{R_1(t)e^{-2\kappa_1 y}}{1 + \frac{R_1(t)}{2\kappa_1}e^{-2\kappa_1 y}} \right).$$

Combining (5.2.11) and (5.2.12) we obtain the one-soliton solution:

$$x(y, t) = \frac{2y}{M_0} + 2 \ln \left(1 - \frac{\kappa_1 M_0 e^{-\kappa_1 \chi_1}}{(1 + \kappa_1 M_0) \cosh \kappa_1 \chi_1} \right), \quad (5.4.3)$$

$$M(y, t) = \frac{M_0}{1 + \frac{\kappa_1^2 M_0^2 \operatorname{sech}^2 \kappa_1 \chi_1}{1 - M_0 \kappa_1 \tanh \kappa_1 \chi_1}}. \quad (5.4.4)$$

The extremum (minimum) of m occurs when $\chi_1 = \frac{1}{4\kappa_1} \ln \left(\frac{1-M_0\kappa_1}{1+M_0\kappa_1} \right)$. This is a constant value, e.g. the soliton moves with a velocity $\frac{f(\kappa_1)}{2\kappa_1}$ that depends on the dispersion law (i.e. the chosen equation from the hierarchy). The profile of the dark soliton is given in Figure 5.2.

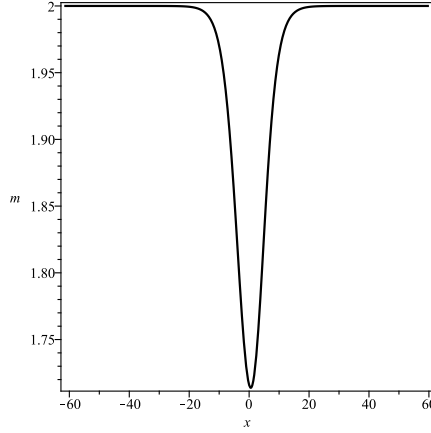


Figure 5.2: One soliton profile with $M_0 = 2$ & $\kappa_1 = 0.2$.

5.4.2 Example: Two-soliton solution

In the case of two discrete eigenvalues we compute

$$\xi_+(y, t, \frac{-i}{M_0}) = e^{\frac{y}{M_0}} \frac{1 + \nu_1 e^{-2\kappa_1 \chi_1} + \nu_2 e^{-2\kappa_2 \chi_2} + \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 \nu_1 \nu_2 e^{-2\kappa_1 \chi_1 - 2\kappa_2 \chi_2}}{1 + e^{-2\kappa_1 \chi_1} + e^{-2\kappa_2 \chi_2} + \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 e^{-2\kappa_1 \chi_1 - 2\kappa_2 \chi_2}} \quad (5.4.5)$$

where the following notation is introduced:

$$\nu_j = \frac{\frac{1}{M_0} - \kappa_j}{\frac{1}{M_0} + \kappa_j}, \quad j = 1, 2.$$

Combining (5.2.11) and (5.2.12) we obtain the two-soliton solution

$$x(y, t) = \frac{2y}{M_0} + 2 \ln \frac{\Delta_1}{\Delta_2} \quad (5.4.6)$$

$$M(y, t) = \frac{M_0}{1 + \frac{M_0 \Delta_3}{\Delta_1 \Delta_2}}. \quad (5.4.7)$$

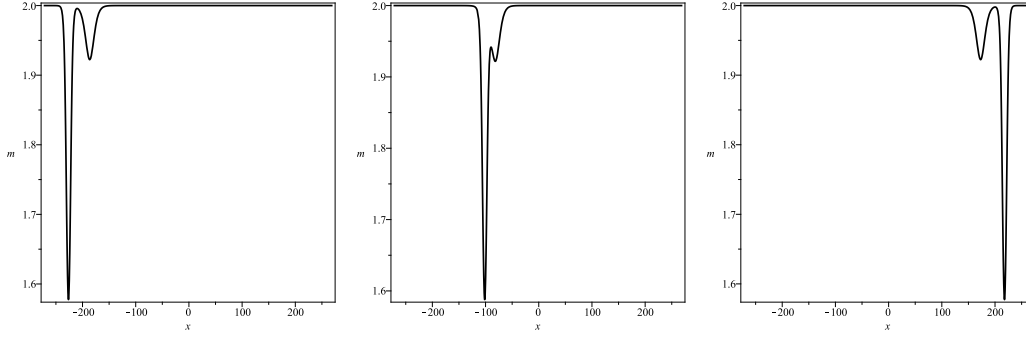


Figure 5.3: Snapshots of the two-soliton solution of the Qiao equation (5.0.1), for three values of t : -30 , -12 and 30 . The other parameters are $M_0 = 2$, $\kappa_1 = 0.1$, $\kappa_2 = 0.25$.

where the following notation is introduced:

$$\begin{aligned} \Delta_1(y, t) &= 1 + e^{-2\kappa_1\chi_1} + e^{-2\kappa_2\chi_2} + \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2 e^{-2\kappa_1\chi_1 - 2\kappa_2\chi_2} \\ \Delta_2(y, t) &= 1 + \nu_1 e^{-2\kappa_1\chi_1} + \nu_2 e^{-2\kappa_2\chi_2} + \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2 \nu_1 \nu_2 e^{-2\kappa_1\chi_1 - 2\kappa_2\chi_2}. \\ \Delta_3(y, t) &= \frac{4\kappa_1^2}{M_0^{-1} + \kappa_1} e^{-2\kappa_1\chi_1} + \frac{4\kappa_2^2}{M_0^{-1} + \kappa_2} e^{-2\kappa_2\chi_2} \\ &\quad + \frac{8(\kappa_1 - \kappa_2)^2}{M_0(M_0^{-1} + \kappa_1)(M_0^{-1} + \kappa_2)} e^{-2\kappa_1\chi_1 - 2\kappa_2\chi_2} \\ &\quad + \frac{4\kappa_2^2 \nu_1}{M_0^{-1} + \kappa_2} \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2 e^{-4\kappa_1\chi_1 - 2\kappa_2\chi_2} \\ &\quad + \frac{4\kappa_1^2 \nu_2}{M_0^{-1} + \kappa_1} \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2 e^{-2\kappa_1\chi_1 - 4\kappa_2\chi_2}. \end{aligned}$$

The interaction of two dark solitons is illustrated in Figure 5.3.

5.5 Conclusion

In this chapter we have reviewed the construction of peakon and soliton solutions for a PDE with cubic nonlinearities referred to in the literature as the Qiao equation. The method developed to construct soliton solutions relied on the realisation that the spectral problem for the system with constant boundary conditions was isomorphic to the IST

for the KdV equation. In the paper [GLOQ2013] the authors develop the weak solutions of this modified Camassa-Holm type equation. In that paper the authors considers solution $u \in W_{\text{loc}}^{3,1}$. The solutions considered differ from those found in [HW2008]. In [HW2008] the authors also presented a reformulation of the spectral problem for the Qiao equation which rendered the spectral problem of the KdV, with a discussion of the relationship between this problem and the negative of the KdV/mKdV negative flow also being presented.

Chapter 6

The Kaup-Boussinesq equation

6.1 Introduction

The Boussinesq equation was originally derived in [Bou1871] as two component system modelling shallow water waves of long wavelength, where the system originally proposed was written as

$$\begin{aligned}\pi_t &= \Phi_{xx} + \beta^2 \Phi_{xxxx} - \varepsilon(\Phi_x \pi)_x + \mathcal{O}(\delta^4, \varepsilon \delta^2) \\ \pi &= \Phi_t + \frac{1}{2} \varepsilon \Phi_x^2,\end{aligned}\tag{6.1.1}$$

with the fluid velocity u being obtained from the potential Φ as $u = \Phi_x$. the parameter ε is that ratio of wave amplitude to fluid depth, $\frac{a}{h}$. In [Kau1975], it was shown by Kaup that the classical Boussinesq equation in (6.1.1) was completely integrable, and as such the system became the Kaup-Boussinesq equation. Following the work of Hirota [Hir1985], the work by Freeman et al. [FGN1990] investigates the Wronskian solutions of the KP hierarchy and subsequently show that Kaup-Boussinesq equation is a reduction of the two component KP hierarchy.

Moreover the last decades have witnessed an explosion in the complexity and sophistication of mathematical theories for fluids and in particular for water waves. The

soliton theory has always been at the centre of these developments, such as from its early days when it transformed and enhanced enormously the mathematical description of nonlinear wave propagation. The simplest and best known integrable water-wave equations belong to the Korteweg-de Vries family. For some classical and modern aspects of the theory of water waves, nonlinear waves and soliton theory we refer to the following monographs and the references therein: [AS1981, Con2011, FT1987, GVY2008, HSS2009, Joh1997, New1985, PS2011, Whi1980, ZMNP1984].

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There are classes of soliton equations whose associated spectral problems are polynomial in the spectral parameter. They too are known as soliton equations with 'energy dependent potentials', due to the analogy with the Schrödinger equation in Quantum Mechanics, whose spectrum represents the energy levels of the Quantum Mechanical system. Some of these integrable systems appear as water wave models, most notably the Kaup-Boussinesq equation [Kau1975, Whi1980, EGP2001] and the two-component Camassa-Holm equation [CI2008, HSS2009, HI2011, Iva2009]. Other systems of this type are studied in e.g. [AF1989, AFL1991, Iva2006, BPZ2001, Fok1995].

In Chapter 1 it was shown that in the shallow water regime the both the two component Camassa-Holm and the Kaup-Boussinesq equation arise as approximate fluid models. In [HI2011] the inverse scattering transform was developed to construct the N -soliton solution of the CH2 equation. In this chapter, we develop a similar technique

to solve a class of partial differential equations with cubic nonlinearities, of which the Kaup-Boussinesq equation is a particular example. The work of this chapter is based on the results in [IL2012c], wherein the construction of the N -soliton solution of the Kaup-Boussinesq equation was investigated, and an explicit breather type solution for the system was also presented.

In what follows we study an integrable system which arises as a compatibility condition of the following pair of linear operators (Lax pair):

$$\Psi_{xx}(x; \lambda) = \left(-\lambda^2 + \lambda u + \frac{\kappa}{2} u^2 + \eta \right) \Psi(x; \lambda) \quad (6.1.2)$$

$$\Psi_t(x; \lambda) = - \left(\lambda + \frac{1}{2} u \right) \psi_x(x; \lambda) + \frac{1}{4} u_x \Psi(x; \lambda). \quad (6.1.3)$$

Here κ is an arbitrary constant, while λ is the spectral parameter. The functions u and η are both functions of the x and t variables, while the spectral functions Ψ are dependant on the variables x , t and the spectral parameter λ . In what follows the explicit dependence of a function will be introduced only as necessary, but it should be understood that the dependence is implicit. The consistency condition $\Psi_{xxt}(x, t; \lambda) = \Psi_{txx}(x, t; \lambda)$ produces a system of equations for the functions $u(x, t)$ and $\eta(x, t)$:

$$\begin{aligned} u_t + \eta_x + \left(\frac{3}{2} + \kappa \right) uu_x &= 0, \\ \eta_t - \frac{1}{4} u_{xxx} + (u\eta)_x - \left(\frac{1}{2} + \kappa \right) u\eta_x - \kappa \left(\frac{1}{2} + \kappa \right) u^2 u_x &= 0. \end{aligned} \quad (6.1.4)$$

Upon choosing $\kappa = -\frac{1}{2}$ these simplify to the well known Kaup-Boussinesq equation:

$$\begin{aligned} u_t + \eta_x + uu_x &= 0, \\ \eta_t - \frac{1}{4} u_{xxx} + (u\eta)_x &= 0. \end{aligned} \quad (6.1.5)$$

The Kaup-Boussinesq equation is introduced as a water-wave model in Chapter 1. We saw that in the shallow water environment that KB equation is an approximates model derived from the Euler equations. It was previously mentioned that the spectral problems of the CH2 equation and the Kaup-Boussinesq equation share several features

cf.[HI2011], and hence the introduction of the CH2 system in Chapter 1. In the current chapter, having derived the KB equation as an approximate hydrodynamical model, we now intend to obtain a solution via the inverse scattering transform. As a water-wave model the Kaup-Boussinesq equation also appears in [Whi1980, Iva2009, EGP2001], while the hierarchy of Hamiltonian structures are given in [Pav2001]. In addition specific solutions are studied in [KKU2003, EGK2005, MY1979]. In relation to the current chapter, energy-dependant spectral problems like that presented in (6.1.2) are also studied in [Jau1972, JJ1972, JJ1976, JM1976, JJ1981, Alo1980, MY1979, AS2002, SS1996, LSS2007].

In addition there is a well know transformation between the solutions of the Kaup-Boussinesq equation and the nonlinear Schrödinger equation. The transformation between NLS and the classical Boussinesq equation was examined in [Hir1985]. The NLS is given by

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} \pm |\psi|^2\psi = 0, \quad (6.1.6)$$

where ϵ is an arbitrary parameter not related to ε used elsewhere in this thesis, see [AS1981, New1985, FT1987, ZMNP1984] for further discussion. Under the Madelung transform

$$\psi = \sqrt{\eta}e^{\frac{i}{\epsilon}\int^x u d\xi}, \quad (6.1.7)$$

along with the transformation $t \rightarrow -t$, the NLS is equivalent to the following coupled system of PDE

$$\begin{aligned} \eta_t &= (u\eta)_x \\ u_t &= \partial_x \left[\frac{u^2}{2} + \eta + \epsilon^2 \left(\frac{\eta_{xx}}{4\eta} - \frac{\eta_x^2}{8\eta^2} \right) \right]. \end{aligned} \quad (6.1.8)$$

With the change of variables $u = v \pm \frac{\epsilon}{2}(\ln \eta)_x$ we find

$$\begin{aligned} \eta_t &= \left(v\eta \pm \epsilon \frac{\eta_x}{2} \right)_x \\ v_t &= \partial_x \left(\frac{v^2}{2} + \eta \mp \epsilon \frac{v_x}{2} \right), \end{aligned} \quad (6.1.9)$$

which is known as the Broer-Kaup system. Finally under the change of variable given by

$$\eta = \rho \pm \epsilon \frac{\eta_x}{2} \quad (6.1.10)$$

and under $t \rightarrow -t$, we obtain the Kaup-Boussinesq equation

$$\begin{aligned} v_t + vv_x + \rho_x &= 0 \\ \rho_t + (\rho v)_x + \frac{\epsilon^2}{4} v_{xxx} &= 0. \end{aligned} \quad (6.1.11)$$

This transformation between Kaup-Boussinesq and NLS does not preserve the reality of solutions.

In this chapter we formulate the inverse scattering transform as a Riemann-Hilbert Problem (RHP) in the case where u and η are real, rapidly decaying functions as $x \rightarrow \pm\infty$. Then taking into account the underlying reductions we obtain the simplest solution.

6.2 The spectral problem

Introducing the auxiliary function

$$w = \frac{\kappa}{2} u^2 + \eta \quad (6.2.1)$$

we consider the following two "conjugate" spectral problems related to (6.1.2):

$$\Psi_{xx}(x; \lambda; \sigma) = (-\lambda^2 + \sigma \lambda u + w) \Psi(x; \lambda; \sigma), \quad (6.2.2)$$

where $\sigma = \pm 1$. The time dependence will be suppressed where possible for the sake of brevity and clarity.

We specify that u , w and η belong to the Schwartz class of functions (the space of rapidly decreasing functions) $\mathcal{S}(\mathbb{R})$. It follows from this requirement that eigenfunc-

tions $\psi_1(x; \lambda; \sigma)$ and $\psi_2(x; \lambda; \sigma)$ exist such that

$$\left. \begin{aligned} \psi_1(x; \lambda; \sigma) &\rightarrow e^{-i\lambda x} \\ \psi_2(x; \lambda; \sigma) &\rightarrow e^{+i\lambda x} \end{aligned} \right\}, \quad x \rightarrow +\infty \quad (6.2.3)$$

Similarly we define a basis of eigenfunctions for (6.2.2) according to

$$\left. \begin{aligned} \phi_1(x; \lambda; \sigma) &\rightarrow e^{-i\lambda x} \\ \phi_2(x; \lambda; \sigma) &\rightarrow e^{+i\lambda x} \end{aligned} \right\}, \quad x \rightarrow -\infty \quad (6.2.4)$$

These eigenfunctions are called Jost Solutions. Since the Jost solutions oscillate when $\lambda \in \mathbb{R}$ the continuous spectrum fills in the real line.

The bases $\{\psi_1(x; \lambda; \sigma), \psi_2(x; \lambda; \sigma)\}$ and $\{\phi_1(x; \lambda; \sigma), \phi_2(x; \lambda; \sigma)\}$ constitute independent bases of solutions to (6.2.2) and as such we may write

$$\begin{pmatrix} \phi_1(x; \lambda; \sigma) \\ \phi_2(x; \lambda; \sigma) \end{pmatrix} = \begin{pmatrix} T_{11}(\lambda; \sigma) & T_{12}(\lambda; \sigma) \\ T_{21}(\lambda; \sigma) & T_{22}(\lambda; \sigma) \end{pmatrix} \begin{pmatrix} \psi_1(x; \lambda; \sigma) \\ \psi_2(x; \lambda; \sigma) \end{pmatrix}. \quad (6.2.5)$$

The matrix

$$\mathbf{T}(\lambda; \sigma) = \begin{pmatrix} T_{11}(\lambda; \sigma) & T_{12}(\lambda; \sigma) \\ T_{21}(\lambda; \sigma) & T_{22}(\lambda; \sigma) \end{pmatrix} \quad (6.2.6)$$

is the scattering matrix for spectral problem (6.2.2).

Under the involution $(\lambda; \sigma) \rightarrow (-\lambda; -\sigma)$, the potential in (6.2.2) remains invariant. Therefore the eigenfunctions $\psi(x; \lambda; \sigma)$ and $\psi(x; -\lambda; -\sigma)$ are solutions to the same spectral problem. Since the asymptotic behaviour of these solutions does not depend on σ it follows that

$$\begin{aligned} \psi_1(x; \lambda; \sigma) &= \psi_2(x; -\lambda; -\sigma) \\ \phi_1(x; \lambda; \sigma) &= \phi_2(x; -\lambda; -\sigma). \end{aligned} \quad (6.2.7)$$

Thus each bases may be expressed using just one of its members, for example we have $\psi(x; \lambda; \sigma) \equiv \psi_1(x; \lambda; \sigma)$ and $\phi(x; \lambda; \sigma) \equiv \phi_1(x; \lambda; \sigma)$. We also have $\psi_2(x; \lambda; \sigma) = \psi(x; -\lambda; -\sigma)$ and $\phi_2(x; \lambda; \sigma) \equiv \phi(x; -\lambda; -\sigma)$.

Since u and η are real, the spectral problem (6.2.2) is invariant under the \mathbb{Z}_2 reduction group [Mik1981], i.e. it has the following property: if $\psi(x; \lambda; \sigma)$ is an eigenfunction so is $\bar{\psi}(x; \bar{\lambda}; \sigma)$. Comparing the asymptotic behaviour of both, we conclude that this coincides with the second Jost solution, i.e.

$$\bar{\psi}(x; \bar{\lambda}; \sigma) = \psi(x; -\lambda; -\sigma). \quad (6.2.8)$$

Thus for $\lambda \in \mathbb{R}$ we have $\psi(x; \lambda; \sigma) = \bar{\psi}(x; -\lambda; -\sigma)$. With this and (6.2.5), it follows that the scattering matrix $\mathbf{T}(\lambda; \sigma)$ may be written in the form

$$\mathbf{T}(\lambda; \sigma) = \begin{pmatrix} a(\lambda; \sigma) & b(\lambda; \sigma) \\ \bar{b}(\lambda; \sigma) & \bar{a}(\lambda; \sigma) \end{pmatrix}, \quad (6.2.9)$$

where $\lambda \in \mathbb{R}$.

We may now write the following relationship between $\phi(x; \lambda; \sigma)$ and the Jost solutions $\psi(x; \lambda; \sigma), \bar{\psi}(x; \bar{\lambda}; \sigma)$:

$$\phi(x; \lambda; \sigma) = a(\lambda; \sigma)\psi(x; \lambda; \sigma) + b(\lambda; \sigma)\bar{\psi}(x; \bar{\lambda}; \sigma). \quad (6.2.10)$$

Furthermore for any pair of solutions $f_1(x; \lambda; \sigma)$ and $f_2(x; \lambda; \sigma)$ of equation (6.2.2), the Wronskian of the pair is independent of x

$$\frac{d}{dx} \mathbf{W}[f_1, f_2] = \partial_x(f_1 \partial_x f_2 - f_2 \partial_x f_1) = 0.$$

In particular it follows that the Jost solutions satisfy the following condition

$$\mathbf{W}[\phi(x; \lambda; \sigma), \bar{\phi}(x; \bar{\lambda}; \sigma)] = \mathbf{W}[\psi(x; \lambda; \sigma), \bar{\psi}(x; \bar{\lambda}; \sigma)] = 2i\lambda, \quad (6.2.11)$$

which is a result of the asymptotic behaviour of $\{\psi, \bar{\psi}\}$ and $\{\phi, \bar{\phi}\}$ as $|x| \rightarrow \infty$. It follows from (6.2.10) and (6.2.11) that

$$\det \mathbf{T}(\lambda; \sigma) = |a(\lambda; \sigma)|^2 - |b(\lambda; \sigma)|^2 = 1, \quad \lambda \in \mathbb{R}. \quad (6.2.12)$$

6.3 Asymptotic behaviour of the Jost solutions

Since the functions u and w are Schwartz class it follows that the Jost solutions $\psi(x; \lambda; \sigma)$ have asymptotic behaviour such that

$$\psi_{xx}(x; \lambda; \sigma) \rightarrow -\lambda^2 e^{-i\lambda x}, \quad x \rightarrow +\infty. \quad (6.3.1)$$

Consequently we make the following ansatz for the asymptotic expansion of the Jost solutions as $|\lambda| \rightarrow \infty$, namely

$$\psi(x; \lambda; \sigma) = \left[X_0(x; \sigma) + \frac{1}{\lambda} X_1(x; \sigma) + \mathcal{O}(\lambda^{-2}) \right] e^{-i\lambda x}. \quad (6.3.2)$$

The asymptotic behaviour of the Jost solutions requires that the functions $X_0(x; \sigma)$ and $X_1(x; \sigma)$ have asymptotic behaviour

$$\left. \begin{aligned} X_0(x; \sigma) &\rightarrow 1 \\ X_1(x; \sigma) &\rightarrow 0 \end{aligned} \right\}, \quad x \rightarrow +\infty. \quad (6.3.3)$$

The substitution of (6.3.2) into (6.2.2) gives

$$\frac{\partial_x X_0(x; \sigma)}{X_0(x; \sigma)} = \frac{i}{2} \sigma u, \quad (6.3.4)$$

$$\sigma u X_1(x; \sigma) + 2i \partial_x X_1(x; \sigma) = -w \cdot X_0(x; \sigma) + \partial_x^2 X_0(x; \sigma). \quad (6.3.5)$$

Using the conditions in (6.3.3) we readily solve (6.3.4) and (6.3.5) to obtain the following expressions for $X_0(x; \sigma)$ and $X_1(x; \sigma)$:

$$\begin{aligned} X_0(x; \sigma) &= \exp \left\{ -\frac{i\sigma}{2} \int_x^\infty u dx' \right\}, \\ X_1(x; \sigma) &= X_0(x; \sigma) \cdot \left[\frac{\sigma}{4} u - \frac{i}{8} \int_x^\infty (u^2 + 4w) dx' \right]. \end{aligned} \quad (6.3.6)$$

Similarly the analogous expression for $\phi(x; \lambda; \sigma)$ is given by

$$\psi(x; \lambda; \sigma) = e^{-i(\lambda x + \frac{\sigma}{2} \int_x^\infty u dx')} \left[1 + \frac{1}{\lambda} \xi_1(x, \sigma) + \dots \right], \quad (6.3.7)$$

$$\phi(x; \lambda; \sigma) = e^{-i(\lambda x - \frac{\sigma}{2} \int_{-\infty}^x u dx')} \left[1 + \frac{1}{\lambda} \zeta_1(x, \sigma) + \dots \right], \quad (6.3.8)$$

where the functions ξ and ζ

$$\begin{aligned}\xi_1(x, \sigma) &= \frac{\sigma}{4}u - \frac{i}{8} \int_x^\infty (u^2 + 4w)dx', \\ \zeta_1(x, \sigma) &= \frac{\sigma}{4}u + \frac{i}{8} \int_{-\infty}^x (u^2 + 4w)dx',\end{aligned}\tag{6.3.9}$$

are introduced for later convenience.

6.4 Analytic behaviour of the Jost solutions

To determine the analytic behaviour of the Jost solutions we introduce the modified Jost solutions as follows

$$\chi^{(+)}(x; \lambda; \sigma) = e^{i\lambda x} \phi(x; \lambda; \sigma) \rightarrow 1, \quad \text{as } x \rightarrow -\infty.\tag{6.4.1}$$

Differentiating (6.4.1) once with respect to x we find

$$e^{i\lambda x} \phi_x(x; \lambda; \sigma) = \chi_x^{(+)}(x; \lambda; \sigma) - i\lambda \chi^{(+)}(x; \lambda; \sigma).$$

Combined with the spectral problem in (6.2.2), it follows that

$$\chi_{xx}^{(+)}(x; \lambda; \sigma) = (\lambda\sigma u + w)\chi^{(+)}(x; \lambda; \sigma) + 2i\lambda \chi_x^{(+)}(x; \lambda; \sigma).\tag{6.4.2}$$

The spectral problem (6.4.2) and the asymptotic expansion in λ appearing in (6.3.8) suggest the following integral representation for $\chi^{(+)}(x; \lambda; \sigma)$

$$\chi^{(+)}(x; \lambda; \sigma) = 1 + \int_{-\infty}^x \frac{e^{2i\lambda(x-x')} - 1}{2i\lambda} P(x'; \lambda; \sigma) \chi^{(+)}(x'; \lambda; \sigma) dx'.\tag{6.4.3}$$

The kernel $P(x; \lambda; \sigma)$ is defined such that $P(x; \lambda; \sigma) \in \mathcal{S}(\mathbb{R})$.

Differentiating this integral representation twice with respect to x we obtain

$$\chi_{xx}^{(+)}(x; \lambda; \sigma) = P(x; \lambda; \sigma) \chi^{(+)}(x; \lambda; \sigma) + 2i\lambda \chi_x^{(+)}(x; \lambda; \sigma).\tag{6.4.4}$$

Combining (6.4.2) and (6.4.4) we determine

$$P(x; \lambda; \sigma) = \lambda\sigma u + w.\tag{6.4.5}$$

Thus we may write

$$\chi^{(+)}(x; \lambda; \sigma) = 1 + \int_{-\infty}^x \frac{e^{2i\lambda(x-x')} - 1}{2i\lambda} (\lambda\sigma u + w) \chi^{(+)}(x'; \lambda; \sigma) dx', \quad (6.4.6)$$

leaving us with an integral representation for $\chi^{(+)}(x; \lambda; \sigma)$ in terms of functions whose analytic properties are obvious. The analyticity of $\chi^{(+)}(x; \lambda; \sigma)$ is of particular importance and clear from (6.4.6). We see that for all values of x , the kernel of the integral above is finite for all values of λ when $\text{Im}\lambda > 0$. Therefore $\chi^{(+)}(x; \lambda; \sigma)$ and $\phi(x; \lambda; \sigma)$ are analytic in the upper half-plane \mathbb{C}_+ . It follows that $\bar{\chi}^{(+)}(x; \bar{\lambda}; \sigma)$ is analytic for $\lambda \in \mathbb{C}_-$.

In a similar manner we define

$$\chi^{(-)}(x; \lambda; \sigma) = e^{i\lambda x} \psi(x; \lambda; \sigma) \rightarrow 1, \quad \text{as } x \rightarrow +\infty \quad (6.4.7)$$

from which it follows ultimately that

$$\chi^{(-)}(x; \lambda; \sigma) = 1 - \int_x^{\infty} \frac{e^{2i\lambda(x-x')} - 1}{2i\lambda} (\lambda\sigma u + w) \chi^{(-)}(x'; \lambda; \sigma) dx'. \quad (6.4.8)$$

It is immediately clear from (6.4.8) that $\chi^{(-)}(x; \lambda; \sigma)$ and therefore $\psi(x; \lambda; \sigma)$ are analytic throughout \mathbb{C}_- .

We introduce a new variable for later convenience, namely

$$\omega_- = \frac{1}{2} \int_{-\infty}^x u dx' \quad \text{and} \quad \omega_+ = \frac{1}{2} \int_x^{\infty} u dx'. \quad (6.4.9)$$

The asymptotic expansion of the bases of Jost solutions given in (6.3.7) and (6.3.8) can be rewritten using (6.4.9), with

$$\begin{aligned} \underline{\psi}(x; \lambda; \sigma) &= \psi(x; \lambda; \sigma) e^{i(\lambda x + \sigma \omega_+(x))} = 1 + \frac{1}{\lambda} \xi_1(x; \sigma), \\ \underline{\phi}(x; \lambda; \sigma) &= \phi(x; \lambda; \sigma) e^{i(\lambda x - \sigma \omega_-(x))} = 1 + \frac{1}{\lambda} \zeta_1(x; \sigma). \end{aligned} \quad (6.4.10)$$

Meanwhile to obtain the analytic properties of $\underline{\psi}(x; \lambda; \sigma)$ and $\underline{\phi}(x; \lambda; \sigma)$, we note that

$$\underline{\psi}(x; \lambda; \sigma) = \chi^{(-)}(x; \lambda; \sigma) e^{i\sigma \omega_+} \quad \text{and} \quad \underline{\phi}(x; \lambda; \sigma) = \chi^{(+)}(x; \lambda; \sigma) e^{-i\sigma \omega_-},$$

which expresses $\underline{\phi}$ and $\underline{\psi}$ in terms of functions whose analytic properties are obvious. The function u has been restricted to $\mathcal{S}(\mathbb{R})$ and furthermore is independent of λ . The analytic behaviour of $\chi^{(\pm)}(x; \lambda; \sigma)$ throughout \mathbb{C}_+ & \mathbb{C}_- have already been determined, and so it follows that, $\underline{\phi}(x; \lambda; \sigma)$ and $\underline{\psi}(x; \lambda; \sigma)$ are also analytic throughout \mathbb{C}_+ and \mathbb{C}_- respectively.

6.5 Time dependence of the scattering data

We may rewrite the time component of the spectral problem in terms of u and in addition add a term with a constant factor γ , without effecting the outcome of the consistency condition. With these changes implemented the spectral problem becomes

$$\Psi_t(x; \lambda; \sigma) = - \left(\sigma\lambda + \frac{1}{2}u \right) \Psi_x(x; \lambda; \sigma) + \left(\gamma + \frac{1}{4}u_x \right) \Psi(x; \lambda; \sigma). \quad (6.5.1)$$

Specifically the Jost solution $\phi(x; \lambda; \sigma)$ satisfies

$$\phi_t(x; \lambda; \sigma) = - \left(\sigma\lambda + \frac{1}{2}u \right) \phi_x(x; \lambda; \sigma) + \left(\gamma + \frac{1}{4}u_x \right) \phi(x; \lambda; \sigma). \quad (6.5.2)$$

However we also note that along the continuous spectrum we have the scattering relation (6.2.10), from which the asymptotic behavior of $\lim_{x \rightarrow +\infty} \phi_t(x; \lambda; \sigma)$ follows, namely

$$\phi_t(x; \lambda; \sigma) \rightarrow a_t(\lambda; \sigma)e^{-i\lambda x} + b_t(\lambda; \sigma)e^{+i\lambda x}. \quad (6.5.3)$$

Replacing (6.2.10) on the right hand side of (6.5.2) and letting $x \rightarrow +\infty$, we find

$$\begin{aligned} \phi_t(x; \lambda; \sigma) \rightarrow & -\sigma\lambda[-i\lambda a(\lambda; \sigma)e^{-i\lambda x} + i\lambda b(\lambda; \sigma)e^{i\lambda x}] \\ & + \gamma[a(\lambda; \sigma)e^{-i\lambda x} + b(\lambda; \sigma)e^{+i\lambda x}], \end{aligned} \quad (6.5.4)$$

where we have made use of the fact that u is Schwartz class and vanishes as $x \rightarrow \pm\infty$. As in the case of the KdV spectral problem, the scattering coefficient $a(\lambda; \sigma)$ is independent of time and as such $\gamma = -i\sigma\lambda^2$, ensuring the time derivative of $a(\lambda; \sigma)$ vanishes.

The time evolution of the two scattering functions $a(\lambda; \sigma)$ and $b(\lambda; \sigma)$ is governed by the following pair of ODE

$$\begin{aligned} a_t(\lambda; \sigma) = 0 &\Rightarrow a(\lambda; \sigma) = a_0(\lambda; \sigma), \\ b_t(\lambda; \sigma) = -2i\lambda^2 b(\lambda; \sigma) &\Rightarrow b(\lambda; \sigma) = b_0(\lambda; \sigma)e^{-2i\sigma\lambda^2 t}. \end{aligned} \quad (6.5.5)$$

where we define

$$a_0(\lambda; \sigma) = a(\lambda; \sigma)|_{t=0} \quad b_0(\lambda; \sigma) = b(\lambda; \sigma)|_{t=0}. \quad (6.5.6)$$

Along the discrete spectrum we have $a(\lambda_n, \sigma) = 0$, and so as with the KdV spectral problem, relation (6.5.3) along with the x -independence of the Wronskian yields

$$\phi(x; \lambda_n; \sigma) = b_n \bar{\psi}(x; \bar{\lambda}_n; \sigma),$$

where b_n is independent of x . It follows that

$$\lim_{x \rightarrow \infty} \phi_t(x; \lambda_n; \sigma) = b_{n,t}(\sigma) e^{i\lambda_n x}. \quad (6.5.7)$$

Analogously instead of (6.5.4) we have

$$\lim_{x \rightarrow +\infty} \phi_t(x; \lambda_n; \sigma) = -\sigma \lambda_n [i\lambda_n b_n(\sigma) e^{i\lambda_n x}] + \gamma(\lambda_n) b_n(\sigma) e^{i\lambda_n x}, \quad (6.5.8)$$

and as such

$$b_{n,t}(\sigma) = -2i\sigma \lambda_n^2 b_n(\sigma). \quad (6.5.9)$$

Consequently

$$b_n(\sigma, t) = b_n(\sigma, 0) e^{-2i\sigma \lambda_n^2 t}, \quad (6.5.10)$$

describes the time evolution of the coefficient scattering b_n . With these two results the time dependence of all the scattering data is available to us.

6.6 Conservation Laws

In order to solve the inverse problem, that is to obtain $u(x, t)$ and $\eta(x, t)$ from the scattering data, it is necessary to obtain a series of conservation laws from the spectral problem (6.1.2). However it will prove more convenient to derive these same conservation laws from (6.2.2). To proceed we define the function

$$\rho(x; \lambda; \sigma) = \frac{\psi_x(x; \lambda; \sigma)}{\psi(x; \lambda; \sigma)}. \quad (6.6.1)$$

Differentiating once with respect to x we find

$$\rho^2(x; \lambda; \sigma) + \rho_x(x; \lambda; \sigma) = -\lambda^2 + \sigma \lambda u + w. \quad (6.6.2)$$

Using this result along with the Lax pair in (6.5.1), we find upon differentiating (6.6.1) with respect to t that

$$\rho_t(x; \lambda; \sigma) = \left[\frac{1}{4}u_x - \left(\sigma \lambda + \frac{1}{2}u \right) \rho(x; \lambda; \sigma) \right]_x. \quad (6.6.3)$$

Using the fact u and w are Schwartz class, we see from (6.6.3) that

$$\int_{-\infty}^{+\infty} \rho_t(x; \lambda; \sigma) dx = -\sigma \lambda \rho(x; \lambda; \sigma) \Big|_{x=-\infty}^{x=+\infty} = 0, \quad (6.6.4)$$

that is to say

$$\mathcal{I}(\lambda; \sigma) = \int_{-\infty}^{+\infty} \rho(x; \lambda; \sigma) dx \quad (6.6.5)$$

is a generating function for the conserved quantities. We may write it as a power series in λ according to

$$\mathcal{I}(\lambda; \sigma) = \sum_{n=1}^{\infty} \lambda^{n-2} \mathcal{I}_n(\sigma), \quad (6.6.6)$$

where $\mathcal{I}_1(\sigma)$, $\mathcal{I}_2(\sigma)$, etc. are individually conserved quantities. Next we write a series expansion in λ for $\rho(x; \lambda; \sigma)$ as follows

$$\rho(x; \lambda; \sigma) = \sum_{n=0}^{\infty} \lambda^{-n} \rho_n(x; \sigma), \quad (6.6.7)$$

and replace it in (6.6.2). Comparing the terms of equivalent order in λ in each expression gives to leading order

$$\rho_0(x; \sigma) = -\frac{i\sigma}{2}u. \quad (6.6.8)$$

As a result of (6.6.6) it follows that

$$\mathcal{I}_2(\sigma) = \int_{-\infty}^{\infty} \rho_0(x; \lambda; \sigma) dx = -\frac{i\sigma}{2} \int_{-\infty}^{\infty} u dx. \quad (6.6.9)$$

So we see that

$$\alpha_1 \equiv \frac{1}{2} \int_{-\infty}^{\infty} u dx \quad (6.6.10)$$

is an integral of motion. Following a similar procedure we find the next conserved quantities to be

$$\begin{aligned} \mathcal{I}_3(\sigma) &= -\frac{i}{8} \int_{-\infty}^{+\infty} (u^2 + 4w) dx, \\ \mathcal{I}_4(\sigma) &= -\frac{i\sigma}{16} \int_{-\infty}^{+\infty} u(u^2 + 4w) dx. \end{aligned} \quad (6.6.11)$$

One may continue a process of iteration indefinitely, whereby an infinite series of such conserved quantities is generated from the u and w , and therefore from the physical variables u and η .

6.7 Analytic continuation of $a(\lambda; \sigma)$

Returning to (6.2.10) we see that we may re-write the scattering coefficient $a(\lambda; \sigma)$ in terms of the x -independent Wronskian, that is

$$a(\lambda; \sigma) = \frac{W[\phi(x; \lambda; \sigma), \psi(x; -\lambda; -\sigma)]}{2i\lambda}. \quad (6.7.1)$$

Since the two eigenfunctions in (6.7.1) are analytic for $\lambda \in \mathbb{C}_+$, $a(\lambda; \sigma)$ allows an analytic continuation in the upper half complex plane. From (6.7.1) with (6.3.7) and (6.3.8) we obtain the asymptotic behaviour of the scattering coefficient,

$$\lim_{|\lambda| \rightarrow \infty} a(\lambda; \sigma) = e^{i\sigma\alpha_1} \quad (6.7.2)$$

where α_1 is the conserved quantity in (6.6.10). We make the further assumption that $a(\lambda; \sigma)$ has a finite number of *simple* zeros $\lambda_n \in \mathbb{C}_+$, $n = 1, 2, 3, \dots, N$. This assumption is of course an additional restriction to the classes of the possible solutions. Our experience with a similar (but simpler) weighted spectral problem associated to the Camassa-Holm equation [BSS1998, CM1999, Con2011, MKST2009] shows that infinitely many zeros are possible. However, in our examples we will confine ourselves with considering only finitely many zeros.

We introduce the auxiliary function

$$A(\lambda; \sigma) = e^{-i\sigma\alpha_1} \prod_{n=1}^N \frac{\lambda - \bar{\lambda}_n}{\lambda - \lambda_n} a(\lambda, \sigma), \quad (6.7.3)$$

which is analytic and without zeroes in \mathbb{C}_+ . It follows from (6.7.3) that

$$|A(\lambda; \sigma)| = |a(\lambda; \sigma)|, \quad \lambda \in \mathbb{R}, \quad (6.7.4)$$

Next, we also see from (6.7.2) and (6.7.3) that

$$\lim_{|\lambda| \rightarrow \infty} \ln A(\lambda; \sigma) = 0, \quad (6.7.5)$$

and so, $\ln A(\lambda; \sigma)$ is analytic throughout \mathbb{C}_+ and vanishes as $|\lambda| \rightarrow \infty$.

We also have from (6.7.4);

$$\ln A(\lambda; \sigma) = \ln |A(\lambda; \sigma)| + i \arg A(\lambda; \sigma) = \ln |a(\lambda; \sigma)| + i \arg A(\lambda; \sigma),$$

for $\lambda \in \mathbb{R}$. We make use of the Kramers-Kronig dispersion relations,

$$\begin{aligned} \ln |a(\lambda; \sigma)| &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\arg A(\lambda'; \sigma)}{\lambda' - \lambda} d\lambda' \\ \arg A(\lambda; \sigma) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(\lambda'; \sigma)|}{\lambda' - \lambda} d\lambda', \end{aligned} \quad (6.7.6)$$

for $\lambda \in \mathbb{R}$, where the dashed integral denoted the principal value part of the integral.

Then from (6.7.6) we have

$$\begin{aligned} \ln A(\lambda; \sigma) &= \ln |a(\lambda, \sigma)| - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\ln |a(\lambda'; \sigma)|}{\lambda' - \lambda} d\lambda' \\ &= \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(\lambda'; \sigma)|}{\lambda' - \lambda - i0^+} d\lambda', \quad \lambda \in \mathbb{R}. \end{aligned} \quad (6.7.7)$$

Meanwhile, (6.7.3) gives

$$\ln A(\lambda; \sigma) = -i\sigma\alpha_1 - \sum_{n=1}^N \frac{\lambda - \lambda_n}{\lambda - \bar{\lambda}_n} + \ln a(\lambda; \sigma). \quad (6.7.8)$$

Using (6.7.7) and (6.7.8), we find that for real values of λ we may write

$$\ln a(\lambda; \sigma) = i\sigma\alpha_1 + \sum_{n=1}^N \frac{\lambda - \lambda_n}{\lambda - \bar{\lambda}_n} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(\lambda'; \sigma)|}{\lambda' - \lambda - i0^+} d\lambda' \quad (6.7.9)$$

and for $\lambda \in \mathbb{C}_+$ the analytical continuation is

$$\ln a(\lambda; \sigma) = i\sigma\alpha_1 + \sum_{n=1}^N \frac{\lambda - \lambda_n}{\lambda - \bar{\lambda}_n} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\ln |a(\lambda'; \sigma)|}{\lambda' - \lambda} d\lambda'. \quad (6.7.10)$$

6.8 The Riemann-Hilbert problem

For $\lambda \in \mathbb{R}$ we may use (6.2.8) to re-write the expression (6.2.10) in terms of the new analytic functions $\underline{\phi}(x; \lambda; \sigma)$ and $\underline{\psi}(x; \lambda; \sigma)$ to find

$$\frac{\underline{\phi}(x; \lambda; \sigma)e^{i\sigma\alpha_1}}{a(\lambda; \sigma)} = \underline{\psi}(x; \lambda; \sigma) + r(\lambda; \sigma)\bar{\underline{\psi}}(x; \lambda; \sigma)e^{2i(\lambda x + \sigma\omega_+)}, \quad (6.8.1)$$

where $r(\lambda; \sigma) = b(\lambda; \sigma)/a(\lambda; \sigma)$. The function $\frac{\underline{\phi}(x; \lambda; \sigma)e^{i\sigma\alpha_1}}{a(\lambda; \sigma)}$ is analytic for $\text{Im } \lambda > 0$ (except for the finitely many simple poles at λ_n), while $\underline{\psi}(x; \lambda; \sigma)$ is analytic for $\text{Im } \lambda < 0$. Thus, equation (6.8.1) represents an additive Riemann-Hilbert Problem (RHP) with a jump on the real line, given by

$$r(\lambda; \sigma)\bar{\underline{\psi}}(x; \lambda; \sigma)e^{2i(\lambda x + \sigma\omega_+)}$$

and a normalization condition $\lim_{|\lambda| \rightarrow \infty} \underline{\psi}(x; \lambda; \sigma) = 1$.

In this section we will follow the standard technique for solving RHP. We integrate the two analytic functions with respect to $\oint \frac{d\lambda'}{\lambda' - \lambda}(\cdot)$ over the boundary of their analyticity domains using the normalization condition. In our case the domains (the upper \mathbb{C}_+ and the lower \mathbb{C}_- complex half-planes) have the real line as a common boundary and there we relate the integrals using the jump condition. The RHP approach for various equation is presented in [GVY2008, HI2011, LSS2007, SS1996].

We now choose some $\lambda \in \mathbb{C}_-$ and integrate the left-hand side as follows

$$\frac{1}{2\pi i} \oint_{C_+} \frac{\underline{\phi}(x; \lambda'; \sigma) e^{i\sigma\alpha_1}}{a(\lambda'; \sigma) \cdot (\lambda' - \lambda)} d\lambda' = \sum_{n=1}^N \frac{\underline{\phi}^{(n)}(x; \sigma) e^{i\sigma\alpha_1}}{\dot{a}_n(\sigma) \cdot (\lambda_n - \lambda)} \quad (6.8.2)$$

where C_+ is the contour in the upper half plane shown in Fig. 1,

$$\dot{a}_n(\sigma) \equiv \left. \frac{da(\lambda; \sigma)}{d\lambda} \right|_{\lambda=\lambda_n} \neq 0, \quad \underline{\phi}^{(n)}(x; \sigma) \equiv \underline{\phi}(x; \lambda_n; \sigma).$$

We may write the integral as such because $\lambda \in \mathbb{C}_-$ and so $\frac{1}{\lambda - \lambda'}$ is analytic throughout \mathbb{C}_+ . Furthermore, $a(\lambda; \sigma)$ is analytic with finite number of simple zeros, λ_n in \mathbb{C}_+ , and the function $\underline{\phi}(x; \lambda; \sigma)$ is analytic throughout \mathbb{C}_+ . Alternatively we may expand the integral as follows

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{C_+} \frac{\underline{\phi}(x; \lambda'; \sigma) e^{i\sigma\alpha_1}}{a(\lambda'; \sigma) \cdot (\lambda' - \lambda)} d\lambda' = \\ & \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\underline{\phi}(x; \lambda'; \sigma) e^{i\sigma\alpha_1}}{a(\lambda'; \sigma) \cdot (\lambda' - \lambda)} d\lambda' + \frac{1}{2\pi i} \int_{\Gamma_+} \frac{\underline{\phi}(x; \lambda'; \sigma) e^{i\sigma\alpha_1}}{a(\lambda', \sigma) \cdot (\lambda' - \lambda)} d\lambda'. \end{aligned} \quad (6.8.3)$$

Using the asymptotic properties of $a(\lambda; \sigma)$ and $\underline{\phi}(x; \lambda; \sigma)$ along with the relationship (6.8.1), we find

$$\begin{aligned} \sum_{n=1}^N \frac{\underline{\phi}^{(n)}(x; \sigma) e^{i\sigma\alpha_1}}{\dot{a}(\lambda_n) \cdot (\lambda_n - \lambda)} &= \frac{1}{2\pi i} \int_{\Gamma_+} \frac{1}{\lambda' - \lambda} d\lambda' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\underline{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda' \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(\lambda'; \sigma) e^{2i(\lambda'x + \sigma\omega_+)} \bar{\underline{\psi}}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda'. \end{aligned}$$

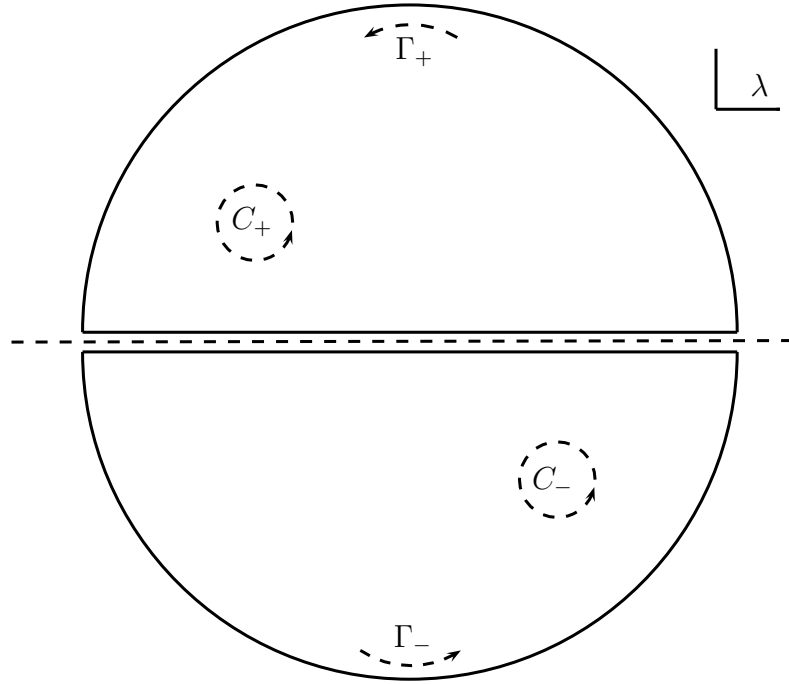


Figure 6.1: The integration contours C_{\pm} are the closed paths in the upper and lower half planes while Γ_{\pm} are the semicircles with infinite radii.

Next we obtain an expression for the line-integral

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\underline{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda',$$

by considering the integral over the contour C_- , shown in Fig. 1. Since C_- is counter-clockwise and $\lambda \in \mathbb{C}_-$ with $\underline{\psi}(x; \lambda; \sigma)$ analytic therein it follows that

$$\frac{1}{2\pi i} \oint_{C_-} \frac{\underline{\psi}(x; \lambda; \sigma)}{\lambda' - \lambda} d\lambda' = \underline{\psi}(x; \lambda; \sigma). \quad (6.8.4)$$

Expanding the integral, we have

$$\frac{1}{2\pi i} \oint_{C_-} \frac{\underline{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda' = \frac{1}{2\pi i} \int_{-\infty}^{-\infty} \frac{\underline{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda' + \frac{1}{2\pi i} \int_{\Gamma_-} \frac{\underline{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda'. \quad (6.8.5)$$

Using the asymptotic properties of $\underline{\psi}(x, \lambda, \sigma)$ as $|\lambda| \rightarrow \infty$ with $\lambda \in \mathbb{C}_-$, we have

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\underline{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda' = \underline{\psi}(x; \lambda; \sigma) - \frac{1}{2\pi i} \int_{\Gamma_-} \frac{1}{\lambda' - \lambda} d\lambda' \quad (6.8.6)$$

We can also make use of the following result when it comes to substituting this expression in (6.8.4)

$$\int_{\Gamma_+} \frac{1}{\lambda' - \lambda} d\lambda' + \int_{\Gamma_-} \frac{1}{\lambda' - \lambda} d\lambda' = 2\pi i.$$

Upon making these substitutions we find the following integral representation for $\underline{\psi}(x; \lambda; \sigma)$, $\lambda \in \mathbb{C}_-$:

$$\underline{\psi}(x; \lambda; \sigma) = 1 - \sum_{n=1}^N \frac{\phi^{(n)}(x; \sigma) e^{i\alpha_1}}{\dot{a}_n(\sigma)(\lambda_n - \lambda)} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(\lambda') e^{2i(\lambda'x + \sigma\omega_+)} \bar{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} d\lambda'.$$

At the points of the discrete spectrum $\phi(x; \lambda_n; \sigma) = b_n(\sigma) \bar{\psi}(x; \bar{\lambda}_n; \sigma)$ we have

$$\frac{\phi^{(n)}(x; \sigma) e^{i\alpha_1}}{\dot{a}_n(\sigma)} = iR_n(\sigma) e^{2i(\lambda_n x + \sigma\omega_+)} \bar{\psi}(x; \bar{\lambda}_n; \sigma) \quad (6.8.7)$$

where we define

$$R_n(\sigma) = \frac{b_n(\sigma)}{i\dot{a}_n(\sigma)}.$$

The Riemann-Hilbert problem is reduced to the linear singular integral equation for $\underline{\psi}(x; \lambda; \sigma)$

$$\begin{aligned} \underline{\psi}(x; \lambda; \sigma) &= 1 - i \sum_{n=1}^N \frac{R_n(\sigma) \bar{\psi}(x; \bar{\lambda}_n; \sigma)}{(\lambda_n - \lambda)} e^{2i(\lambda_n x + \sigma\omega_+)} \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(\lambda'; \sigma) \bar{\psi}(x; \lambda'; \sigma)}{\lambda' - \lambda} e^{2i(\lambda' x + \sigma\omega_+)} d\lambda'. \end{aligned} \quad (6.8.8)$$

In addition to (6.8.8) we have an analogous system written at the points $\lambda = \bar{\lambda}_p \in \mathbb{C}_-$, $p = 1, 2, \dots, N$:

$$\begin{aligned} \underline{\psi}(x; \bar{\lambda}_p; \sigma) &= 1 - i \sum_{n=1}^N \frac{R_n(\sigma) \bar{\psi}(x; \bar{\lambda}_n; \sigma)}{(\lambda_n - \bar{\lambda}_p)} e^{2i(\lambda_n x + \sigma\omega_+)} \\ &+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{r(\lambda'; \sigma) \bar{\psi}(x; \lambda'; \sigma)}{\lambda' - \bar{\lambda}_p} e^{2i(\lambda' x + \sigma\omega_+)} d\lambda'. \end{aligned} \quad (6.8.9)$$

Finally, the fact that at $\lambda = 0$ the Jost solution $\psi(x; 0; \sigma)$ does not depend on σ gives $\psi(x; 0; \sigma) = \psi(x; 0; -\sigma)$ or an algebraic equation for $e^{2i\sigma\omega_+}$, namely

$$e^{2i\sigma\omega_+} = \frac{\underline{\psi}(x; 0; \sigma)}{\underline{\psi}(x; 0; -\sigma)} = \frac{\overline{\psi}(x; 0; \sigma)}{\overline{\psi}(x; 0; \sigma)}. \quad (6.8.10)$$

The system (6.8.8), (6.8.9) and (6.8.10) allows for the determination of both the Jost solution and the potential functions of the spectral problem in terms of the scattering data. Note that the time-dependence of the scattering data is known from (6.5.5), (6.5.10):

$$r(\lambda; \sigma) = r_0(\lambda\sigma)e^{-2i\sigma\lambda^2 t}, \quad R_n(\sigma) = R_n^0(\sigma)e^{-2i\sigma\lambda_n^2 t}. \quad (6.8.11)$$

Thus the complete set of scattering data is

$$r(\lambda; \sigma), \quad \lambda_n, \quad R_n(\sigma) \quad (n = 1, 2, \dots, N). \quad (6.8.12)$$

Also it is sufficient to know the scattering data for $\sigma = 1$, because of the \mathbb{Z}_2 involution, which holds on the scattering data too:

$$r(\lambda; -\sigma) = \bar{r}(-\lambda; \sigma), \quad R_n(-\sigma) = \bar{R}_n(\sigma). \quad (6.8.13)$$

6.9 Reflectionless potentials and soliton solutions

The so-called reflectionless potentials are a subclass which corresponds to a restricted set of scattering data: $r(\lambda; \sigma) = 0$; $\lambda \in \mathbb{R}$. Then the system (6.8.8), (6.8.9), (6.8.10) is algebraic, and the solutions of the PDE are called solitons.

The simplest case is the $N = 1$ -soliton solution, so we start with this case first. From (6.8.8) we have

$$\underline{\psi}(x; \lambda; \sigma) = 1 - i \frac{R_1(\sigma) \bar{\underline{\psi}}(x; \bar{\lambda}_1; \sigma)}{\lambda_1 - \lambda} e^{2i(\lambda_1 x + \sigma\omega_+)}, \quad \lambda \in \mathbb{C}_-. \quad (6.9.1)$$

We notice that $\bar{\underline{\psi}}(x; \bar{\lambda}; \sigma)$ has an unique pole at $\bar{\lambda}_1$ and $\underline{\psi}(x; -\lambda; -\sigma)$ has an unique pole at $-\lambda_1$. Due to (6.2.8) these two poles coincide, i.e. $\bar{\lambda}_1 = -\lambda_1$ and therefore $\lambda_1 = i\nu$ is purely imaginary, $\nu > 0$ is real.

Solving for $\underline{\psi}(x, \bar{\lambda}_1, \sigma)$ we find

$$\underline{\psi}(x; \bar{\lambda}_1; \sigma) = \frac{1 - i \frac{\bar{R}_1(\sigma)}{2\lambda_1} e^{2i(\bar{\lambda}_1 x + \sigma \omega_+)}}{1 + \frac{|R_1(\sigma)|^2 e^{4i\lambda_1 x}}{4\lambda_1^2}}. \quad (6.9.2)$$

Then (6.9.1) takes the form

$$\underline{\psi}(x; \lambda; \sigma) = 1 + \frac{2i\nu}{\lambda - i\nu} \cdot \frac{\frac{R_1^0(\sigma)}{2\nu} e^{-2\nu x + 2i\sigma\nu^2 t + 2i\sigma\omega_+} - \frac{|R_1^0(\sigma)|^2}{4\nu^2} e^{-4\nu x}}{1 - \frac{|R_1^0(\sigma)|^2}{4\nu^2} e^{-4\nu x}} \quad (6.9.3)$$

Furthermore we can relate the real and imaginary parts of the complex constant $\frac{R_1^0(\sigma)}{2\nu}$ to two new constants, say x_0 and t_0 as follows:

$$\frac{R_1^0(\sigma)}{2\nu} = e^{4\nu x_0 - 2i\sigma\nu^2 t_0}.$$

Now $\underline{\psi}(x; \lambda; \sigma)$ in (6.9.3) depends only on $x - x_0$ and $t - t_0$. Due to the translational invariance of the problem and without loss of generality we can choose $x_0 = 0$ and $t_0 = 0$. This simplifies (6.9.3) which yields

$$\underline{\psi}(x; \lambda; \sigma) = 1 + \frac{2i\nu}{\lambda - i\nu} \cdot \frac{e^{-2\nu x + 2i\sigma\nu^2 t + 2i\sigma\omega_+} - e^{-4\nu x}}{1 - e^{-4\nu x}} \quad (6.9.4)$$

Then (6.8.10) gives

$$e^{2i\sigma\omega_+} = \frac{1 + 2e^{-2\nu x - 2i\sigma\nu^2 t} + e^{-4\nu x}}{1 + 2e^{-2\nu x + 2i\sigma\nu^2 t} + e^{-4\nu x}} \quad (6.9.5)$$

Using (6.4.9) we can recover $u(x)$ from ω_+ :

$$u = \nu \frac{\sin(2\nu^2 t) \sinh(2\nu x)}{\cosh^4(\nu x) \cos^2(\nu^2 t) + \sinh^4(\nu x) \sin^2(\nu^2 t)} \quad (6.9.6)$$

On the other hand, we also have (6.4.10),

$$\underline{\psi}(x; \lambda; \sigma) = 1 + \frac{1}{\lambda} \left[\frac{\sigma}{4} u + \frac{i}{8} \int_x^\infty (u^2 + 4w) dx' \right] + \mathcal{O}\left(\frac{1}{\lambda^2}\right),$$

which can be compared to (6.9.4):

$$\underline{\psi}(x; \lambda; \sigma) = 1 + \frac{2i\nu}{\lambda} \cdot \frac{e^{-2\nu x + 2i\sigma\nu^2 t + 2i\sigma\omega_+} - e^{-4\nu x}}{1 - e^{-4\nu x}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \quad (6.9.7)$$

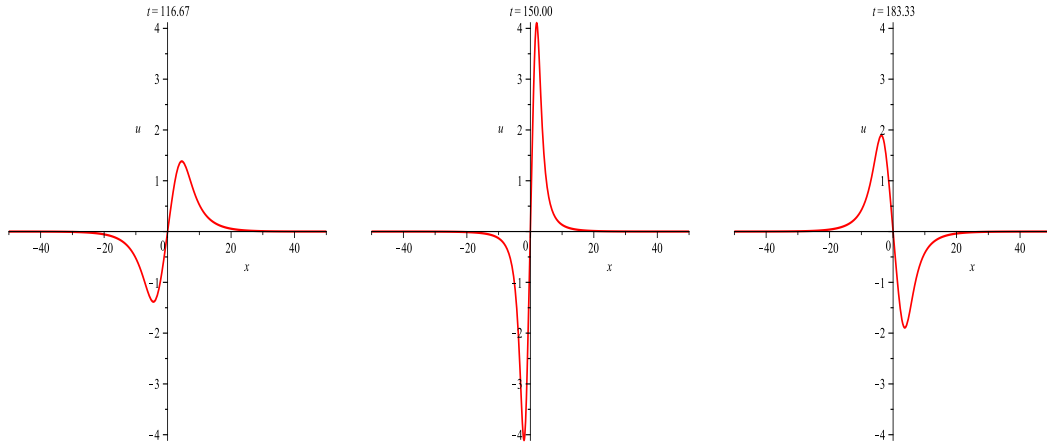


Figure 6.2: Snapshots of the solution for the u -component of the Kaup-Boussinesq equation (6.9.6), (6.9.8) for three values of t .

Since ω_+ and u are already known we can find w and consequently η . With (6.2.1) we compute

$$u^2 + 4w = 2\left(\kappa + \frac{1}{2}\right)u^2 + 4\eta.$$

For the Kaup-Boussinesq case, namely $\kappa = -\frac{1}{2}$ and

$$u^2 + 4w = 4\eta = -4\partial_x^2 \ln \left[(1 + e^{-2\nu x})^4 + (1 - e^{-2\nu x})^4 \tan^2 \nu^2 t \right],$$

in which case we have

$$\eta = -2\nu^2 \frac{\cosh^6(\nu x) \cos^4(\nu^2 t) + \frac{3}{4} \sin^2(2\nu^2 t) \sinh^2(2\nu x) - \sinh^6(\nu x) \sin^4(\nu^2 t)}{[\cosh^4(\nu x) \cos^2(\nu^2 t) + \sinh^4(\nu x) \sin^2(\nu^2 t)]^2}. \quad (6.9.8)$$

The solution (6.9.6), (6.9.8) is presented on Fig. 6.2. Note that u is an odd function and η is an even function of x . The solution is of 'breather' type and develops singularities 'infinitesimally' close to $x = 0$ at countably many isolated values of t .

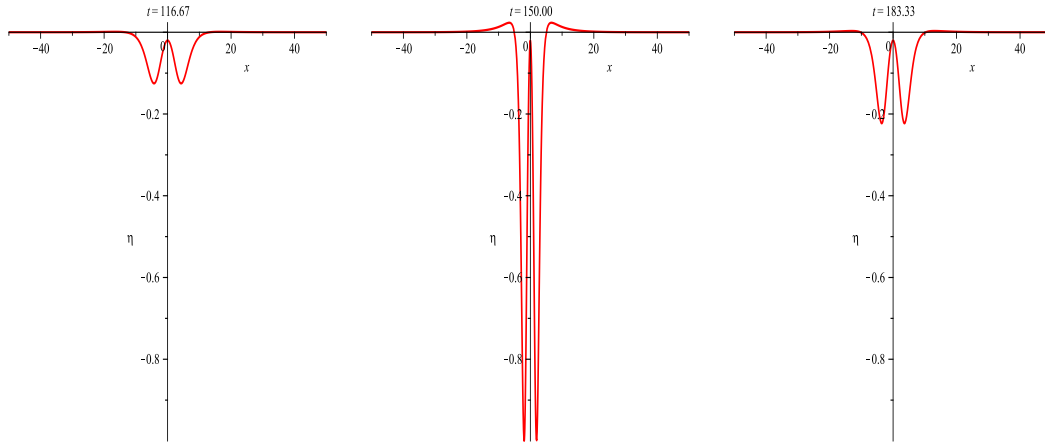


Figure 6.3: Snapshots of the solutions for the $\eta(x, t)$ -component of the Kaup-Boussinesq equation (6.9.6),(6.9.8) for three values of t . The first panel is before, the third panel is after the blowup.

6.10 Conclusion

The next case is a solution with $N = 2$ discrete eigenvalues. Due to (6.2.8) there are the following situations:

(i) Both eigenvalues are on the imaginary axis: $\lambda_1 = i\nu_1$, $\lambda_2 = i\nu_2$ for some real and positive ν_1 and ν_2 ;

(ii) $\lambda_2 = -\bar{\lambda}_1$, $R_2(\sigma) = \bar{R}_1(-\sigma)$. For the (ii) case from (6.8.8) we have

$$\underline{\psi}(x; \lambda; \sigma) = 1 + ie^{2i\sigma x} \left[\frac{R_1(\sigma)e^{2i\lambda_1 x} \underline{\psi}(x; \bar{\lambda}_1; \sigma)}{\lambda - \lambda_1} + \frac{\bar{R}_1(-\sigma)e^{-2i\bar{\lambda}_1 x} \underline{\psi}(x; \bar{\lambda}_1; -\sigma)}{\lambda + \bar{\lambda}_1} \right], \quad (6.10.1)$$

From (6.10.1) we obtain a linear system of four equations for the quantities $\underline{\psi}(x; \bar{\lambda}_1; \pm\sigma)$ and their complex conjugates by writing (6.10.1) for $\lambda = \bar{\lambda}_1$, the same with σ replaced by $-\sigma$ and their complex conjugates.

The case with $N > 2$ eigenvalues is always a combination between (i) and (ii) - in general it involves eigenvalues on the imaginary axis as well as conjugate pairs λ_k and

$-\bar{\lambda}_k$.

We have outlined the inverse scattering for spectral problems of the form (6.2.2) with real functions in the potential, which necessitates the \mathbb{Z}_2 reduction (6.2.8). The soliton solution in the case of a single pole of the eigenfunction does not have the form of a travelling wave and develops singularities with time. This solution is probably not relevant for the theory of water waves. There is another feature of this type of equations which points in the direction that the purely soliton solutions are probably not the ones which are observed in the context of water waves. Indeed, since η is the deviation from the equilibrium surface, then one expects that its space-average value is zero, $\int_{-\infty}^{\infty} \eta(x, t) dx = 0$. However, the trace identities which can be derived easily (see e.g. [LSS2007]) for the N -soliton solution of the Kaup-Boussinesq equation lead to the following result:

$$\int_{-\infty}^{\infty} \eta(x, t) dx = \frac{1}{4} \int_{-\infty}^{\infty} (u^2 + 4w) dx = -4 \sum_{k=1}^N \text{Im} \lambda_k.$$

By assumption $\text{Im} \lambda_k > 0$ since λ_k are in the upper half complex plane. Thus, we have the following 'mostly negative' result for the N -soliton solution:

$$\int_{-\infty}^{\infty} \eta(x, t) dx < 0.$$

This results indicates that the water wave solutions are related only to the continuous spectrum and are therefore unstable. This agrees with the fact that the travelling wave solutions to the Euler's equation with zero surface tension are unstable.

Results and Future Work

This thesis has examined several aspects of fluids arising in various physical circumstances. We presented a brief overview of several fluid models which have played an important role in the continuing development of the mathematical analysis of nonlinear phenomena as they arise in the context of fluid dynamics. Among the features of such nonlinear systems which we encountered were the phenomenon of wave breaking, the drift of fluid particles along streamlines, along with stable solutions such as peakon and soliton solutions. All of these results relied on the nonlinearity of the systems from which they were derived. These models in turn were derived from the basic principles of Newtonian mechanics, applied to continuous media of constant density and neglecting any viscous effects. The basic equations to follow from these physical laws constituted the Euler equations, and were the starting point for the derivation of the models we wished to investigate.

To ensure a coherent structure it was necessary to include both original results and well established ones, thereby allowing the material to be presented with a somewhat natural flow. As such it is necessary to emphasise that the material in Chapter 1 and Chapter 4 has been borrowed from other sources, e.g. [Joh1997], [Iva2009], [ZMNP1984] among others. The aim of Chapter 1 was to provide background and motivation for the investigation of systems of nonlinear PDE. Meanwhile Chapter 4 was necessary, as it not only provided an introduction to the methods of the IST, but the inverse scattering transform method of solving KdV was explicitly used when solving

Qiao's equation in Chapter 5. In addition the derivation of KdV as a model of arterial blood flow relied on that presented in [DP2006], and was included as further motivation for the study of KdV and moreover provided a bridge between the quasi-linear models examined in Chapter 2 and the fully nonlinear models studied in Part 2 of this thesis.

In Chapter 2 we presented an original derivation of the quasi-linear Hopf equation as it arises in the theory of arterial blood flow, cf. [Lyo2012]. In addition we presented an original demonstration of the occurrence of wave breaking for this system which relied on the method of characteristics for solving such systems. This wave breaking phenomena was interpreted physiologically as the pistol shot pulse, while at the same time we provided the necessary criteria for solutions of the system to remain bounded. Chapter 3 was in a sense a diversion from the main material investigated in the rest of the thesis, but was included to demonstrate an entirely different approach to tackling the complexities presented by the nonlinearities encountered in fluid dynamics. The results presented in this section relate to [Lyo2014] and concerned the drift of particles along streamlines in a fluid body of infinite depth and whose free surface assumed the form of an extreme Stokes wave. The result is an extension of those found in [Con2006, Con2012, Hen2008]. This chapter of the thesis addressed two issues left answered in these publication, namely that it was possible to extend the techniques of conformal mapping theory and the application of maximum principles to a semi-infinite fluid domain with free boundary which is continuous but not continuously differentiable.

The result of Chapter 5 and Chapter 6 were original derivations of two types of stable solutions of Qiao's equation and the Kaup-Boussinesq equation respectively, and were based on the publications [IL2012c, IL2012a]. In deriving the peakon solutions we managed to circumvent a well known problem with their construction for the Qiao equation, namely the occurrence of the double delta function. Having done so we then

provided an alternative derivation of the travelling W/M -wave solutions for this system. In addition in this chapter we presented an entirely new method of solving the Qioa equation via the IST, which involved reformulating the spectral problem for solutions with constant boundary values, to yield the well understood spectral problem of the KdV. At this point it was then possible to construct the soliton solutions by solving the associated Riemann-Hilbert problem, thereby allowing us to present the one and two soliton solutions. The results of Chapter 6 were again an original result found in [IL2012b], but relied on the method employed in [HI2011] among others. Essentially Chapter 6 presented an extension of the IST for the KdV equation, thus allowing us to apply similar techniques to the Kaup-Boussinesq equation. The major obstacle to overcome in this instance was the energy dependence of the spectral problem, which was done by introducing a pair of conjugate spectral problems. The construction of the Riemann-Hilbert problem was then possible, which in turn allows one to construct the soliton solutions of the Kaup-Boussinesq equation. In this thesis we presented an explicit expression for the one soliton solution of this system, which was found to be a breather type solution.

The work being continued on from the research presented in this thesis focuses on the two soliton solution of the Kaup-Boussinesq equation. In contrast to the breather type solutions presented in chapter 6, it appears from the most recent research that the two soliton solution is a travelling wave type solution. In addition to the construction of this two soliton solution, there are several question to be answered in relation to the IST for the system, in particular is it possible to demonstrate that the scattering coefficient $a(\lambda)$ has a finite number of simple zeros, as is the case with the KdV equation. This was an issue which was not addressed during the the analysis presented here and provides another line of research to be pursued from this thesis.

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