2011-01-01

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Recommended Citation

Gerdjikov, V., Grahovski, G., Ivanov, R.: On the integrability of KdV hierarchy with self-consistent sources, Commun. Pure Appl. Analysis, vol. 11 (2012), pp. 1439-1452.

On the (Non)-Integrability of KdV Hierarchy with Self-consistent Sources

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Abstract

Nonholonomic deformations of integrable equations of the KdV hierarchy are studied by using the expansions over the so-called “squared solutions” (squared eigenfunctions). Such deformations are equivalent to a perturbed model with external (self-consistent) sources. In this regard, the KdV6 equation is viewed as a special perturbation of KdV. Applying expansions over the symplectic basis of squared eigenfunctions, the integrability properties of the KdV6 equation are analysed. This allows for a formulation of conditions on the perturbation terms that preserve its integrability. The perturbation corrections to the scattering data and to the corresponding action-angle (canonical) variables are studied. The analysis shows that although many nontrivial solutions of KdV6 can be obtained by the Inverse Scattering Transform (IST), there are solutions that in principle can not be obtained via IST. Thus the equation in general is not completely integrable.

1 Introduction

Nonholonomic deformations of integrable equations attracted the attention of the scientific community in the last few years. In [20], based on the Painlevé test, applied to a class of sixth-order nonlinear wave equations, a list of four equations that pass the test was obtained. Among the three known ones, there was a new equation in the list (later known as sixth-order KdV equation, or just KdV6):

$$\left(-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}\right)(v_t + v_{xxx} - 3v_x^2) = 0. \tag{1}$$

Recognising the potential KdV (PKdV) equation in the second factor in (1) and the Nijenhuis recursion operator for PKdV $\mathcal{N}_{\mathrm{PKdV}} = (-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx})\partial_x^{-1}$, after change of variables $v = v_x$ and $w = v_t + v_{xxx} - 3v_x^2$, one can convert (1) into a “potential” form:

$$u_t + u_{xxx} - 6uu_x - w_x = 0, \tag{2}$$

$$-\frac{1}{4}w_{xxx} + uw_x + \frac{1}{2}wu_x = 0, \tag{3}$$

or equivalently

$$\left(-\frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial^{-1}\right)(u_t + u_{xxx} - 6uu_x) = 0. \tag{4}$$

Here ∂_x^{-1} is a notation for the left-inverse of ∂_x: $\partial_x^{-1}f(x) = \int^x f(y)\,\mathrm{d}y$. Note also, that ∂_x^{-1} is a Hamiltonian operator and $-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}$ is a symplectic operator. Since both operators $-\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}$ and $\mathcal{N}_{\mathrm{PKdV}}$ are weakly nonlocal, then eqn. (1) possesses the same recursion operator as the PKdV equation [43] (as noted in [20] as well) and therefore infinitely many integrals of motion. In the same paper [20], a Lax pair and auto-Bäcklund transformations are obtained for KdV6 equation, (1). In [41], applying Hirota bilinear method to KdV6 equation, another class (much simpler) of auto-Bäcklund transformation is obtained.

In [31] B. Kupershmidt described (2) and (3) as a nonholonomic of the KdV equation, written in a bi-Hamiltonian form:

$$v_t = B^1\left(\frac{\delta H_{n+1}}{\delta v}\right) - B^1(w) = B^2\left(\frac{\delta H_n}{\delta v}\right) - B^1(w), \qquad B^2(w) = 0 \tag{5}$$

where

$$B^1 = \partial_x, \qquad B^2 = -\frac{1}{4}\partial_x^3 + v_x\partial_x + \frac{1}{2}v_{xx}$$

are the two standard Hamiltonian operators of the KdV hierarchy, and

$$H_1 = u, \qquad H_2 = \frac{u^2}{2}, \qquad H_3 = \frac{u^3}{3} - \frac{u_x^2}{2}, \ldots$$

are the conserved densities for the same hierarchy. The recursion operator for the KdV hierarchy is given (after a suitable rescaling) by $\Lambda = B^2 \circ (B^1)^{-1}$.

Later on, it was shown in [45, 46] that the KdV6 equation is equivalent to a Rosochatius deformation of the KdV equation with self-consistent sources. Soliton equations with self-consistent sources have many physical applications [37, 38, 34, 46]. For example, they describe the interaction of long and short capillary-gravity waves [37]. Different classes of exact solutions of (1) are obtained in [20, 27, 28, 29, 30]. A geometric interpretation of the KdV6 equation is given in [15].

In the present paper, exploiting the potential form of the KdV6 equation (2), we study the class of inhomogeneous equations of KdV type

$$u_t + u_{xxx} - 6uu_x = W_x[u](x), \tag{6}$$

with an inhomogeneity/perturbation that presumably belongs to the same class of functions as the field $u(x)$ (i.e. decreasing fast enough, when $|x| \to \infty$).

Generally speaking, the perturbation, as a rule destroys the integrability of the considered nonlinear evolution equation (NLEE). The idea of perturbation through nonholonomic deformation, however, is to perturb an integrable NLEE with a driving force (deforming function), such that under suitable differential constraints on the perturbing function(s) the integrability of the entire system is preserved. In the case of local NLEE's, having a constraint given through differential relations (not by evolutionary equations) is equivalent to a nonholonomic constraint.

To the best of our knowledge, the most natural and efficient way for studying inhomogeneities/perturbations of NLEE integrable by the inverse scattering method is by using the expansions over the so-called "squared solutions" (squared eigenfunctions). The squared eigenfunctions of the spectral problem associated to an integrable equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform (IST) for the corresponding hierarchy as a Generalised Fourier transform (GFT). The Fourier modes for the GFT are the Scattering data. Thus all the fundamental properties of an integrable equation such as the integrals of motion, the description of the equations of the whole hierarchy and their Hamiltonian structures can be naturally expressed making use of the completeness relation for the squared eigenfunctions and the properties of the corresponding recursion operator. For more details, we refer to [16, 12] and the references therein.

The expansions over the squared solutions [11] have been used by Kaup and Newell in [24, 39, 25] for the study of the perturbations of various completely integrable systems. As important particular cases these authors analyzed NLEE with singular dispersion relations and have shown that whenever the dispersion relation has a pole located on a given eigenvalue λ of the corresponding Lax operator L, then the corresponding evolution is no longer isospectral (i.e. λ depends on the time explicitly). For all these NLEE the evolution of the scattering data is given by homogeneous equations, which makes it easier to be solved. Further generalisations are made in [10, 9].

An alternative approach to the same type of equations has been proposed by Leon in [32, 33, 35, 36]; it is based on the equivalence of the Zakharov-Shabat system to a $\bar{\partial}$-problem.

The structure of the present paper is as follows: In Section 2 we give some background material about the direct scattering problem for KdV hierarchy. In Section 3 we describe briefly the generalised Fourier transform for the equations of KdV hierarchy as a key point for their integrability. This includes the minimal set of scattering data, their time evolution, the expansions over the complete set of squared solutions and over the symplectic basis, action-angle variables, etc. In the next Section 4 we formulate general conditions on the perturbations (driving forces) for preserving integrability. Then, in Section 5 we apply these results to the KdV6 equation and its hierarchy. In Section 6 we indicate examples for solutions that can be obtaines via IST.

2 Preliminaries: Integrability of the Equations of KdV Hierarchy

The spectral problem for the KdV hierarchy is given by the Hill's operator [40, 16]

$$-\Psi_{xx} + u(x)\Psi = k^2\Psi, \tag{7}$$

where $u(x)$ is a real-valued (Schwartz-class) potential on the whole axia and $k \in \mathbb{C}$ is spectral parameter. The continuous spectrum under these conditions corresponds to real k. We assume that the discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \dots, N$ where κ_n is real.

The direct scattering problem for (7) is based on the so-called "Jost solutions" $f^+(x,k)$ and $\bar{f}^+(x,\bar{k})$,given by their asymptotics: $x \to \infty$ for all real $k \neq 0$ [40]:

$$\lim_{x\to\pm\infty} e^{-ikx} f^{\pm}(x,k) = 1, \qquad k \in \mathbb{R}\backslash\{0\}. \tag{8}$$

From the reality condition for $u(x)$ it follows that $\bar{f}^{\pm}(x,\bar{k}) = f^{\pm}(x,-k)$.

On the continuous spectrum of the problem (7) and the vectors of the two Jost solutions are linearly related [1]:

$$f^-(x,k) = a(k)f^+(x,-k) + b(k)f^+(x,k), \qquad \text{Im } k = 0. \tag{9}$$

In addition, for real $k \neq 0$ we have:

$$\bar{f}^{\pm}(x,k) = f^{\pm}(x,-k). \tag{10}$$

The quantities $\mathcal{R}^{\pm}(k) = b(\pm k)/a(k)$ are known as reflection coefficients (to the right with superscript $(+)$ and to the left with superscript $(-)$ respectively). It is sufficient to know $\mathcal{R}^{\pm}(k)$ only on the half line $k > 0$. Furthermore, $\mathcal{R}^{\pm}(k)$ uniquely determines $a(k)$ [40]. At the points κ_n of the discrete spectrum, $a(k)$ has simple zeroes i.e.:

$$a(k) = (k - i\kappa_n)\dot{a}_n + \frac{1}{2}(k - i\kappa_n)^2\ddot{a}_n + \cdots, \tag{11}$$

Here, the dot stands for a derivative with respect to k and $\dot{a}_n \equiv \dot{a}(i\kappa_n)$, $\ddot{a}_n \equiv \ddot{a}(i\kappa_n)$, etc. At the points of the discrete spectrum f^- and f^+ are again linearly dependent:

$$f^-(x,i\kappa_n) = b_n f^+(x,i\kappa_n). \tag{12}$$

The discrete spectrum is simple, there is only one (real) linearly independent eigenfunction, corresponding to each eigenvalue $i\kappa_n$, say $f_n^-(x) \equiv f^-(x,i\kappa_n)$.

The sets

$$\mathcal{S}^{\pm} \equiv \left\{ \mathcal{R}^{\pm}(k) \quad (k>0), \quad \kappa_n, \quad R_n^{\pm} \equiv \frac{b_n^{\pm 1}}{i\dot{a}_n}, \quad n = 1, \dots N \right\} \tag{13}$$

[1] According to the notations used in [40] $f^+(x,k) \equiv \bar{\psi}(x,\bar{k})$, $f^-(x,k) \equiv \varphi(x,k)$.

are known as scattering data. Each set – $\mathcal{S}^+$ or $\mathcal{S}^-$ of scattering data uniquely determines the other one and also the potential $u(x)$ [40, 16, 47].

KdV equation appeared initially as models of the propagation of two- dimensional shallow water waves over a flat bottom. More about the physical relevance of the KdV equation can be found e.g. in [3, 17, 19, 6, 18].

3 Generalized Fourier Transforms

A key role in the interpretation of the inverse scattering method as a generalized Fourier transform plays the so-called 'generating' (recursion) operator: for the KdV hierarchy it has the form

$$L_{\pm} = -\frac{1}{4}\partial^2 + u(x) - \frac{1}{2}\int_{\pm\infty}^{x} \mathrm{d}\tilde{x} u'(\tilde{x}) \cdot . \tag{14}$$

The eigenfunctions of the recursion operator are the squared eigenfunctions of the spectral problem (7):

$$F^{\pm}(x,k) \equiv (f^{\pm}(x,k))^2, \qquad F_n^{\pm}(x) \equiv F(x, i\kappa_n), \tag{15}$$

From the completeness of the squared eigenfunctions it follows that every function $g(x)$, belonging to the same class of functions as the potential $u(x)$ of the Lax operator, can be expanded over the two complete sets of "squared solutions"[16]:

$$g(x) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \tilde{g}^{\pm}(k) F_x^{\pm}(x,k)\mathrm{d}k \mp \sum_{j=1}^{N}\left(g_{1,j}^{\pm}\dot{F}_{j,x}^{\pm}(x) + g_{2,j}^{\pm}F_{j,x}^{\pm}(x)\right). \tag{16}$$

where $\dot{F}_j^{\pm}(x) \equiv [\frac{\partial}{\partial k}F^{\pm}(x,k)]_{k=k_j}$ and the Fourier coefficients are

$$\begin{aligned}
\tilde{g}^{\pm}(k) &= \frac{1}{ka^2(k)}\left(g, F^{\mp}\right), \quad \text{where} \quad (g,F) \equiv \int_{-\infty}^{\infty} g(x)F(x)\mathrm{d}x, \\
g_{1,j}^{\pm} &= \frac{1}{k_j\dot{a}_j^2}\left(g, F_j^{\mp}\right), \qquad g_{2,j}^{\pm} = \frac{1}{k_j\dot{a}_j^2}\left[\left(g, \dot{F}_j^{\mp}\right) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j}\right)\left(g, F_j^{\mp}\right)\right].
\end{aligned}$$

Here we assume that in addition f^+ and f^- are not linearly dependent at $x = 0$. The details of the derivation can be found e.g. in [7, 16].

In particular one can expand the potential $u(x)$, the coefficients are given through the scattering data [7, 16]:

$$u(x) = \pm\frac{2}{\pi i}\int_{-\infty}^{\infty} k\mathcal{R}^{\pm}(k)F^{\pm}(x,k)\mathrm{d}k + 4\mathrm{i}\sum_{j=1}^{N} k_j R_j^{\pm} F_j^{\pm}(x), \tag{17}$$

The variation $\delta u(x)$ under the assumption that the number of the discrete eigenvalues is conserved is

$$\delta u(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \delta\mathcal{R}^{\pm}(k)F_x^{\pm}(x,k)\mathrm{d}k \pm 2\sum_{j=1}^{N}\left[R_j^{\pm}\delta k_j \dot{F}_{j,x}^{\pm}(x) + \delta R_j^{\pm} F_{j,x}^{\pm}\right]. \tag{18}$$

An important subclass of variations are due to the time-evolution of u, i.e. effectively we consider a one-parametric family of spectral problems, allowing a dependence of an additional parameter t (time). Then $\delta u(x,t) = u_t\delta t + Q((\delta t)^2)$, etc. The equations of the KdV hierarchy can be written as

$$u_t + \partial_x \Omega(L_\pm) u(x,t) = 0, \tag{19}$$

where $\Omega(k^2)$ is a rational function specifying the dispersion law of the equation. The substitution of (18) and (17) in (19) gives a system of trivial linear ordinary differential equations for the scattering data:

$$\begin{aligned} \mathcal{R}^\pm_t \pm 2ik\Omega(k^2)\mathcal{R}^\pm &= 0, \\ R^\pm_{j,t} \pm 2ik_j\Omega(k_j^2)R^\pm_j &= 0, \\ k_{j,t} &= 0. \end{aligned} \tag{20}$$

The KdV equation $u_t - 6uu_x + u_{xxx} = 0$ can be obtained for $\Omega(k^2) = -4k^2$.

Once the scattering data are determined from (20) – (20) one can recover the solution from (18). Thus the Inverse Scattering Transform can be viewed as a GFT.

For our purposes, it is more convenient to adopt a special set of "squared solutions", called symplectic basis [11, 16]. It has the property that the expansion coefficients of the potential $u(x)$ over the symplectic basis are the so-called action-angle variables for the corresponding NLEE.

For the KdV hierarchy, the symplectic basis is given by:

$$\mathcal{P}(x,k) = \mp\left(\mathcal{R}^\pm(k)F^\pm(x,k) - \mathcal{R}^\pm(-k)F^\pm(x,-k)\right), \tag{21}$$

$$\mathcal{Q}(x,k) = \mathcal{R}^-(k)F^-(x,k) + \mathcal{R}^+(k)F^+(x,k), \tag{22}$$

$$P_n(x) = -R_n^\pm F_n^\pm(x), \qquad Q_n(x) = -\frac{1}{2k_n}\left(R_n^+\dot{F}_n^-(x) - R_n^-\dot{F}_n^+(x)\right). \tag{23}$$

Its elements satisfy the following canonical relations:

$$\begin{aligned} [\![\mathcal{P}(k_1),\mathcal{Q}(k_2)]\!] &= \delta(k_1-k_2), \qquad [\![\mathcal{P}(k_1),\mathcal{P}(k_2)]\!] = [\![\mathcal{Q}(k_1),\mathcal{Q}(k_2)]\!] = 0, \\ [\![P_m,Q_n]\!] &= \delta_{mn}, \quad [\![P_m,P_n]\!] = [\![Q_m,Q_n]\!] = 0, \quad k_{1,2} > 0 \end{aligned} \tag{24}$$

with respect to the skew-symmetric product

$$[\![f,g]\!] \equiv \frac{1}{2}\int_{-\infty}^{\infty}(f(x)g_x(x) - g(x)f_x(x))\mathrm{d}x = \int_{-\infty}^{\infty} f(x)g_x(x)\mathrm{d}x, \tag{25}$$

The symplectic basis satisfies the completeness relation [16]:

$$\begin{aligned} \frac{\theta(x-y)-\theta(y-x)}{2} &= \frac{1}{2\pi}\int_0^\infty \Big(\mathcal{P}(x,k)\mathcal{Q}(y,k) - \mathcal{Q}(x,k),\mathcal{P}(y,k)\Big)\frac{\mathrm{d}k}{\beta(k)} \\ &- \sum_{n=1}^{N}\Big(P_n(x)Q_n(y) - Q_n(x)P_n(y)\Big), \end{aligned} \tag{26}$$

where $\beta(k) = 2ikb(k)b(-k)$. Notice that the integration over k is from 0 to ∞. It follows that every function $X(x)$ from the same class as the potential $u(x)$ (i.e. smooth and vanishing fast enough when $x \to \pm\infty$) can be expanded over the symplectic basis:

$$\begin{aligned} X(x) &= \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{\beta(k)} \left(\mathcal{P}(x,k)\phi_X(k) - \mathcal{Q}(x,k)\rho_X(k)\right) \\ &- \sum_{n=1}^{N} \left(P_n(x)\phi_{n,X} - Q_n(x)\rho_{n,X}\right). \end{aligned} \tag{27}$$

The expansion coefficients can be recovered from the so-called inversion formulas:

$$\begin{aligned} \phi_X(k) &= [\![\mathcal{Q}(y,k), X(y)]\!], \qquad \rho_X(k) = [\![\mathcal{P}(y,k), X(y)]\!], \\ \phi_{n,X} &= [\![Q_n(y), X(y)]\!], \qquad \rho_{n,X} = [\![P_n(y), X(y)]\!]. \end{aligned} \tag{28}$$

In particular, if $X(x) = u(x)$ is a solution of the spectral problem, one can compute [16]:

$$[\![\mathcal{P}(y,k), u(y)]\!] = 0, \qquad [\![\mathcal{Q}(y,k), u(y)]\!] = -4ik\beta(k), \tag{29}$$

$$[\![P_n(y), u(y)]\!] = 0, \qquad [\![Q_n(y), u(y)]\!] = 4ik_n. \tag{30}$$

Thus, from (29) one gets:

$$u(x) = \frac{2}{\pi i}\int_0^\infty \mathcal{P}(x,k)\mathrm{d}k - \sum_{n=1}^{N} 4ik_n P_n(x). \tag{31}$$

The expression for the variation of the potential is

$$\begin{aligned} \delta u(x) &= \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{\beta(k)} \left(\mathcal{P}_x(x,k)\delta\phi(k) - \mathcal{Q}_x(x,k)\delta\rho(k)\right) \\ &- \sum_{n=1}^{N} \left(P_{n,x}\delta\phi_n - Q_{n,x}\delta\rho_n\right) \end{aligned} \tag{32}$$

with expansion coefficients

$$\rho(k) \equiv -2ik\ln|a(k)| = -2ik\ln(1 - \mathcal{R}^-(k)\mathcal{R}^-(-k)), \quad k > 0, \tag{33}$$

$$\phi(k) \equiv 2i\beta(k)\arg b(k) = \beta(k)\ln\frac{\mathcal{R}^-(k)}{\mathcal{R}^+(k)}, \tag{34}$$

$$\rho_n = -\lambda_n = -k_n^2, \qquad \phi_n = 2\ln b_n = \ln\frac{R_n^-}{R_n^+}. \tag{35}$$

These are known as action ($\rho(k)$) - angle ($\phi(k)$) variables for KdV equation [47, 40]. They satisfy the canonical relations

$$\begin{aligned} \{\rho(k_1), \phi(k_2)\} &= \delta(k_1 - k_2) \quad \{\rho(k_1), \rho(k_2)\} = \{\phi(k_1), \phi(k_2)\} = 0, \\ \{\rho_m, \phi_n\} &= \delta_{mn}, \quad \{\rho_m, \rho_n\} = \{\phi_m, \phi_n\} = 0 \quad k_{1,2} > 0. \end{aligned} \tag{36}$$

with respect to the Poisson bracket:

$$\{A, B\} = \left[\!\left[\frac{\delta A}{\delta u}, \frac{\delta B}{\delta u} \right]\!\right].$$

One can verify that the scattering data determines the canonical variables and vice-versa. The symplectic basis consists of the eigenfunctions of the operator $\Lambda = \frac{1}{2}(L_+ + L_-)$:

$$\begin{aligned} \Lambda \mathcal{P}(x,k) &= k^2 \mathcal{P}(x,k), & (37) \\ \Lambda \mathcal{Q}(x,k) &= k^2 \mathcal{Q}(x,k), & (38) \\ \Lambda P_n(x) &= k_n^2 P_n(x), & (39) \\ \Lambda Q_n(x) &= k_n^2 Q_n(x). & (40) \end{aligned}$$

Again, for variations due to the time-evolution of u, $\delta u(x,t) = u_t \delta t + Q((\delta t)^2)$, etc. the equations of the KdV hierarchy

$$u_t + \partial_x \Omega(\Lambda) u(x,t) = 0, \tag{41}$$

with (31) and (32) are equivalent to a system of trivial linear ordinary differential equations for the canonical variables (which can be considered as scattering data):

$$\begin{aligned} \phi_t &= 4i\beta(k)\Omega(k^2), & (42) \\ \rho_t(k) &= 0, & (43) \\ \phi_{n,t} &= 4ik_n \Omega(k_n^2), & (44) \\ \rho_{n,t} &= 0. & (45) \end{aligned}$$

4 Perturbations to the equations of the KdV hierarchy

Let us consider a general perturbation $W_x[u]$ to an equation from the KdV hierarchy:

$$u_t + \partial_x \Omega(\Lambda) u(x,t) = W_x[u]. \tag{46}$$

The function $W_x[u]$ is assumed to belong to the class of admissible potentials for the associated spectral problem (7) (Schwartz class functions, in our case).

The expansion of the perturbation over the symplectic basis is:

$$\begin{aligned} W_x[u] &= \frac{1}{2\pi} \int_0^\infty \frac{\mathrm{d}k}{\beta(k)} \left(\mathcal{P}_x(x,k)\phi_W(k) - \mathcal{Q}_x(x,k)\rho_W(k) \right) \\ &- \sum_{n=1}^{N} \left(P_{n,x}(x)\phi_{n,W} - Q_{n,x}(x)\rho_{n,W} \right). \end{aligned} \tag{47}$$

The substitution of the above expansion (47) in (46) together with (31) and (32) leads to a modification of the time evolution (42) – (45) of the scattering data as follows:

$$\begin{aligned}
\phi_t &= 4i\beta(k)\Omega(k^2) + \phi_W(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), & (48)\\
\rho_t(k) &= \rho_W(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), & (49)\\
\phi_{n,t} &= 4ik_n\Omega(k_n^2) + \phi_{n,W}(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)), & (50)\\
\rho_{n,t} &= \rho_{n,W}(k,t;\rho(k,t),\phi(k,t),\rho_n(t),\phi_n(t)). & (51)
\end{aligned}$$

Since $W = W[u]$ and u depend on the scattering data, we observe that the expansion coefficients of the perturbation ($\phi_W(k) = [\![\mathcal{Q}(y,k), W(y)]\!]$ etc.) also depend on the scattering data. Thus for generic W the new dynamical system (48) – (51) for the scattering data can be extremely complicated and non-integrable in general. This reflects the obvious fact that the perturbed integrable equations are, in general, not integrable.

5 (Non)-Integrability of the KdV Hirarchy with Self-consistent Sources

Let us investigate the integrability of the following equation:

$$\Lambda^*(u_t + \partial_x\Omega(\Lambda)u(x,t)) = 0, \tag{52}$$

where the star is a notation for a Hermitian conjugation. KdV6 in (4) is a particular case of this equation with $\Omega(\Lambda) = -4\Lambda$. In order to simplify our further analysis, instead of the equation (52) we study the following one:

$$(\Lambda^* - \lambda_1)(u_t + \partial_x\Omega(\Lambda)u(x,t)) = 0, \tag{53}$$

where λ_1 is a constant. The corresponding analogue for KdV6 is

$$\begin{aligned}
\frac{1}{4}v_{6x} + \frac{1}{4}v_{txxx} - \frac{1}{2}v_t v_{xx} - v_x v_{xt} &- \frac{5}{2}v_x v_{4x} - 5v_{xx}v_{xxx} + \frac{15}{2}v_x^2 v_{xx}\\
&+ \lambda_1(v_{xt} + v_{xxxx} - 6v_x v_{xx}) = 0.
\end{aligned}$$

Equation (53) follows from (52) after linear (Galilean) change of the variables:

$$x' = x + 6\lambda_1 t, \qquad t' = t, \qquad u' = u - \lambda_1, \tag{54}$$

i.e. $u'_{t'} = u_t - 6\lambda_1 u_x$.

The transformation (54) changes the functional class of u, but it does not affect the integrability of the equation.

Due to the identity $\partial\Lambda = \Lambda^*\partial$ we can represent (53) in the form

$$\partial(\Lambda - \lambda_1)((\partial_x^{-1}u_t) + \Omega(\Lambda)u(x,t)) = 0. \tag{55}$$

Since the operator ∂ does not have a kernel when u is Schwartz class, (55) is equivalent to

$$u_t + \partial_x \Omega(\Lambda) u(x,t) = \begin{cases} (c_1 P_1(x,t) + c_2 Q_1(x,t))_x & \text{for} \quad \lambda_1 = k_1^2 < 0, \\ (c_1 \mathcal{P}(x,k_1,t) + c_2 \mathcal{Q}(x,k_1,t))_x & \text{for} \quad \lambda_1 = k_1^2 > 0, \end{cases} \tag{56}$$

where $c_{1,2} = c_{1,2}(t, k_1; \rho(k_1,t), \phi(k_1,t), \rho_n(t), \phi_n(t))$ are x-independent functions, but the important observation is that the time-dependence could be implicit through the scattering data of the potential $u(x,t)$. Equation (56) is a perturbed equation from the KdV hierarhy. The perturbation in the right-hand side of (56) is in the eigenspace of the recursion operator corresponding to the eigenvalue λ_1, i.e. it is given by 'squared' eigenfunctions of the spectral problem (7) at λ_1. For simplicity we use the symplectic basis, see the precise definitions (21) – (23). Such a special perturbation is often called 'self-consistent sources' perturbation.

If $\lambda_1 < 0$ is a discrete eigenvalue, the corresponding dynamical system (48) – (51) for the scattering data is

$$\phi_t = 4i\beta(k)\Omega(k^2), \tag{57}$$
$$\rho_t(k) = 0, \tag{58}$$
$$\phi_{n,t} = 4ik_n\Omega(k_n^2) - c_1(t, k_1; \rho(k_1,t), \phi(k_1,t), \rho_n(t), \phi_n(t))\delta_{n,1}, \tag{59}$$
$$\rho_{n,t} = c_2(t, k_1; \rho(k_1,t), \phi(k_1,t), \rho_n(t), \phi_n(t))\delta_{n,1}. \tag{60}$$

If $\lambda_1 > 0$ is a continuous spectrum eigenvalue, the dynamical system (48) – (51) has the form

$$\phi_t = 4i\beta(k)\Omega(k^2) + 2\pi\beta(k)c_1(t, k_1; \rho(k_1,t), \phi(k_1,t), \rho_n(t), \phi_n(t))\delta(k-k_1), \tag{61}$$
$$\rho_t(k) = -2\pi\beta(k)c_2(t, k_1; \rho(k_1,t), \phi(k_1,t), \rho_n(t), \phi_n(t))\delta(k-k_1), \tag{62}$$
$$\phi_{n,t} = 4ik_n\Omega(k_n^2), \tag{63}$$
$$\rho_{n,t} = 0. \tag{64}$$

It is clear, that dynamical systems like (57) – (60) or (61) – (64) can not be integrable for a general functional dependence of $c_{1,2}$ on the scattering data. Thus the equations (52), including KdV6, are not **completely** integrable. In other words, there are solutions, which can not be obtained via the Inverse Scattering Method, since the aforementioned dynamical systems for the scattering data are not always integrable.

6 Examples for exact solutions

As we have seen, KdV6 in general is not completely integrable system. However, there are many solutions of (52) or (55) that can be written explicitly (apart from the KdV solutions, which are an obvious subclass of solutions).

One such example is when $c_{1,2}$ depend only on t. Then the system (57) – (60) is integrable, with new canonical variables

$$\begin{aligned} \tilde{\phi}_n &= \phi_n(0) + 4ik_n\Omega(k_n^2)t - \delta_{n,1}\int_0^t c_1(\tau)d\tau, \\ \tilde{\rho}_n &= \rho_n(0) + \delta_{n,1}\int_0^t c_2(\tau)d\tau. \end{aligned}$$

Since $c_{1,2}(t)$ do not depend on the scattering data and therefore on $u(x,t)$, the Poisson brackets between $\tilde{\phi}_n$ and $\tilde{\rho}_n$ are canonical.

This is most often the situation that many authors assume about KdV6 (i.e. $c_{1,2} = c_{1,2}(t)$ simply given functions of t) when discuss its integrability.

There are, however other exactly solvable cases. Indeed, if

$$\begin{aligned} c_1 &= \sum_{k=1}^{N}(\alpha_{1k}\phi_k + \beta_{1k}\rho_k), \\ c_2 &= \sum_{k=1}^{N}(\alpha_{2k}\phi_k + \beta_{2k}\rho_k) \end{aligned}$$

are linear combinations of the scattering data with some constants $\alpha_{p,q}$, $\beta_{p,q}$ the system (57) – (60) becomes a system of linear ordinary ODEs and therefore is integrable.

There are of course many other nontrivial integrable examples that one can construct by various choices of the functions $c_{1,2}$ – when (57) – (60) is an integrable system of ODEs. In general, however this system is not integrable and therefore there exist solutions that can not be constructed by the means of IST.

7 Discussions and Conclusions

Here we have used the expansion over the eigenfunctions of the recursion operator for the KdV hierarchy for studying nonholonomic deformations of the corresponding NLEE from the hierarchy. We have shown, that in the case of self-consistent sources, the corresponding perturbed NLEE is integrable, but not completely integrable.

The perturbation results for the Zakharov-Shabat (ZS) type spectral problems have been obtained firstly in [22] and for KdV in [21]. As it has been explained, the perturbation theory is based on the completeness relations for the squared eigenfunctions. For the Sturm-Liouville spectral problem such relations apparently have been studied as early as in 1946 [1] and then by other authors, e.g. [26, 16]. The completeness relation for the eigenfunctions of the ZS spectral problem is derived in [23] and generalisations are studied further in [8, 10, 11, 9, 13, 42], see also [12] (and the references therein).

The approach presented in this article can be applied also to the study of inhomogeneous versions of NLEE, related to other linear spectral problems, e.g.

the Camassa-Holm equation [4, 5], various difference and matrix generalizations of KdV-like and Zakharov - Shabat spectral problems, etc.

Acknowledgments

The authors have the pleasure to thank Prof. Adrian Constantin, Prof. Nikolay Kostov, Prof. Anjan Kundu and Dr. Jing Ping Wang for numerous useful discussions. VSG and GGG are thankful to the organizers of the international workshop: "Nonlinear Physics: Theory and Experiment. VI" for the financial support and the warm hospitality in Gallipoli. This material is based upon works supported by the Science Foundation Ireland (SFI), under Grant No. 09/RFP/MTH2144.

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