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MULTIPLE SOLUTIONS OF THE QUASIRELATIVISTIC CHOQUARD EQUATION

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Abstract. We prove existence of multiple solutions to the quasirelativistic Choquard equations with a scalar potential.

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1. Introduction

We study the nonlocal and nonlinear problem

$$L\phi + V\phi - |\phi|^2 * W\phi = -\lambda\phi, \tag{1.1}$$

$$\|\phi\|_{L^2(\mathbb{R}^3)} = 1, \tag{1.2}$$

for a large class of potentials V and W, and $L = \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}$ (the quasirelativistic Laplacian) with α being Sommerfeld's fine structure constant. This Hartree-like Choquard equation arises as the Euler-Lagrange equation associated with a energy functional $\mathcal{E}(\cdot)$ introduced in (3.2). We prove the existence of multiple solutions for two separate cases. Theorem 3.2 concerns the unconstrained problem (1.1), and Theorem 3.4 treats the constrained problem (1.1)-(1.2).

By replacing L by the negative Laplacian and by choosing $V = 0$, and $W(x) = 1/|x|$, we obtain the nonrelativistic Choquard equation which models an electron trapped in its own hole and was proposed by Choquard in 1976 as an approximation to Hartree-Fock theory of a one-component plasma [6]. In a meson nucleon theory a system similar to this equation, but with $W(x) = \frac{e^{-\mu|x|}}{|x|}$, arises when one includes the nucleon recoil caused by surrounding mesons [9]; this classical model provides solitary waves. A quantum theory of gravitating particles yields another example [2]. Furthermore, the Choquard equation has become a prototype of nonlocal problems, which arise in many situations [17].

For the nonrelativistic Choquard equation (in the special case $W(x) = 1/|x|$) Lieb proved existence and uniqueness (modulo translations) of a minimizer (for some λ) by using symmetric decreasing rearrangement inequalities. His existence proof can be extended to more general W provided W is symmetric decreasing which, in some sense, has to be considered a severe restriction; regularity of the solution was subsequently studied by Menzala [14].

Within the same setting, always for the negative Laplacian, Lions [13] proved existence of infinitely many spherically symmetric solutions by application of abstract critical point theory both without the constraint (here it suffices that W is spherical symmetric) and with the constraint (more severe restrictions on W must be assumed). Zhang [18, 19] has

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studied existence of solutions for the nonhomogenous Choquard equation; considering $\lambda = 1$, a negative V which tends to zero at infinity, and adding a positive function g on the right-hand side of (1.1). Küpper, Zhang, and Xia [10] have studied positive solutions and the bifurcation problem arising when one adds a term $\mu f(x)$ to the (1.1); $\mu > 0$ and f being nonnegative. Furthermore, Zhang, Küpper, Hu and Xia have studied existence of solutions, when the right-hand side is multiplied by a positive function which tends to a constant at infinity [20].

For $V = 0$ and $W = 1/|x|$, the first rigorous study of (1.1) was performed by Lieb and Yau [12] in a slightly different context, when the constraint is replaced by $\|\phi\|_{L^2} = N$. They established the existence of a symmetric decreasing minimizer provided $N < N_{\mathrm{b}}$ for some number N_{b}.

We prove existence of multiple solutions, including a minimizer of the corresponding energy functional $\mathcal{E}$. Moreover, we prove some additional properties of the solutions. Our proofs are based upon two classic theorems of critical point theory: in the unconstrained case we apply the mountain pass theorem by Ambrosetti and Rabinowitz [3], and for the constrained case, we apply a suitable variant due to Berestycki and Lions [5].

2. Preliminaries

Throughout the paper we denote by C (with or without indices) various constants whose precise value is of no importance. Let $\mathbb{R}^N$ be the N-dimensional Euclidean space. We set

$$B_R = \{\, x \in \mathbb{R}^N \,:\, |x| < R \,\}, \quad B(x,R) = \{\, y \in \mathbb{R}^N \,:\, |x-y| < R \,\}.$$

By $\mathbb{S}^{N-1}$ we will denote the unit sphere in $\mathbb{R}^N$.
Functions. By C_0^∞, C^∞, and L^p we refer to the standard function spaces. For a measure space $\langle M, \mu\rangle$, μ being a σ-finite measure, the weak L^p space (or Marcinkiewicz space) is defined as the space of measurable functions ϕ such that

$$\sup_{t>0} t\,\mu\left(\{\, x \,:\, |\phi(x)| > t \}\right)^{1/p} < \infty.$$

The space of bounded measures is denoted $\mathcal{M}_{\mathrm{b}}$.
Sobolev spaces. Denoting the Fourier-Plancherel transform of $u \in L^2(\mathbb{R}^3)$ by $\hat{u}$, we define

$$\mathbf{H}^{1/2}(\mathbb{R}^3) = \{\, \phi \in L^2(\mathbb{R}^3) \,:\, (1+|\xi|)^{1/2}\hat{\phi} \in L^2(\mathbb{R}^3) \,\}, \tag{2.1}$$

which, equipped with the scalar product

$$\langle \phi, \psi \rangle_{\mathbf{H}^{1/2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1+|\xi|)\hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\,\mathrm{d}\xi,$$

becomes a Hilbert space; evidently, $\mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}(\mathbb{R}^3)$. We have that $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and the continuous embedding $\mathbf{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ holds whenever $r \in [2,3]$ [1, Theorem 7.57]. Moreover, we shall use that any weakly convergent sequence in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ has a pointwise convergent subsequence. The space of radial (i.e., spherically symmetric) functions belonging to $\mathbf{H}^{1/2}(\mathbb{R}^3)$ will be denoted $\mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3)$.
Auxiliary results. We need the following "radial" lemma by Lions [4].

Lemma 2.1. *If $u \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$, is a radial nonincreasing function (i.e., $0 \le u(x) \le u(y)$ whenever $|x| \ge |y|$), then*

$$|u(x)| \le |x|^{-N/p} \left(\frac{N}{|\mathbb{S}^{N-1}|} \right)^{1/p} \|u\|_{L^p(\mathbb{R}^N)}, \quad x \neq 0.$$

Moreover, we will apply the following compactness lemma due to Strauss [15].

Lemma 2.2. *Let P and $Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying $P(s)/Q(s) \to 0$ as $s \to +\infty$. Let (u_n) be a sequence of measurable functions from $\mathbb{R}^N$ into $\mathbb{R}$ such that*

$$\sup_n \int_{\mathbb{R}^N} |Q(u_n(x))| \, dx < \infty$$

and

$$P(u_n(x)) \to v(x) \text{ a.e. in } \mathbb{R}^N, \quad \text{asn} \to +\infty.$$

Then for any bounded Borel set Ω one has

$$\int_\Omega |P(u_n(x)) - v(x)| \, dx \longrightarrow 0 \text{ as } n \to +\infty.$$

If, moreover, one assumes that $P(s)/Q(s) \to 0$ as $s \to 0$ and $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n, then $P(u_n)$ converges to v in $L^1(\mathbb{R}^N)$ as $n \to \infty$.

Genus. The genus of any compact symmetric subset A of $\mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3) \setminus \{0\}$ will be denoted by $\gamma(A)$. Bear in mind that the boundary ∂A of a symmetric bounded neighborhood of 0 in a d-dimensional space has a genus equal to d. For the definition and properties of the genus, we refer to Struwe [16].

3. ASSUMPTIONS AND MAIN THEOREMS

Functionals. The kinetic energy is defined by

$$\tilde{\mathfrak{l}}_0[\phi] := \alpha^{-1} \|\hat{\phi}(k)\|^2_{L^2(\mathbb{R}^3, (\sqrt{(2\pi|k|)^2 + \alpha^{-2}} - \alpha^{-1}) \mathrm{d}x)}$$

on $\mathbf{H}^{1/2}(\mathbb{R}^3)$. It is convenient to introduce

$$\mathfrak{l}_0[\phi] := \alpha^{-1} \|\hat{\phi}(k)\|^2_{L^2(\mathbb{R}^3, \sqrt{(2\pi|k|)^2 + \alpha^{-2}} \mathrm{d}x)}.$$

Moreover, we introduce

$$\mathfrak{s}_V : \mathbf{H}^{1/2}(\mathbb{R}^3) \to \mathbb{R} \text{ by } \phi \mapsto \int_{\mathbb{R}^3} V(x) |\phi(x)|^2 \mathrm{d}x \tag{3.1}$$

along with (arising from the direct Coulomb energy)

$$\mathcal{J}_W(\psi, \phi) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi(x) \phi(y) W(x-y) \mathrm{d}x \mathrm{d}y,$$

whenever it makes sense. We consider the following functional $\mathcal{E} : \mathbf{H}^{1/2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\phi \mapsto \frac{1}{2} \mathfrak{l}_0[\phi] + \frac{1}{2} \mathfrak{s}_V[\phi] + \frac{1}{2} (\lambda - \alpha^{-2}) \|\phi\|^2_{L^2} - \frac{1}{4} \mathcal{J}_W(|\phi|^2, |\phi|^2), \tag{3.2}$$

At this place we do not focus on whether the functionals are well-defined or not, this will be discussed in detail in the sequel.

Assumptions. We impose the following conditions.

Assumption 3.1. Let V be a real-valued measurable function on $\mathbb{R}^3$ such that V is nonnegative, the associated form $\mathfrak{s}_V$ is $\mathfrak{l}_0$-bounded with bound less than one, and $\mathfrak{l}_0+\mathfrak{s}_V$ is weakly lower semicontinuous on $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Let W be a nonnegative, nonzero, spherically symmetric measure such that there exist $K \geq 1$, $p_k \in (1,\infty)$, with $k \in [1,K]$, and functions W_k satisfying

$$\begin{cases} W = \nu + \sum_{k=1}^{K} W_k, \\ \nu \in \mathcal{M}_{\mathrm{b}}(\mathbb{R}^3), \quad W_k \in L^{p_k}_{\mathrm{w}}(\mathbb{R}^3). \end{cases}$$

We have:

Theorem 3.2. *Let Assumption 3.1 be satisfied. Then, for $\lambda > \alpha^{-2}$, there exists a sequence of (nontrivial) solutions $(u_j)_{j\geq 1}$ of (1.1) satisfying:*
1. The functions u_j are radial and non-increasing.
2. The function u_1 is positive and decreasing provided W is non increasing and V is nonnegative and bounded from above.
3. One has

$$0 < \mathcal{E}(u_{j-1}) \leq \mathcal{E}(u_j) \xrightarrow[j\to\infty]{} \infty.$$

The general case. We introduce, for $N > 0$, the set

$$\mathcal{C} = \{\, u \in \mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3) \,:\, \|u\|_{L^2} = N \,\}.$$

We seek critical points of $\mathcal{E}$ restricted to $\mathcal{C}$.

Assumption 3.3. Let V satisfy the hypotheses in Assumption 3.1. Let W be a nonnegative, nonzero, spherically symmetric measure such that there exist $K \geq 1$, $p_k \in (3/2,\infty)$, with $k \in [1,K]$, and functions W_k satisfying

$$W = \sum_{k=1}^{K} W_k, \quad W_k \in L^{p_k}_{\mathrm{w}}(\mathbb{R}^3).$$

The main result is:

Theorem 3.4. *Let Assumption 3.3 be satisfied and let $d \geq 1$. Suppose there exists a compact symmetric set Ω such that*

$$\Omega \subset \mathcal{C}\,; \quad \gamma(\Omega) \geq d, : \quad \mathcal{E}(u) < 0 \text{ for } u \in \Omega. \tag{3.3}$$

Then there exists a sequence of pairs $(\lambda_j, u_j)_{1\leq j\leq d}$ satisfying

$$\begin{cases} \alpha^{-2} < \lambda_j < \infty \\ u_j \text{ is a solution of } (1.1) \text{ with } \lambda = \lambda_j \end{cases}$$

and, furthermore, one has:
1. The function u_1 *is positive and*

$$\mathcal{E}(u_1) = \min_{\phi \in \mathcal{C}} \mathcal{E}(\phi) < 0.$$

2. The functions u_j *belong to* $\mathcal{C}$.
3. One has $\mathcal{E}(u_1) \le \mathcal{E}(u_2) \le \cdots \le \mathcal{E}(u_j) < 0$.
4. All u_j *are distinct.*
If (3.3) holds for all d, *then assertions 1-3 are valid for* $j \ge 1$ *and* $\mathcal{E}(u_j) \nearrow 0$ *as* $j \to \infty$.

4. UNCONSTRAINED PROBLEM. PROOF OF THEOREM 3.2

We begin with the following auxiliary result.

Lemma 4.1. *For every* $u \in \mathbf{H}^{1/2}(\mathbb{R}^3)$ *we have*

$$\frac{1}{2}\|u\|^2_{\mathbf{H}^{1/2}} \le \langle u, (\sqrt{-\Delta + \alpha^{-2}})u\rangle \le \alpha^{-1}\|u\|^2_{\mathbf{H}^{1/2}}. \tag{4.1}$$

Proof. For every real $a \ge 0$ and $b \ge 1$ we have the following inequality

$$\frac{a+1}{2} \le \sqrt{a^2+b^2} \le b(a+1). \tag{4.2}$$

Letting $a = 2\pi|k|$ and $b = \alpha^{-1}$ in (4.2) we get

$$\frac{2\pi|k|+1}{2} \le \sqrt{(2\pi|k|)^2 + \alpha^{-1}} \le \alpha^{-1}(2\pi|k|+1),$$

and, consequently,

$$\frac{1}{2}\langle(2\pi|k| + \alpha^{-1})\hat{u}, \hat{u}\rangle_{L^2} \le \langle\sqrt{(2\pi|k|)^2 + \alpha^{-2}}\hat{u}, \hat{u}\rangle_{L^2} \le \alpha^{-1}\langle(2\pi|k| + \alpha^{-1})\hat{u}, \hat{u}\rangle_{L^2}.$$

Since $\langle u, (\sqrt{-\Delta + \alpha^{-2}})u\rangle_{L^2} = \langle\sqrt{(2\pi|k|)^2 + \alpha^{-2}}\hat{u}, \hat{u}\rangle_{L^2}$ we obtain (4.1). □

Proof of Theorem 3.2. We apply Theorems 2.1 and 2.8 of Ambrosetti and Rabinowitz [3]. For this purpose we need to verify several conditions. We divide the proof into three steps but first we fix some notation. Let $\mathcal{K} = \mathbf{H}^{1/2}(\mathbb{R}^3)$ and make the decomposition $\mathcal{K} = \mathcal{X} \oplus \mathcal{V}$, where $\mathcal{V}$ is a finite dimensional subspace of $\mathcal{K}$. Moreover, we let $B_\rho = \{u \in \mathcal{X} : \|u\|_{\mathbf{H}^{1/2}} = \rho\}$.
1. First we show that there exist $\rho, \sigma > 0$ such that $\mathcal{E}_{|\partial B_\rho \cap \mathcal{X}} > \sigma$. For any $u \in \mathcal{X}$, the weak Young inequality implies that

$$\mathcal{J}_W(u^2, u^2) \le \|W\|_{L^p_{\mathrm{w}}}\|u^2\|_{L^1}\|u^2\|_{L^r} = \|W\|_{L^p_w}\|u\|^2_{L^2}\|u\|^2_{L^{2r}},$$

with $1/p + 1/r + 1 = 2$ and $r \in [1, 3/2]$; the latter is a consequence of the Sobolev embedding $\mathbf{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ valid for $s \in [2,3]$. In particular, $\|u\|_{L^2} \le C_1\|u\|_{\mathbf{H}^{1/2}}$ and $\|u\|_{L^r} \le C_2\|u\|_{\mathbf{H}^{1/2}}$ and, therefore,

$$\mathcal{J}_W(u^2, u^2) \le C\|u\|^4_{\mathbf{H}^{1/2}} \tag{4.3}$$

From the latter inequality, Lemma 4.1, and $\lambda > \alpha^{-2}$, we get that

$$\begin{aligned}\mathcal{E}(u) &\geq \frac{\alpha^{-1}}{4}\|u\|^2_{\mathbf{H}^{1/2}} + \frac{1}{2}(\lambda - \alpha^{-2})\|u\|^2_{L^2} - C\|u\|^4_{\mathbf{H}^{1/2}} \\ &\geq \frac{\alpha^{-1}}{4}\|u\|^2_{\mathbf{H}^{1/2}} - C\|u\|^4_{\mathbf{H}^{1/2}} \\ &\geq \|u\|^2_{\mathbf{H}^{1/2}}\left(\frac{\alpha^{-1}}{4} - C\|u\|^2_{\mathbf{H}^{1/2}}\right).\end{aligned}$$

Next we choose codim $\mathcal{X}$ such that, for every $u \in \mathcal{X}$, $\|u\|^2_{\mathbf{H}^{1/2}} < \frac{\alpha^{-1}}{4C}$. Then, for every $u \in \partial B_\rho \cap \mathcal{X}$, we conclude that $\mathcal{E}(u) > \sigma > 0$ with $\sigma = \rho^2(\frac{\alpha^{-1}}{4} - C\rho^2)$.

2. For each finite dimensional subspace $\mathcal{V}$ of $\mathcal{K}$ there exists $R = R(\mathcal{V})$ such that $\mathcal{E} < 0$ on $\mathcal{V} \setminus B_R$; B_R is defined similarly to B_ρ above. With a slight abuse of notation we let $\mathsf{J}(u) = \mathcal{J}_W(u^2, u^2)$. Then we see that $\mathsf{J}'(u)u = 4\mathsf{J}(u)$ for all $u \in \mathcal{K}$. Let $\mathcal{V}$ be a finite dimensional subspace of $\mathcal{K}$. For every $u \in \mathcal{K}$ with $\|u\|_{\mathbf{H}^{1/2}} \geq 1$ and, for any $t > 0$, let $g(t) = \mathsf{J}(tu/\|u\|_{\mathbf{H}^{1/2}})$. Then $g(t) > 0$ and

$$\begin{aligned}g'(t) &= \mathsf{J}'\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right)\frac{u}{\|u\|_{\mathbf{H}^{1/2}}} = \frac{1}{t}\mathsf{J}'\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right)\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}} \\ &= \frac{4}{t}\mathsf{J}\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right) = 4t^{-1}g(t).\end{aligned}$$

Thus

$$\frac{g'(t)}{g(t)} = \frac{4}{t} \Rightarrow \int_1^{\|u\|_{\mathbf{H}^{1/2}}} \frac{g'(t)}{g(t)}\,dt = \int_1^{\|u\|_{\mathbf{H}^{1/2}}} \frac{4}{t}\,dt,$$

and, consequently,

$$\begin{aligned}\ln[\mathsf{J}(u)] - \ln[\mathsf{J}(tu/\|u\|_{\mathbf{H}^{1/2}})] &= \ln[\|u\|^4_{\mathbf{H}^{1/2}}] \\ \Rightarrow \qquad \mathsf{J}(u) &= \|u\|^4_{\mathbf{H}^{1/2}}\mathsf{J}\left(\frac{tu}{\|u\|_{\mathbf{H}^{1/2}}}\right). \qquad (4.4)\end{aligned}$$

Let $\delta = \inf\{\mathsf{J}(u) : \|u\|_{\mathbf{H}^{1/2}} = 1,\, u \in \mathcal{V}\}$ and let $\mathcal{S}_\mathcal{V}$ be the unit sphere of $\mathcal{V}$, and let $(u_j)_{j\geq 1}$ be a sequence in $\mathcal{S}_\mathcal{V}$. Then (u_j) is bounded and therefore there exists a subsequence of (u_j) still denoted by (u_j) that converges weakly to u in $\mathcal{K}$. Since $\dim \mathcal{V} < \infty$ we can assume that (u_j) is a minimizing sequence of $\mathsf{J}(\cdot)$ and also (u_j) converges strongly to u in $\mathcal{V}$. The weakly lower semicontinuity of $\mathsf{J}(\cdot)$ implies that

$$\delta = \inf_{v\in\mathcal{S}_\mathcal{V}} \mathsf{J}(v) = \liminf_j \mathsf{J}(u_j) \geq \mathsf{J}(u) > 0, \quad \text{because } u \neq 0.$$

From (4.4) and above it follows that

$$\mathsf{J}(u) \geq \|u\|^4_{\mathbf{H}^{1/2}} \inf_{\mathcal{S}_\mathcal{V}} \mathsf{J}(u) \text{ i.e. } \mathsf{J}(u) \geq \delta\|u\|^4_{\mathbf{H}^{1/2}}.$$

This, in conjunction with Lemma 4.1, gives us that

$$\mathcal{E}(u) \leq \alpha^{-1}\|u\|^2_{\mathbf{H}^{1/2}} + (\lambda - \alpha^{-2})\|u\|^2_{L^2} - \delta\|u\|^4_{\mathbf{H}^{1/2}}$$

It is not hard to see that $\mathcal{E}(u) \to -\infty$ as $\|u\|_{\mathbf{H}^{1/2}} \to +\infty$. This ends step 2.

3. Within the framework of Ambrosetti and Rabinowitz we look for critical points of $\mathcal{E}(\cdot)$ in $\mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3)$. It is easy to see that $\mathcal{E} \in C^1(\mathbf{H}^{1/2}(\mathbb{R}^3);\mathbb{R})$. It remains to check the

Palais-Smale (PS) condition, i.e., if $(u_j)_{j\geq 1}$ is a sequence of non increasing functions in $\mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3)$ such that

$$\begin{cases} \mathcal{E}(u_j) \text{ is bounded,} \\ \mathcal{E}'(u_j) = \left(\alpha^{-1}\sqrt{-\Delta+\alpha^{-2}}\right)u_j + (\lambda-\alpha^{-2})u_j + Vu_j - (W * |u_j|^2)u_j \xrightarrow[\mathbf{H}^{-1/2}]{} 0. \end{cases}$$

then there exists a subsequence of (u_j) which converges in $\mathbf{H}^{1/2}(\mathbb{R}^3)$.

Let $(u_j)_{j\geq 1}$ be such a sequence and let $\epsilon_j = \mathcal{E}'(u_j)$. We begin by proving that $(u_j)_{j\geq 1}$ is a bounded sequence in $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Now,

$$\mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] - \mathcal{J}_W(u_j^2,u_j^2) = \langle \epsilon_j, u_j\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} \tag{4.5}$$

Since, by hypothesis, $\mathcal{E}(u_j)$ is bounded, we have that

$$\begin{aligned} \mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] &= 2\mathcal{E}(u_j) + \frac{1}{2}\mathcal{J}_W(u_j^2,u_j^2) \\ &\leq C + \frac{1}{2}\mathcal{J}_W(u_j^2,u_j^2) \end{aligned} \tag{4.6}$$

On the other hand,

$$\langle \mathcal{E}'(u_j), u_j\rangle = \mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] + \mathcal{J}_W(u_j^2,u_j^2),$$

i.e.,

$$\langle \mathcal{E}'(u_j), u_j\rangle = 2\mathcal{E}(u_j) - \frac{1}{2}\mathcal{J}_W(u_j^2,u_j^2),$$

which implies that

$$\langle \epsilon_j, u_j\rangle + \frac{1}{2}\mathcal{J}_W(u_j^2,u_j^2) = 2\mathcal{E}(u_j) \leq C$$

and, consequently,

$$|\langle \mathcal{E}'(u_j), u_j\rangle| \leq C \text{ and } \frac{1}{2}\mathcal{J}_W(u_j^2,u_j^2) \leq C.$$

This, in conjunction with (4.6) implies that

$$\mathfrak{t}_0[u_j] + (\lambda-\alpha^{-2})\|u_j\|^2_{L^2} + \mathfrak{s}[u_j] \leq C,$$

whence

$$\mathfrak{t}_0[u_j] + \alpha(\lambda-\alpha^{-2})\|u_n\|^2_{L^2} \leq C$$

because V is nonnegative. Then by (4.1) we obtain

$$\frac{1}{2}\|u_j\|^2_{\mathbf{H}^{1/2}} + \alpha(\lambda-\alpha^{-2})\|u_j\|^2_{L^2} \leq C$$

Since $\lambda-\alpha^{-2}\geq 0$, then we immediately conclude that $\|u_j\|_{\mathbf{H}^{1/2}} \leq C$.

Now, by the Banach-Alaoglu theorem there exists a subsequence of u_j (still denoted u_j) such that $u_j \rightharpoonup u$ in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and a.e. on $\mathbb{R}^3$. It is worth to mention that u is radial and non increasing because all u_j are. Since u_j is radial and non increasing, Lemma 2.1 implies that

$$|u_j(x)| \leq c|x|^{-3/2}, \quad x \neq 0.$$

Therefore $\lim_{|x|\to\infty} u_j(x) = 0$ and, consequently, $\lim_{|x|\to\infty} u(x) = 0$. Let $v_j = u_j - u$. Then it is not hard to see that $(v_j)_{j\geq 1}$ is bounded in $\mathbf{H}^{1/2}$ and $\lim_{|x|\to\infty} v_j(x) = 0$. An application of Sobolev's embedding theorem shows that each v_j belongs to $L^p(\mathbb{R}^3)$, $p \in [2,3]$. Hence we can apply Lemma 2.2, i.e., Strauss' compactness principle [15], wherein we choose $P(s) = |s|^r$ and $Q(s) = |s|^2 + |s|^3$, and $v = 0$. It follows that

$$\int_{\mathbb{R}^3} |v_n|^r \, dx \xrightarrow[n\to\infty]{} 0, \text{ i.e. } \|u_j - u\|_{L^r} \xrightarrow[n\to\infty]{} 0, \quad r \in [2,3].$$

Next we show that $\mathcal{E}'(u_j) \to \mathcal{E}'(u)$ in $\mathbf{H}^{-1/2}(\mathbb{R}^3)$. We have $(u_j^2)_{j\geq 1}$ bounded in $L^s(\mathbb{R}^3)$, $s \in [1, \frac{3}{2}]$ since u_j is bounded in $L^r(\mathbb{R}^3)$, $r \in [2,3]$ and, together with $W \in L^{p_k}_{\mathrm{w}}(\mathbb{R}^3)$ and the generalized Young inequality, we deduce that $W * u_j^2$ is bounded in $L^q(\mathbb{R}^3)$ with $3/2 < q < \infty$. Moreover, by the dominated convergence theorem we infer that $W * u_j^2$ converges strongly to $W * |u|^2$ in $L^q(\mathbb{R}^3)$. Let $\psi_j = W * |u_j|^2$, and $w \in \mathbf{H}^{1/2}$. Then

$$\begin{aligned} |\langle \psi_j u_j - \psi u, w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}}| &= |\langle \psi_j u_j - \psi_j u + \psi_j u - \psi u, w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}}| \\ &\leq C\left[\|\psi_j(u_j - u)\|_{L^2} + \|(\psi_j - \psi)u\|_{L^2}\right] \end{aligned}$$

By Hölder's inequality we have that

$$\|\psi_j(u_j - u)\|_{L^2} \leq \|\psi_j^2\|_{L^l}\|(u_j - u)^2\|_{L^m}$$

with $(1/l) + (1/m) = 1$; valid because $m \in [1, 3/2]$ and $l \in (3/4, \infty)$. Then, by the uniform boundedness of ψ_j in $L^q(\mathbb{R}^3)$, $q \in (3/2, \infty)$, and the strong convergence of u_j to u in L^r, $r \in [2,3]$, and the strong convergence of ψ_j to ψ in $L^q(\mathbb{R}^3)$, it follows that $\langle \psi_j u_j - \psi u, w\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} \to 0$ as $j \to \infty$. Hence

$$\psi_j u_j = (W * u_j^2)u_j \underset{\mathbf{H}^{-1/2}}{\rightharpoonup} \psi u = (W * u^2)u. \tag{4.7}$$

On the other hand, by the boundedness of u_j in $\mathbf{H}^{1/2}(\mathbb{R}^3)$ and the boundedness of $W * u_j^2$ in L^q, we have that $(W * u_j^2)u_j^2$ is bounded in L^1. These facts, together with the pointwise convergence of $(W*u_j^2)u_j^2$ to $(W*u^2)u^2$ in $\mathbb{R}^3$ imply that Lebesgue's dominated convergence theorem yields

$$\mathcal{J}_W(u_j^2, u_j^2) \longrightarrow \mathcal{J}_W(u^2, u^2).$$

By passing to the limit in (4.5) as $j \to \infty$, we get that

$$\lim_j \left\{\mathfrak{l}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j]\right\} = \mathcal{J}_W(u^2, u^2).$$

An application of Fatou's lemma yields

$$\begin{aligned} \mathfrak{l}_0[u] + (\lambda - \alpha^{-2})\|u\|_{L^2}^2 + \mathfrak{s}[u] &\leq \liminf_j \left\{\mathfrak{l}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j]\right\} \\ &= \lim_j \left\{\mathfrak{l}_0[u_j] + (\lambda - \alpha^{-2})\|u_j\|_{L^2}^2 + \mathfrak{s}[u_j]\right\} \\ &= \mathcal{J}_W(u^2, u^2). \end{aligned}$$

Moreover, since u_j converges strongly to u in $L^r(\mathbb{R}^3)$, $r \in [2,3]$, we have that

$$\alpha^{-1}\left(\sqrt{-\Delta + \alpha^{-2}} - \alpha^{-1}\right)u_j + \lambda u_j + V u_j \xrightarrow[\mathbf{H}^{-1/2}]{} \alpha^{-1}\left(\sqrt{-\Delta + \alpha^{-2}} - \alpha^{-1}\right)u + \lambda u + V u$$

in the sense of distributions. The latter, in conjunction with (4.7), implies that

$$\mathcal{E}'(u_j) \xrightarrow[\mathbf{H}^{-1/2}]{} \mathcal{E}'(u) = \left(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}\right)u + \lambda u + V u + (W * u^2)u.$$

Then, by hypothesis, we deduce that $\mathcal{E}'(u) = 0$. In particular, $\langle \mathcal{E}'(u), u\rangle = 0$ and we infer that

$$\mathfrak{l}_0[u] + (\lambda - \alpha^{-2})\|u\|_{L^2}^2 + \mathfrak{s}[u] = \mathcal{J}_W(u^2, u^2).$$

Furthermore,

$$\begin{aligned}
&\langle u_j - u, \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u)\rangle \\
&\quad = \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u, u - u_j\rangle - \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j, u - u_j\rangle \\
&\quad = \langle \Big(\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2}\Big) u + \lambda u + Vu - (W * u^2)u, u - u_j\rangle + \int (W * |u|^2)u(u - u_j)\,dx \\
&\qquad + (\alpha^{-2} - \lambda)\langle u, u - u_j\rangle - \langle Vu, u - u_j\rangle - \langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j, u - u_j\rangle.
\end{aligned}$$

The first term on the right-hand side is equal to $\langle \mathcal{E}'(u), u - u_j\rangle_{\mathbf{H}^{-1/2},\mathbf{H}^{1/2}} = 0$, the third term from the right-hand side, viz. $\langle u, u - u_j\rangle$ tends to zero (because u_j converges weakly to u in $\mathbf{H}^{1/2}$), the same argument applies to fourth term. As for the second term we apply Hölder's inequality twice. Since both $W * u^2$ and u are bounded in L^q, $3/2 < q < \infty$ and u_j converges strongly to u in L^r, $r \in [2, 3]$, this implies that the second term tends to zero. For the last term we need the uniform boundedness of $\sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}u_j$ in $L^2(\mathbb{R}^3)$, together with the strong convergence of u_j to u in $L^2(\mathbb{R}^3)$ to conclude. In view of the above, we obtain

$$\langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u), u_j - u\rangle_{L^2} \longrightarrow 0$$

Since $\langle \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}}(u_j - u), u_j - u\rangle \geq (|\nabla|(u_j - u), u_j - u)\rangle$, we have $\langle |\nabla|(u_j - u), u_j - u\rangle \to 0$. We conclude that $\|u_j - u\|_{\mathbf{H}^{1/2}} \to 0$. ☐

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It is worth to mention that Assumption 3.1 is optimal for a nonnegative, radial W because there exists $W \in L^\infty(\mathbb{R}^3)$ such that (1.1) has no $\mathbf{H}^{1/2}(\mathbb{R}^3)$ solutions. For instance, we may choose $W \equiv 1$. Then (1.1), with $V \equiv 0$, takes the form $Lu + (1 - \|u\|_{L^2}^2)u = 0$ and this implies that $u \equiv 0$.

5. Constrained problem. Proof of Theorem 3.4

We prove Theorem 3.4 and we establish two corollaries.

Proof of Theorem 3.4. Without loss of generality we consider $W \in L_w^{p_i}(\mathbb{R}^3)$. The idea is to apply the critical point theory by Berestycki and Lions [5] in the following framework: $\mathcal{H} = L^2(\mathbb{R}^3)$ and $\mathcal{K} = \mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3)$. In order to apply the abstract theorem, we need to establish the following requirements:

1. $\mathcal{E}_{|\mathcal{C}}$ is bounded below;
2. $\mathcal{E}$ is weakly lower semicontinuous on $\mathcal{T} = \{\, u \in \mathcal{C} \,:\, \mathcal{E}(u) \leq 0\}$;
3. $\mathcal{E}_{|\mathcal{C}}$ satisfies the $(\mathrm{PS})_-$ condition.

Verification of item 1. From Lemma 4.1 we find that

$$\mathcal{E}(u) \geq \frac{\alpha^{-1}}{4}(\|u\|_{\mathbf{H}^{1/2}}^2 - \|u\|_{L^2}^2) - 1/4\mathcal{J}_W(u^2, u^2). \tag{5.1}$$

An application of the weak Young inequality and Sobolev's inequality yield

$$\mathcal{J}_W(u^2, u^2) \leq \|W\|_{L_w^p}\|u^2\|_{L^1}\|u^2\|_{L^{r/2}} \leq CN^2\|W\|_{L_w^p}\|u\|_{\mathbf{H}^{1/2}}^2 \tag{5.2}$$

where $1/p+2/r+1=2$, i.e., $1/p+2/r=1$ which is possible to satisfy because $r\in[2,3]$ and $p\geq 3$. Since u belongs to $\mathcal{C}$, it is not hard to see that $\|u^2\|_{L^1}=\|u\|_{L^2}^2=N^2$. Moreover, $\|u^2\|_{L^{r/2}}=\|u\|_{L^r}^2\leq C\|u\|_{\mathbf{H}^{1/2}}^2$. Without loss of generality, we choose $\|W\|_{L^p_w}=1/2\alpha CN^2$. Then inequality (5.2) becomes

$$\mathcal{J}_W(u^2,u^2)\leq\frac{\alpha^{-1}}{2}\|u\|_{H^{1/2}}^2$$

while (5.1) becomes simply $\mathcal{E}(u)\geq -N^2$.

Verification of item 2. Let $(u_j)\subset\mathcal{T}:=\{u\in\mathcal{C}\,:\,\mathcal{E}(u)\leq 0\}$ such that $u_j\rightharpoonup u$ in $\mathbf{H}^{1/2}(\mathbb{R}^3)$. Obviously, as for item 1, it follows that

$$\sup_j\mathcal{J}_W(u_j^2,u_j^2)<\infty$$

and, by Fatou's lemma, we get that

$$\mathcal{J}_W(u^2,u^2)\leq\liminf_j\mathcal{J}_W(u_j^2,u_j^2).$$

Since the remaining terms are obviously weakly lower semicontinuous, it follows that $\mathcal{E}$ is weakly lower semicontinuous on $\mathcal{T}$.

Verification of item 3. Let $(u_j)_{j\geq1}$ be a sequence in $\mathcal{C}$ satisfying

$$\begin{cases}-\infty<\beta\leq\mathcal{E}(u_j)\leq\sigma<0\\ (\sqrt{-\alpha^{-2}\Delta-\alpha^{-4}}-\alpha^{-2})u_j+Vu_j-(W*u_j^2)u_j+\lambda_ju_j=\epsilon_j\xrightarrow[\mathbf{H}^{-1/2}]{}0,\end{cases}$$

where

$$-\lambda_j=\mathcal{E}(u_j)=\frac{1}{2}\mathfrak{t}_0[u_j]+\frac{1}{2}\mathfrak{s}[u_j]-\frac{1}{4}\mathcal{J}_W(u_j^2,u_j^2)$$

We have

$$\frac{1}{2}\langle(\sqrt{-\alpha^{-2}\Delta-\alpha^{-4}}-\alpha^{-2})u_j,u_j\rangle+\frac{1}{2}(\lambda-\alpha^{-2})\|u_j\|_{L^2}^2$$
$$+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|u_j(x)|^2\,dx-\frac{1}{4}\int\int W(x-y)|u_j(x)|^2|u_j(y)|^2\,dxdy\leq\sigma.$$

Since we have already proved that, for any $v\in\mathcal{C}$, $\mathcal{J}_W(v^2,v^2)\leq C$, we obtain

$$\frac{1}{2}\langle\sqrt{-\alpha^{-2}\Delta-\alpha^{-4}}u_j,u_j\rangle+\frac{1}{2}(\lambda-\alpha^{-2})\|u_j\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|u_j(x)|^2\,dx\leq C,$$

whence

$$C\geq\frac{1}{2}\langle(\sqrt{-\alpha^{-2}\Delta-\alpha^{-4}})u_j,u_j\rangle\geq\|u_j\|_{\mathbf{H}^{1/2}}^2.$$

Therefore, $C\geq\|u_j\|_{\mathbf{H}^{1/2}}^2$, i.e., (u_j) is bounded in $\mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3)$. Furthermore,

$$-\lambda_j\leq 2\mathcal{E}(u_j)\leq 2\sigma,\qquad -2\sigma\leq\lambda_j\leq\lambda.$$

Indeed,

$$\frac{-1}{2}\lambda_j=\frac{1}{2}\langle(\sqrt{-\alpha^{-2}\Delta+\alpha^{-4}}-\alpha^{-2})u_j,u_j\rangle+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|u_j(x)|^2\,dx$$
$$-\frac{1}{4}\int\int W(x-y)|u_j(x)|^2|u_j(y)|^2\,dxdy-\frac{1}{4}\int\int W(x-y)|u_n(x)|^2|u_n(y)|^2\,dxdy$$

i.e.

$$\frac{-1}{2}\lambda_j = \mathcal{E}(u_j) - \frac{1}{4}\int\int W(x-y)|u_j(x)|^2|u_j(y)|^2\,dxdy,$$

This shows that $\frac{-1}{2}\lambda_j \leq \mathcal{E}(u_j)$ and then $-\lambda_j \leq 2\mathcal{E} \leq 2\sigma$.

On the other hand, since $\mathcal{J}_W(u_j^2, u_j^2)$ is uniformly bounded with respect to j and from the facts above we conclude that $\lambda_j \leq \lambda$. Now we can follow the proof of Theorem 3.2 and conclude that u_j converges strongly to u in $\mathbf{H}_{\mathrm{r}}^{1/2}(\mathbb{R}^3)$. This verifies item 3. Then the assertions of the theorem follows immediately from Berestycki and Lions [5, Theorems 7 and 9].

□

Corollary 5.1. *Let the hypotheses of Theorem 3.4 be satisfied. Then there there exists a nondecreasing and positive sequence $(N_d)_{d\geq 1}$ such that, if $N \geq N_d$, then the conclusions of Theorem 3.4 hold.*

Proof. Let $(\mathcal{V}_d)_{d\geq 1}$ be a sequence of d-dimensional subspaces of $\mathbf{H}_{\mathrm{r}}^{1/2}$ such that $\mathcal{V}_d \subset \mathcal{V}_{d+1}$ and let $\mathcal{C}_1 = \{u \in \mathbf{H}_{\mathrm{r}}^{1/2} \,:\, \|u\|_{L^2} = 1\}$. By definition of the genus, $\gamma(\mathcal{C}_1 \cap \mathcal{V}_d) = d$. For any positive real number N and any $u \in \mathcal{C}_1 \cap \mathcal{V}_d$, we have that

$$\begin{aligned}\mathcal{E}(Nu) &\leq \frac{N^2}{2}\mathfrak{l}_0[u] + \frac{N^2}{2}\mathfrak{s}[u] - \frac{N^4}{4}\mathcal{J}_W(u^2,u^2)\\ &\leq \frac{N^2}{2}\left\{\sup_{u\in\mathcal{C}_1\cap\mathcal{V}_d}(\mathfrak{l}_0[u]+\mathfrak{s}[u]) - \frac{N^2}{2}\inf_{u\in\mathcal{C}_1\cap\mathcal{V}_d}\mathcal{J}_W(u^2,u^2))\right\}.\end{aligned}$$

Then there exists N_d such that for $N \geq N_d$ the right-hand side is negative and, therefore, $\mathcal{E}$ is negative. Thus, for $N \geq N_d$, $\tilde{A} = \{Nu \,:\, u \in \mathcal{C}_1 \cap \mathcal{V}_d\}$ satisfies (3.3) and, consequently, the assertions of Theorem 3.4 hold true. □

Corollary 5.2. *Let the hypotheses of Theorem 3.4 be satisfied. If, moreover,*

$$\liminf_{r\to+\infty} r^2 W(r) \geq L, \tag{5.3}$$

then there exists L_d such that (3.3) holds true provided $L \geq L_d$. If $L = +\infty$, then (3.3) holds true for all $d \geq 1$. In particular, the assertions of Theorem 3.4 are valid.

Proof. Without loss of generality we may suppose $N = 1$. Let $A = \mathcal{C}_1 \cap \mathcal{V}_d$ where $(\mathcal{V}_d)_{d\geq 1}$ is a sequence of d-dimensional subspaces of $\mathbf{H}_{\mathrm{r}}^{1/2}$ (to be specified below) such that $\mathcal{V}_d \subset \mathcal{V}_{d+1}$.

Choose $u \in A$ and let $u_\kappa(x) = u(x/\kappa)$. Then $\|\kappa^{-3/2}u_\kappa\|_{L^2} = 1$ and

$$\mathcal{E}(\kappa^{-3/2}u_\kappa) \leq \frac{1}{2}\mathfrak{l}_0[\kappa^{-3/2}u_\kappa] + \frac{1}{2}\int_{R^3} V(\kappa x)|u(x)|^2\,dx - \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u^2(x)u^2(y)W(\kappa|x-y|)\,dxdy.$$

Using that $\mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}$ and, specifically,

$$\mathfrak{l}_0[\phi] \leq C\|\phi\|^2_{\mathbf{H}^1}, \quad \forall \phi \in \mathbf{H}^1(\mathbb{R}^3),$$

in conjunction with

$$\int\int_{\frac{1}{2}\leq|x-y|\leq 1} u^2(x)u^2(y)W(\kappa|x-y|)\,dxdy \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u^2(x)u^2W(\kappa|x-y|)\,dxdy$$

we have that

$$\begin{aligned}\mathcal{E}(\kappa^{-3/2}u_\kappa) &\leq \frac{C}{2}\lambda^{-2}\int_{\mathbb{R}^3}|\nabla u|^2 + 1 + \frac{1}{2}\int_{\mathbb{R}^3}V(\kappa x)|u(x)|^2\,dx \\ &\quad -\frac{1}{4}\int\int_{1/2\leq|x-y|\leq 1}u^2(x)u^2(y)W(\kappa|x-y|)\ dxdy \\ &\leq \frac{C_1}{2}\kappa^{-2}\left\{\int_{\mathbb{R}^3}|\nabla u|^2 - \frac{\kappa^2}{2}\int\int_{1/2\leq|x-y|\leq 1}u^2(x)u^2(y)W(\kappa|x-y|)\ dxdy\right\} + C_2 \\ &\leq \frac{C_1}{2}\kappa^{-2}\left\{\int_{\mathbb{R}^3}|\nabla u|^2 - \frac{L}{2}\int\int_{1/2\leq|x-y|\leq 1}u^2(x)u^2(y)\ dxdy\right\} + C_2.\end{aligned}$$

where, in the last inequality, we used the assumption in (5.3). For $u \in A$ we may suppose that $u^2(x) > 0$ for $\Xi = \{|x| \leq 2\}$. Indeed, we may choose $\mathcal{V}_d$ to be the subspace spanned by the first d eigenfunctions u_n of $-\Delta$ with Dirichlet boundary conditions on $\partial\Xi$. Since each $u_n \in \mathbf{H}^1(\mathbb{R}^3) \subset \mathbf{H}^{1/2}(\mathbb{R}^3)$ is radial, we have that $u_n \in \mathbf{H}^1_{\mathrm{r}}(\mathbb{R}^3) \subset \mathbf{H}^{1/2}_{\mathrm{r}}(\mathbb{R}^3)$ as required. This choice of $\mathcal{V}_d$ will ensure that

$$\inf_{u\in\mathcal{C}_1\cap\mathcal{V}_d}\int\int_{1/2\leq|x-y|\leq 1}u^2(x)u^2(y)\ dxdy > 0$$

and, by taking L large enough, we find that

$$\sup_{u\in\mathcal{C}_1\cap\mathcal{V}_d}\mathcal{E}(\kappa^{-3/2}u_\kappa) < 0 \quad \text{for } \kappa \geq \kappa_0.$$

Finally, with $\tilde{A} = \{\kappa_0^{-3/2}u_{\kappa_0}\ :\ u \in \mathcal{C}_1 \cap \mathcal{V}_d\}$ we conclude that $\gamma(\tilde{A}) = \gamma(A) = d$ and, therefore, (3.3) is satisfied for $\tilde{A}$.

□

If one takes $W(x) = 1/|x|^\alpha$, $2 < \alpha < 4$, then $\mathcal{E}$ is not even bounded below; this observation alone shows that Assumption 3.3 is necessary.

A posteriori it can be shown that solutions u_j of (1.1) satisfy the following properties:

- (i) $u_j \in C^\infty(\mathbb{R}^3 \setminus \{0\})$;
- (ii) For all $R > 0$ and $\beta < \nu := \sqrt{\lambda(2\alpha^{-2} - \lambda)}$, there exists $C = C(\beta, R) > 0$ such that

$$|u_j(x)| \leq Ce^{-\beta|x|}, \quad \text{for} \quad |x| \geq R.$$

Indeed, the proof of properties (i) and (ii) for the quasirelativistic Choquard equation (1.1) is carried over, with minor changes, from the proof of similar properties, valid for the quasirelativistic Hartree-Fock equations, found in Dall-Aqua *et al.* [7].

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