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Some transitivity results for torsion Abelian groups

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SOME TRANSITIVITY RESULTS FOR TORSION ABELIAN GROUPS

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Abstract. We introduce a new class of fully transitive and transitive Abelian p-groups and study the new concept of weak transitivity which is the missing link between full transitivity and transitivity.

Introduction

An Abelian p-group G is called (fully) transitive if for all $x, y \in G$ with $U_G(x) = U_G(y)$ ($U_G(x) \leq U_G(y)$), where $U_G(g)$ denotes the so-called Ulm sequence of the element $g \in G$, there exists an automorphism (endomorphism) of G which maps x onto y. The notions of transitivity and full transitivity originated in the book "Infinite Abelian Groups" by I. Kaplansky [16]. The definitions in [16] came with an "emphatic warning": *It is by no means clear that full transitivity implies transitivity*. There is, however, an obvious relaxation of the notion of transitivity, which we shall designate as *Krylov transitivity*: in the definition of transitivity require only that there exists an appropriate *endomorphism* mapping x onto y. Strangely this is not discussed by Kaplansky and we shall see why in Section 2. Extensive classes of abelian p-groups which are both transitive and fully transitive were found in [12] and [15]. Moreover, p-groups of cardinality the continuum with neither property were constructed in [18] and for larger cardinalities in [10]. In [8] Files and the first author proved that a p-group G is fully transitive if and only if its square $G \oplus G$ is transitive. This was rather surprising as the independence of both concepts had been shown by Corner in [3] and in his original work, Kaplansky had shown that for $p > 2$, transitivity always implies

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full transitivity. Therefore it is natural to ask which fully transitive non-transitive p-groups can occur. By a fundamental observation of Corner [3] one can reduce the decision of whether or not a p-group G is (fully) transitive to consideration of the action of the endomorphism ring on the first Ulm subgroup $p^\omega G$: G is (fully) transitive if and only if $\mathrm{End}(G) \restriction p^\omega G$ acts (fully) transitively on $p^\omega G$ i.e., for any $x, y \in p^\omega G$ such that $U_{p^\omega G}(x) = U_{p^\omega G}(y)$ ($U_{p^\omega G}(x) \leq U_{p^\omega G}(y)$) there exists an automorphism (endomorphism) α of G such that $x\alpha = y$. In this paper we first introduce a new class of fully transitive and transitive abelian p-groups by considering the torsion subgroup of a direct product of p-groups. Afterwards it is shown that for $p \neq 2$ full transitivity is equivalent to what we call Krylov transitivity, i.e. for all $x, y \in G$ with $U_G(x) = U_G(y)$ there exists an endomorphism of G which maps x onto y. Obviously this surprising result clarifies the relation between full transitivity and transitivity. In section 3 we then introduce the notion of weak transitivity which is the missing link between full transitivity and transitivity. An abelian p-group G is called weakly transitive if for any pair of elements $x, y \in G$ there exists an automorphism ϕ of G mapping x onto y if and only if there exist endomorphisms φ, ψ of G such that $x\varphi = y$ and $y\psi = x$. The independence of the three transitivity notions is shown.

We follow standard notations which can be found in Fuchs [9] and Kaplansky [16].

1. A new class of fully transitive groups

The direct sum of two fully transitive p-groups need not, of course, be fully transitive – see for example [18, Theorem 2.4]. However, if a collection of groups $\{G_i\}_{i\in I}$ has the property that each pair $\{G_i, G_j\}$ is a fully transitive pair in the sense defined below, then it follows from Proposition 1 in [8], that $\bigoplus_{i\in I} G_i$ is also fully transitive. Recall that the groups G_1 and G_2 form a *fully transitive pair* if for every $x \in G_i,\ y \in G_j (i, j \in \{1, 2\})$ satisfying $U_{G_i}(x) \leq U_{G_j}(y)$, there exists $\alpha \in \mathrm{Hom}(G_i, G_j)$ with $x\alpha = y$.

In this section we extend the approach in [8] and study transitivity properties of the torsion part of products of abelian p-groups. This leads us to a new class of fully transitive and transitive p-groups which strictly contains both the class of totally projective p-groups and the groups G which occur as extensions of a fully transitive subgroup $p^\beta G$ by a totally projective group i.e the groups covered by Theorem 4 in Hill's original paper on the topic [15].

Theorem 1.1. *If $\{G_i : i \in I\}$ is a family of p-groups such that for each $i, j \in I, \{G_i, G_j\}$ is a fully transitive pair, then the torsion group $H = t(\prod_{i\in I} G_i)$ is fully transitive.*

Proof. Suppose $U_H(\underline{x}) \leq U_H(\underline{y})$ where $\underline{x} = (..., x_i, ...), \underline{y} = (..., y_i, ...)$ are elements of H. The proof is by induction on the order of $\underline{y}$. Suppose firstly that $p\underline{y} = 0$. Now $ht_H(\underline{x}) = \inf\{ht_{G_i}(x_i) : i \in I\}$ and so there exists $i \in I$ such that $ht_H(\underline{x}) = ht_{G_i}(x_i)$; by suitable re-arranging there is no loss to assume $i = 1$ i.e. $ht_H(\underline{x}) = ht_{G_1}(x_1) \leq ht_H(\underline{y})$. Since $p\underline{y} = 0, U_{G_1}(x_1) \leq U_H(\underline{y}) \leq U_{G_i}(y_i)$ for all i and as $\{G_1, G_i\}$ is a fully transitive pair, there exists $\alpha_i : G_1 \to G_i$ such that $x_1\alpha_i = y_i$ for all $i \in I$. The matrix with first row equal to $(\alpha_1, ..., \alpha_i, ...)$ and all other rows zero is (acting on the right) a well defined endomorphism of $\prod_{i\in I} G_i$ and its restriction to H is an endomorphism of H by the full invariance of the torsion subgroup. Moreover, this matrix maps $\underline{x}$ to $\underline{y}$.
Now assume $\text{ord}(\underline{y}) = p^r$ with $r > 1$ and that we have shown the result for all $p^s, s < r$. Since $U_H(p\underline{x}) \leq U_H(p\underline{y})$ and $\text{ord}(p\underline{y}) < p^r$ we have by induction an endomorphism β of H with $(p\underline{x})\beta = p\underline{y}$. Let $\underline{x}' = \underline{x}\beta$ so that $\underline{y} - \underline{x}' \in H[p]$. Since $ht_H(\underline{y} - \underline{x}') \geq \min\ \{ht_H(\underline{y}), ht_H(\underline{x}')\}$ and β does not decrease heights, we have $U_H(\underline{x}) \leq U_H(\underline{y} - \underline{x}')$ and so the result above for elements of order p, provides an endomorphism α sending $\underline{x}$ onto $\underline{y} - \underline{x}'$. The endomorphism $\beta + \alpha$ has the desired property of mapping $\underline{x}$ to $\underline{y}$. □

In light of Theorem 1.1, it is reasonable to search for classes of fully transitive pairs. Such a search is simplified by the easily-proved observation that the groups G, H are a fully transitive pair, if there is a fully transitive group which has a direct summand isomorphic to $G \oplus H$. Our next lemma is an immediate consequence of this observation.

Lemma 1.2. *If G, H are cocyclic p-groups then $\{G, H\}$ is a fully transitive pair.*

Thus we obtain the following familiar results:

Corollary 1.3. *If $\{C_i : i \in I\}$ is a family of cocyclic p-groups then $H = t(\prod_{i\in I} C_i)$ is fully transitive.*

Corollary 1.4. *If A is a p-local algebraically compact group and T is the torsion subgroup of A, then T is fully transitive. In particular torsion-complete p-groups are fully transitive.*

Proof. Since A is p-local algebraically compact it is a direct summand of a direct product of cocyclic $p-$groups and hence its torsion subgroup T is a summand of the torsion subgroup of a direct product of cocyclic p-groups. It follows from Corollary 1.3 that this latter group is fully transitive. Since summands of a fully transitive group are fully transitive – see e.g. [2, Theorem 3.4.] – we conclude immediately that T is fully transitive. □

Full transitivity and transitivity are defined for torsion groups by reducing to p-primary components: G is (fully)transitive if, and only if for each $p \in \mathbb{P}$, the p-primary component G_p is (fully)transitive. The concepts can be extended to torsion-free groups in the obvious way by replacing the Ulm sequence with the indicator sequence, or in the local situation, with the height. We therefore obtain the following:

Corollary 1.5. *If A is a p-local algebraically compact group then both tA and A/tA are fully transitive.*

Proof. By Corollary 1.4 it is sufficient to show that A/tA is fully transitive. However A/tA is cotorsion since it is the epimorphic image of an algebraically compact group and since it is a torsion-free group it is in fact algebraically compact [9, Corollary 54.5]. The result now follows from Krylov's result [17]. □

Corollary 1.6. *If A is an algebraically compact group, then tA is fully transitive.*

Proof. Since A is algebraically compact it is a summand of a product P of cocyclic groups. Hence tA is a summand of the torsion subgroup T of P. But tA is fully transitive if each of its p-components is fully transitive and this follows from Corollary 1.4, since the p-component of T is the torsion part of P_p, where P_p is the product of all cocyclic p-groups involved in P. □

We now turn our attention to transitivity.

Theorem 1.7. *Suppose $\{G_i : i \in I\}$ is a family of p-groups such that*

(1) *$t(\prod_{i\in I} G_i)$ is fully transitive;*
(2) *For any finite subset J of I, $\bigoplus_{j\in J} G_j$ is transitive,*

then $H = t(\prod_{i\in I} G_i)$ is transitive.

Proof. Suppose $\underline{x} = (x_1, ..., x_i, ...)$ and $\underline{y} = (y_1, ..., y_i, ...)$ are in H such that $U_H(\underline{x}) = U_H(\underline{y})$. Then there is a finite subset J_x of I such that $J_x = \{i_1, ...i_n\}$ and $ht_H(\underline{x}) = ht_{G_{i_1}}(x_{i_1})$; $ht_H(p\underline{x}) = ht_{G_{i_2}}(px_{i_2})$ etc. In a similar way there exists a finite subset J_y of I with corresponding properties for $ht_H(\underline{y})$, $ht_H(p\underline{y})$ etc. Let $J = J_x \cup J_y$. Then we can write $\underline{x} = x_J + x'$ where the H-component of x_J is zero for all $i \notin J$; similarly $\underline{y} = y_J + y'$. Since $J_x \subseteq J$, $U_{\bigoplus_{i\in J} G_i}(x_J) = U_H(\underline{x})$ and similarly $U_{\bigoplus_{i\in J} G_j}(y_J) = U_H(\underline{y})$. By assumption there is an automorphism α of $\bigoplus_{j\in J} G_j$ with $x_J\alpha = y_J$. Moreover, $\underline{x} = x_J + x'$ implies $U_H(x') \geq U_H(\underline{x})$ and similarly for y'. Thus $U_H(y' - x') \geq U_H(\underline{x}) = U_H(x_J)$ and so, by full transitivity of H, we have an endomorphism φ of H with $x_J\varphi = y' - x'$. Let π denote the projection of H onto $\bigoplus_{j\in J} G_j$ along $t(\prod_{i\in I\setminus J} G_i)$ and consider the matrix given by $\left(\begin{smallmatrix} \alpha & \pi\varphi(1-\pi) \\ 0 & 1 \end{smallmatrix}\right)$. Note $\alpha : \bigoplus_J G_j \to \bigoplus_J G_j$ and $\pi\varphi(1-\pi) : \bigoplus_J G_j \to t(\prod_{i\in I\setminus J} G_i)$.

Clearly this represents an endomorphism of H and the image of $\underline{x}$ under this endomorphism is $(x_J, x')\left(\begin{smallmatrix} \alpha & \pi\varphi(1-\pi) \\ 0 & 1 \end{smallmatrix}\right) = (y_J, x_J\pi(\varphi(1-\pi) + x')$.
But $x_J\pi\varphi = x_J\varphi = y' - x' \in t(\prod_{i\in I\setminus J} G_i)$ and so $x_J\pi\varphi(1-\pi) = y' - x'$ and the image of(x_J, x') is (y_J, y'), i.e. the endomorphism maps $\underline{x}$ to $\underline{y}$.
It remains only to verify that this matrix is invertible. But this is immediate since α is invertible - the inverse is $\begin{pmatrix} \alpha^{-1} & -\alpha^{-1}\pi\varphi(1-\pi) \\ 0 & 1 \end{pmatrix}$ as is easily checked. □

Again using classes of fully transitive pairs we obtain the well-known:

Corollary 1.8. *The group $H = t(\prod_{i\in I} C_i)$ where each $C_i (i \in I)$ is a cocyclic p-group, is transitive.*

Proof. It follows from Corollary 1.3 that condition (i) of Theorem 1.7 holds. But any finite direct sum of cocyclics is transitive, since the reduced part is a finite group and hence transitive. Thus condition (ii) of Theorem 1.7 holds and H is therefore transitive. □

We finish off this section by applying Theorems 1.1 and 1.7 to a much wider collection of groups forming fully transitive pairs. Recall that the generalized Prüfer group of length λ, H_λ, is defined inductively as follows: $H_0 = 0$ and if λ is a limit ordinal then $H_\lambda = \bigoplus_{\sigma<\lambda} H_\sigma$; if $\lambda = \sigma + 1$ then $p^\sigma H_{\sigma+1}$ is cyclic of order p and $H_{\sigma+1}/p^\sigma H_{\sigma+1} \cong H_\sigma$. Thus for each finite ordinal n, H_n is cyclic of order p^n.

Further details of this family may be found in [9, §81]. For our purposes it suffices to note that each group H_λ is totally projective and hence both transitive and fully transitive by [15]. Moreover, since the direct sum of two totally projective groups is again totally projective, it is immediate that any family of generalized Prüfer groups forms a collection of fully transitive pairs and, moreover, the direct sum of any finite collection is transitive. Thus we have established:

Corollary 1.9. *If $\{C_i : i \in I\}$ is a family of (generalized) Prüfer groups, then $H = t(\prod_{i\in I} C_i)$ is fully transitive and transitive.*

Note that the class of fully transitive and transitive p-groups constructed above is strictly greater than the class of totally projective p-groups.

In fact we can modify our original argument in Theorem 1.1 to deal with a class of groups introduced by Crawley [5]. Recall that if λ is an ordinal cofinal with ω and $\{B_n\}$ is a sequence of reduced totally projective groups with lengths strictly increasing to λ, then a generalized torsion-complete group G is the λ-*direct product* of the B_n's i.e. the subgroup of the direct product consisting of those functions x of finite order such that $ht\ x(n) \to \lambda$.

Theorem 1.10. *If G is a generalized torsion-complete group, then G is fully transitive.*

Proof. The totally projective groups B_n clearly are pairwise fully transitive. Moreover an examination of the argument used in Theorem 1.1 shows that the key step is the use of an endomorphism which can be represented in the form of a matrix with first row equal to $(\alpha_1, ..., \alpha_i, ...)$ and all other rows zero, where the α_i are appropriate endomorphisms. The restriction of this map to the λ-direct product is readily seen to be an endomorphism of that product and this suffices to prove the result. □

2. Krylov transitivity

Recall that Krylov [17] introduced the following transitivity type notion in the context of his work on torsion-free groups: we shall use the terminology *Krylov transitivity* for the corresponding concept for p-groups.

Definition 2.1. *A p-group G is called Krylov transitive if for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$, there is an endomorphism of G mapping x onto y.*

Whilst this definition is an entirely reasonable relaxation of the usual concept of transitivity, it is, at first sight, surprising that it has not appeared in the literature on p-groups. We can only speculate on the reason for this, but the next Proposition may shed some light on the matter. First we need a preliminary result.

Lemma 2.2. *If G is a p-group such that for all $x, y \in G$ with $y \in G[p]$ and $U_G(x) \leq U_G(y)$, there is an endomorphism φ of G mapping x onto y, then G is fully transitive.*

Proof. Suppose x and y are arbitrary with $U_G(x) \leq U_G(y)$. We prove the result by induction on the order of y. Clearly if $o(y) = p$, then by hypothesis a suitable endomorphism maps x onto y. Assume the result is true for all elements of order p^n and suppose $o(y) = p^{n+1}$. Now $U_G(x) \leq U_G(y)$ implies $U_G(px) \leq U_G(py)$ and since $o(py) \leq p^n$, by induction there is an endomorphism φ mapping px to py, $(px)\varphi = py$. But if $y' = y - x\varphi$ then $y' \in G[p]$ and clearly $U_G(x) \leq U_G(y')$ and there is a mapping $\psi \in \mathrm{End}(G)$ such that $x\psi = y'$. Clearly $\varphi + \psi$ is the required map.

Proposition 2.3. *If G is an abelian p-group ($p \neq 2$) then G is fully transitive if and only if it is Krylov transitive.*

Proof. Full transitivity clearly implies Krylov transitivity so we consider the reverse implication. Suppose $x, y \in G$ and $U_G(x) \leq U_G(y)$. Observe from Lemma 2.2 that there is no loss of generality to assume $y \in G[p]$. If $U_G(x) = U_G(y)$ then by hypothesis there exists $\varphi \in \mathrm{End}(G)$ with $x\varphi = y$ as required. So suppose $U_G(x) < U_G(y)$. If $ht(x) < ht(y)$, then $ht(x) = ht(x + y)$ and so $U_G(x) = U_G(x + y)$. Thus there is a $\varphi \in \mathrm{End}(G)$ with $x\varphi = x + y$ and $\varphi - id_G$ is the required map. Note that if $x \in G[p]$ and $U_G(x) < U_G(y)$ then necessarily $ht(x) < ht(y)$. Thus we may assume that $x \notin G[p]$. The remaining case to be considered is when $ht(x) = ht(y)$. We claim that we can find $y' \in G[p]$ with $ht(y) = ht(y')$ and $ht(x) = ht(x + y')$. Assuming for the moment this claim we have immediately that $U_G(x) = U_G(x + y')$ and $U_G(y) = U_G(y')$ and so there exist endomorphisms $\varphi, \psi \in \mathrm{End}(G)$ such that $x\varphi = x + y'$ and $y'\psi = y$. But then the composition $(\varphi - id_G)\psi$ maps x to y as required. Thus it remains only to complete the proof of the claim.

Suppose $x \in G \backslash G[p]$, $y \in G[p]$ and $ht(x) = ht(y)$. If $ht(x) = ht(x + y)$ it suffices to take $y = y'$. However, if $ht(x) < ht(x + y)$ set $y' = 2y$. Since $p \neq 2$, we have $ht(y) = ht(y')$ and $ht(x + y') = ht((x + y) + y) = ht(y)$ since $ht(y) = ht(x) < ht(x + y)$. This completes the proof.

An immediate consequence of this proposition is that the traditional concepts of transitivity and full transitivity for p-groups ($p \neq 2$) may be rephrased: If G is a p-group and $x, y \in G$ with $U_G(x) = U_G(y)$ then G is transitive (fully transitive) if there exists an automorphism (endomorphism) mapping x to y. Moreover, it is now much more obvious in which way transitivity and full transitivity are related for p-groups with $p \neq 2$. In the next section we will introduce the concept of weak transitivity which will play a significant role in elucidating the relationship between the concepts of full transitivity and transitivity.

However, note that for $p = 2$ Proposition 2.3 fails: Corner [3] has exhibited an Abelian 2-group which is transitive (hence Krylov transitive) but not fully transitive.

3. Weakly transitive torsion groups

We introduce the following notion of weak transitivity for torsion Abelian groups. The definition can easily be generalized to arbitrary Abelian groups (or even more general settings): for example torsion-free weakly transitive groups were considered in [11].

Definition 3.1. *Let G be a torsion group. Then G is called weakly transitive if for any pair of elements $x, y \in G$ there exists an automorphism ϕ of G mapping x onto y if and only if there exist endomorphisms φ, ψ of G such that $x\varphi = y$ and $y\psi = x$.*

Our first proposition can be found in [11] and the proof is straightforward and left to the reader.

Proposition 3.2. *Let $G = G_1 \oplus G_2$ be any torsion group such that* $\mathrm{Hom}(G_1, G_2) = 0$. *If G is weakly transitive, then G_1 and G_2 are weakly transitive.*

The next result shows that it suffices to consider p-groups in order to verify weak transitivity for a torsion group. Let $\mathbb{P}$ denote the set of primes.

Theorem 3.3. *Let T be a torsion group. Then T is weakly transitive if and only if all its primary components are weakly transitive.*

Proof. Let $T = \bigoplus_{p \in \mathbb{P}} T_p$ be the decomposition of T into its primary components. Then each T_p is an invariant subgroup of T and a summand. Thus T_p is weakly transitive if T is weakly transitive by Proposition 3.2. Conversely, assume that each T_p is weakly transitive and let $x, y \in T$ such that $x\varphi = y$ and $y\psi = x$ for some endomorphisms φ and ψ of T. Let $x = (x_p : p \in \mathbb{P})$ and $y = (y_p : p \in \mathbb{P})$ be the representations of x and y in the primary decomposition of T with $x_p = y_p = 0$ for almost all $p \in \mathbb{P}$. By the invariance of T_p in T we conclude that $x_p\varphi = y_p$ and $y_p\psi = x_p$ for all $p \in \mathbb{P}$. Hence there are automorphisms α_p of T_p such that $x_p\alpha_p = y_p$ for every $p \in \mathbb{P}$. Clearly the sum $\alpha := \bigoplus_{p\in\mathbb{P}} \alpha_p$ is an automorphism of T mapping x onto y. $\square$

We will now clarify the relationship between weak transitivity and (full) transitivity in the case of p-groups. Clearly any transitive p-group is weakly transitive but the converse need not be true (see Corollary 3.13).

Lemma 3.4. *If G is a fully transitive, weakly transitive p-group, then G is transitive.*

Proof. Let $x, y \in G$ such that $U_G(x) = U_G(y)$. Then there exist endomorphisms $\alpha, \beta \in \mathrm{End}(G)$ such that $x\alpha = y$ and $y\beta = x$ since G is fully transitive. By the weak transitivity there exists an automorphism ψ of G such that $x\psi = y$. $\square$

In [16, Theorem 26] it was proved that for $p \neq 2$, transitivity implies full transitivity. Thus we obtain our desired connection between the various transitivity-type concepts:

Corollary 3.5. *Let $p \neq 2$ be a prime. A p-group is transitive if and only if it is weakly transitive and fully transitive.*

We can now show that a summand of a weakly transitive p-group need not be weakly transitive.

Corollary 3.6. *For every prime p there is a p-group which is weakly transitive but has a summand that is not weakly transitive.*

Proof. Corner [3] has established the existence of a p-group T which is fully transitive but not transitive. Thus it follows from [8, Corollary 3] that $T \oplus T$ is transitive, and hence weakly transitive. However the summand T can not be weakly transitive for otherwise Lemma 3.4 would imply that T is transitive. $\square$

In [3] and [10] it was shown that there exist large classes of fully transitive, non-transitive p-groups. Thus we have the following:

Lemma 3.7. *There exist large classes of fully transitive p-groups which are not weakly transitive.*

To show the independence of the three notions of transitivity we use the following modified version of a result by A.L.S. Corner: if G is a p-group and $1 \in R$ is a subring of $\operatorname{End}(G)$, then we say that R acts weakly transitively on $p^\omega G$ (weakly transitively on G respectively) if for any pair of elements $x, y \in p^\omega G$ $(x, y \in G)$ and endomorphisms $\varphi, \psi \in R$ such that $x\varphi = y$ and $y\psi = x$ there is an automorphism $\alpha \in R$ with $x\alpha = y$. In a similar way, we define R as acting fully transitively or transitively on $p^\omega G$, G respectively.

Proposition 3.8. *A p-group G is weakly transitive (fully transitive, transitive) if and only if* $\operatorname{End}(G)$ *acts weakly transitively (fully transitively, transitively) on $p^\omega G$.*

Proof. The result is known for full transitivity and transitivity and the proof for weak transitivity carries over verbatim from [3]. □

Recall that a p-group T is said to be of type A if $\operatorname{Aut}(T) \restriction_{p^\omega T} = U(\operatorname{End}(T) \restriction_{p^\omega T})$ (see [2]). Note, for instance that all groups which are constructed using Corner's original realization theorem for p-groups, are of type A.

In [19, Theorem 2.4] the following was proved.

Theorem 3.9. *Let $1 \in R$ be a finite semisimple ring and M a finitely generated R-module. If $u, v \in M$ and $r, s \in R$ such that $ur = v$ and $vs = u$, then there exists a unit $t \in R$ such that $ut = v$.*

An easy application of Theorem 3.9 gives

Theorem 3.10. *Let G be a finite p-group and $1 \in R \subseteq \operatorname{End}(G)$. Then R acts weakly transitively on G.*

Proof. Let $x, y \in G$ and $\varphi, \psi \in R$ such that $x\varphi = y$ and $y\psi = x$. Denote the Jacobson radical of R by J. If $\varphi \in J$, then $x = x\varphi\psi \in xJ$, a contradiction, hence $\varphi \notin J$ and similarly $\psi \notin J$. Thus $(x+xJ)(\varphi+J) = (y+xJ)$ and $(y+xJ)(\psi+J) = (x+xJ)$ and by Theorem 3.9 it follows that $(x+xJ)(\beta+J) = (y+xJ)$ for some unit $(\beta+J) \in R/J$ since R/J is finite semi-simple. By Proposition 2.2. (iii) from [19] we get that $x\alpha = y$ for some automorphism $\alpha \in R$. □

Corollary 3.11. *If G is a p-group of type A with finite first Ulm subgroup, then G is weakly transitive.*

Proof. By Proposition 3.8 it suffices to prove that $\mathrm{End}(G)$ acts weakly transitively on $p^\omega G$. Therefore let $x, y \in p^\omega G$ and $\varphi, \psi \in \mathrm{End}(G)$ such that $x\varphi = y$ and $y\psi = x$. Since $p^\omega G$ is a fully invariant subgroup of G and finite Theorem 3.10 applies and there exists an automorphism α in $R = \mathrm{End}(G) \restriction_{p^\omega G}$ which maps x onto y. Since G is of type A any automorphism of $p^\omega G$ extends to an automorphism of G and this finishes the proof. □

We need an important result by Corner [4, Theorem 6.1].

Theorem 3.12. *Let p be a prime and T be a countable bounded Abelian p-group. If R is a countable subring of* $\mathrm{End}(T)$*, then there exists a p-group G such that $p^\omega G = T$ and* $\mathrm{End}(G) \restriction_T = R$*,* $\mathrm{Aut}(G) \restriction_T = U(R)$ *where $U(R)$ denotes the units of R.*

Corollary 3.13. *There exist weakly transitive p-groups which are neither transitive nor fully transitive.*

Proof. Choose any finite p-group H and a subring $1 \in R$ of $\mathrm{End}(H)$ which acts neither fully transitively nor transitively on H. Note that such groups and rings exist in abundance, see for example [13]. Applying Corner's Theorem 3.12 there is a p-group G such that $p^\omega G = H$ and $\mathrm{End}\, G \restriction_H = R$. Moreover, G is of type A and hence Corollary 3.11 shows that G is weakly transitive but neither fully transitive nor transitive by Proposition 3.8. □

It follows from the results in [8] that the square of a fully transitive p-group is fully transitive but the same is not true for transitive p-groups. Moreover as we saw in the proof of Corollary 3.6, there exist non-weakly transitive groups T whose direct squares $T \oplus T$ are weakly transitive. We therefore pose the following question:

Question 3.14. *Is a direct sum of weakly transitive groups again weakly transitive? Is it even true that a square of an Abelian p-group T is always weakly transitive ?*

Our final theorem shows that in some cases the square of a weakly transitive group is, in fact, weakly transitive.

Theorem 3.15. *Let G be a p-group with endomorphism ring $J_p \oplus small(G)$, where J_p is the ring of p-adic integers and $small(G)$ denotes the ideal of all small homomorphisms of G. Then G and $G \oplus G$ are both weakly transitive.*

Proof. We show firstly that G is weakly transitive. Suppose that $x, y \in p^\omega G$ and $x\varphi = y, y\psi = x$ for $\varphi, \psi \in \mathrm{End}(G)$. Now $\varphi = r + \theta$ and $\psi = s + \mu$ where $r, s \in J_p$ and θ, μ are small. However $p^\omega G$ is contained in the kernel of any small endomorphism and so θ, μ act trivially on x and y. This however forces r, s to be inverse p-adic units and consequently they are inverse automorphisms of G. Thus G is weakly transitive.

Now consider $G \oplus G$. By Proposition 3.8 it is enough to consider $H = p^\omega G \oplus p^\omega G$. Assume that we are given $0 \neq x, y \in p^\omega G \oplus p^\omega G$ and $xA = y, yB = x$ for some endomorphisms A, B of $G \oplus G$. Since $\mathrm{End}(G) \restriction_{p^\omega G} \subseteq J_p$, we may regard A and B as two by two matrices over J_p, hence there are invertible matrices $P, Q, P', Q' \in M_2(R)$ and diagonal matrices $D_1 = diag(d_1, d_2), D_2 = diag(d_1', d_2')$ such that $d_1 | d_2$ and $d_1' | d_2'$ and $A = PD_1Q$ and $B = P'D_2Q'$. Let $\tilde{x} = xP$ and $\tilde{y} = yP'$, then it suffices to find an invertible matrix T_1 such that $\tilde{x}T_1 = \tilde{y}$ since it then follows that $xPT_1{P'}^{-1} = y$ and $PT_1{P'}^{-1}$ is an automorphism of $G \oplus G$. By assumption we get $\tilde{x}D_1QP' = \tilde{y}$ and $\tilde{y}D_2Q'P = \tilde{x}$. Let $S = QP'$ then S is an automorphism of $G \oplus G$, so it suffices to find an invertible matrix T_2 such that $\tilde{x}T_2 = \tilde{x}D_1$ (then $\tilde{x}T_2S = \tilde{y}$ and we are finished). Let $\tilde{x} = (x_1, x_2)$, hence $\tilde{x}D_1 = (x_1d_1, x_2d_2)$. We have $(\tilde{x}D_1)SD_2Q'P = \tilde{x}$ hence letting $W = SD_2Q'P$ it follows that $\tilde{x}D_1W = \tilde{x}$. From now on we regard H as an J_p-module and note that group-homomorphisms are the same as J_p-homomorphisms. Therefore, since $\chi(\tilde{x}) = \chi(\tilde{x}D_1)$ (as elements of the J_p-module $G \oplus G$) and $d_1 | d_2$ we obtain that d_1 is a unit of J_p, without loss of generality $d_1 = 1$. Let $W = \left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right)$ with $a, b, c, e \in J_p$. If d_2 is a unit in J_p there is nothing to prove. Hence assume that d_2 is not a unit. Since J_p is a local ring we obtain that $d_2 + 1$ is a unit of J_p. Now consider the following equations:

$$x_1a + x_2d_2c = x_1 \ \ and \ x_1b + x_2d_2e = x_2.$$

If e is a unit of J_p, then the automorphism $\left(\begin{smallmatrix} 1 & b \\ 0 & e \end{smallmatrix}\right)$ maps (x_1, x_2d_2) onto (x_1, x_2). On the other hand if e is not a unit of J_p, $e+1$ is a unit and then $\left(\begin{smallmatrix} 1 & b \\ 0 & (e+1) \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & (1+d_2)^{-1} \end{smallmatrix}\right)$ is the required mapping. $\square$

A natural starting point in the search for a group whose square is not weakly transitive, is the class of groups whose squares are not transitive. Recall that Corner [3, §4] constructed a transitive 2-group G, with first Ulm subgroup H isomorphic to $\mathbb{Z}(2) \oplus \mathbb{Z}(8)$, such that G was not fully transitive. In our final example we show that for this group G, the square $G \oplus G$ is again weakly transitive but not transitive.

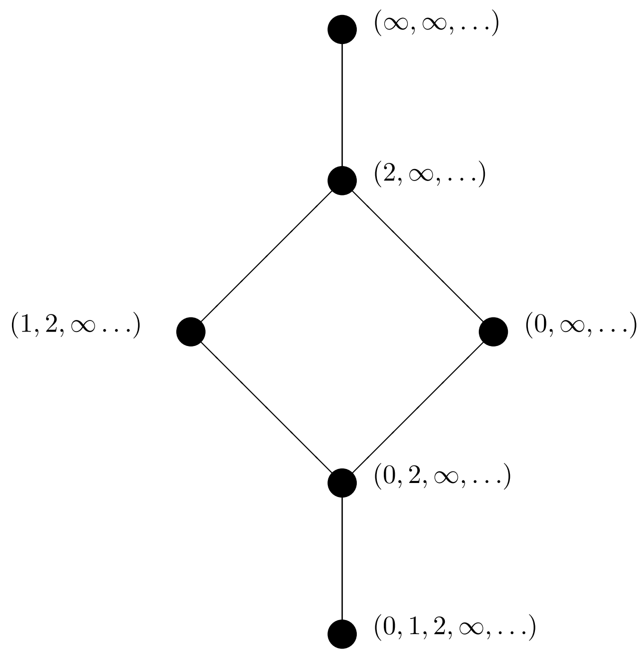
Example 3.16. *If G is the transitive, non fully transitive 2-group constructed by Corner in* [3], *then $G \oplus G$ is weakly transitive but not transitive.*

Proof. It follows immediately from [8, Corollary 3] that $G \oplus G$ is not transitive. The group G has the property that $2^\omega G = H$, $\mathrm{Aut}(G) \restriction 2^\omega G = \mathrm{Aut}(H)$ and $\mathrm{End}(G) \restriction 2^\omega G = \Phi$, where Φ is the subring of H generated by $\mathrm{Aut}(H)$. Moreover there is a semigroup homomorphism $(\quad)^* : \Phi \to \mathrm{End}(G)$ such that $\phi^* \restriction H = \phi$ for all $\phi \in \Phi$. The homomorphism * respects units but is not, of course, additive. We will show that $G \oplus G$ is weakly transitive by showing that $\mathrm{End}(G \oplus G)$ acts weakly transitively on $H \oplus H$. Before establishing this we need some further details about $H, H \oplus H$ and Φ.

The group $H = \langle a \rangle \oplus \langle b \rangle$, where a has order 2 and b has order 8, has six different associated Ulm sequences:

$$(\infty, \infty, \dots); (2, \infty, \dots); (0, \infty, \dots); (1, 2, \infty, \dots); (0, 2, \infty, \dots); (0, 1, 2, \infty, \dots).$$

The associated lattice has just one pair of incomparable Ulm types:



The subring, Φ, of the endomorphism ring of H generated by its automorphism group, has order 32 and consists of eight types of endomorphism, each parameterized by $\lambda = \pm 1, \pm 3$ or $\mu = 0, \pm 1, 2$. These are the mappings given by (i)

$\theta_{1\lambda} : a \mapsto a, b \mapsto \lambda b$ (ii) $\theta_{2\lambda} : a \mapsto a + 4b, b \mapsto \lambda b$ (iii) $\theta_{3\lambda} : a \mapsto a, b \mapsto a + \lambda b$ (iv) $\theta_{4\lambda} : a \mapsto a+4b, b \mapsto a+\lambda b$ (v) $\phi_{1\mu} : a \mapsto 4b, b \mapsto 2\mu b$ (vi) $\phi_{2\mu} : a \mapsto 0, b \mapsto a+2\mu b$ (vii) $\phi_{3\mu} : a \mapsto 4b, b \mapsto a + 2\mu b$ (viii) $\phi_{4\mu} : a \mapsto 0, b \mapsto 2\mu b$.

The mappings $\theta_{i\lambda}$ are automorphisms but the $\phi_{i\mu}$ are not.

Although the group H is not fully transitive under Φ, if $x, y \in H$ are such that $U_H(x) \leq U_H(y)$, then there is an endomorphism $\psi \in \Phi$ with $x\psi = y$ except in the case where $x = a \pm 2b$. A straightforward check shows that all elements of H with the same Ulm sequence lie in a single orbit under the action of the group of units of Φ. A useful consequence of this last statement is that in our calculations it will suffice to take just a single representative of each type of Ulm sequence; representatives of the sequences (in the order listed above) are $0, 4b, a, 2b, a+2b, b$.

A straightforward check shows that the lattice of Ulm types of $H \oplus H$ is identical to that of H. We establish the weak transitivity of $G \oplus G$ by showing that with one exception, for any pair $(x, y), (z, w) \in H \oplus H$ with equal Ulm sequences in $H \oplus H$, there is a 2×2 matrix M over Φ with $(x, y)M = (z, w)$. We will then show that this matrix M may be lifted to a 2×2 invertible matrix M^* over $\mathrm{End}(G)$ with $(x, y)M^* = (z, w)$. In the exceptional case we shall exhibit the appropriate matrix M or show that it is impossible to have mappings ϕ, ψ with $(x, y)\phi = (z, w)$, $(z, w)\psi = (x, y)$. Clearly this will suffice to establish weak transitivity.

Inevitably the argument reduces to a case by case consideration requiring considerable calculations. We shall compromise between the extremes of omitting all details and giving the full calculations, by outlining the more difficult steps in some detail and leaving the straightforward ones to the reader.

Case 1. $U((x, y)) = U((z, w)) = (0, 1, 2, \infty, \dots)$.

In this situation at least one element in each pair has type $(0, 1, 2, \infty, \dots)$, say x, z. Note also that $U(x) \leq U(w - y)$. Then as noted above there is an automorphism $\alpha \in \Phi$ and an element $\beta \in \Phi$ with $x\alpha = z, x\beta = w - y$. Then the matrix $M = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right)$ clearly maps $(x, y) \mapsto (z, w)$. If we set $M^* = \left(\begin{smallmatrix} \alpha^* & \beta^* \\ 0 & 1 \end{smallmatrix}\right)$, then M^* is a 2×2 matrix over $\mathrm{End}(G)$ and is easily seen to be invertible since α^* is a unit.

Case 2. $U((x, y)) = U((z, w)) = (0, \infty, \dots)$.

Again we may assume that $U(x) = U(z) = (0, \infty, \dots)$. Notice that y cannot have Ulm sequence $(1, 2, \infty, \dots)$ because this would force $U((x, y)) = (0, 2, \infty, \dots)$. Thus we must have $U(y) \geq U(x)$. Similarly $U(w) \geq U(z)$. As in case 1, there is an automorphism α with $x\alpha = z$ and a map β with $x\beta = w - y$. The rest of the argument follows exactly as in Case 1.

Case 3. $U((x, y)) = U((z, w)) = (1, 2, \infty, \dots)$ or $(2, \infty, \dots)$ or $(\infty, \dots)$.

Here the first two cases are similar to Case 2 while the final case is trivial.

The remaining case is the core of the argument:

Case 4. $U((x,y)) = U((z,w)) = (0,2,\infty,\dots)$.

This situation can arise in two different ways:

(I) $U(x) = (0,2,\infty,\dots)$, $U(y) \geq U(x), U(z) = U(x), U(w) \geq U(z)$ or

(II) $U(x) = (0,2,\infty,\dots)$ and $U(y) \geq U(x), U(z) = (0,\infty,\dots), U(w) = (1,2,\infty,\dots)$.

Since all terms with a given Ulm sequence lie in the same orbit under the automorphism group of H, we may assume that in (I) $x = z = a + 2b,\ U(y), U(w) \geq U(x)$. In case(II) we may assume that $x = a + 2b,\ U(y) \geq U(x),\ z = a,\ w = 2b$.

We consider firstly the various subcases which arise in (I). Note that there is no loss in generality in assuming that $U(y) \not\leq U(w)$: if $U(y) \leq U(w)$ simply interchange the roles of (x,y) and (z,w).

Subcase (i) $y = a + 2b$, so that $w \in \{2b, a, 4b, 0\}$.

The last two possibilities are straightforward: the 2×2 matrices over Φ $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ map $(a+2b, a+2b)$ to $(a+2b, 4b)$ and $(a+2b, 0)$ respectively, and these matrices clearly lift to automorphisms of $G \oplus G$. In the remaining two possible cases, there can be no mapping from $(a+2b, a+2b)$ to $(a+2b, 2b)$ nor $(a+2b, a)$. This is because the subgroup generated by $a \pm 2b$ is fully invariant and equals $\{0, 4b, a \pm 2b\}$; clearly neither $2b$ nor a belong to this subgroup.

Subcase (ii) $y = a$, so that $w \in \{2b, 4b, 0\}$.

The first possibility is easily handled: the 2×2 matrix over Φ $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ maps $(a + 2b, a)$ to $(a + 2b, 2b)$ and, as before, lifts to an automorphism of $G \oplus G$. The remaining two cases do not arise since it is not possible to have maps from $(a+2b, 0)$ to $(a+2b, a)$ nor from $(a+2b, 4b)$ to $(a+2b, a)$. We outline the argument in the second situation; the first situation is similar and slightly easier. So suppose we have a matrix over Φ, say $\begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix}$ mapping $(a + 2b, 4b)$ to $(a + 2b, a)$. Then $(a+2b)\psi_2 + 4b\psi_4 = a$, or equivalently, $a\psi_2 = a - 2r$, where $r = (b\psi_2 + 2b\psi_4)$. An examination of the possibilities for ψ_2 show that it must be of the form $\theta_{i\lambda}$ for some i, λ; $a\phi_{j\mu}$ never contains a term in a. Examining each $\theta_{i\lambda}$, it is easy to see that either $2r = 0$ or $2r + 4b = 0$. Noting that $2b\psi_2$ must then have a term in b of the form $2\lambda b$, we conclude that either $2\lambda b + 4b\psi_4 = 0$ or $2\lambda b + 4b\psi_4 + 4b = 0$; both are impossible since $(\lambda, 2) = 1$.

Subcase (iii) $y = 2b$ so that $w \in \{a, 4b, 0\}$.

Again the first possibility is easy to handle: the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ maps $(a + 2b, 2b) \mapsto (a + 2b, a)$ and lifts to an automorphism of $G \oplus G$. The remaining possibilities for w do not arise because, again, there are no maps from $(a+2b, 4b)$ nor $(a + 2b, 0)$ to $(a + 2b, 2b)$. The latter possibility cannot occur because $2b$ is not in the fully invariant subgroup generated by $a \pm 2b$. In the former, assuming the same notation as in Subcase (ii), if $(a+2b)\psi_2 + 4b\psi_4 = 2b$, then we know that $(a+2b)\psi_2 \in \{a \pm 2b, 4b, 0\}$. Clearly the first two possibilities $a \pm 2b$ cannot occur, while the remaining possibilities would imply that b is divisible by 2, which is not so.

The final subcase for (I) is :

Subcase (iv) $y = 4b$ so that $w = 0$.

The matrix $\left(\begin{smallmatrix} 1 & \phi_{21} \\ 0 & -1 \end{smallmatrix}\right)$ maps $(a+2b, 4b)$ to $(a+2b, 0)$ and lifts to an automorphism of $G \oplus G$.

Now consider the situation in case (II) where $x = a + 2b,\ U(y) \geq U(x),\ z = a,\ w = 2b$. There are five possibilities to be considered for y: $y \in \{a, 2b, a + 2b, 4b, 0\}$. The first two possibilities are easily handled: the matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ \theta_{13} & 1 \end{smallmatrix}\right)$, which lift to automorphisms of $G \oplus G$, map $(a + 2b, a)$ and $(a + 2b, 2b)$ respectively to $(a, 2b)$. If $y = 0$ then there is no map from $(a + 2b, 0) \mapsto (a, 2b)$ since $2b$ is not in the fully invariant subgroup generated by $a \pm 2b$. So there are just two remaining cases: $y = a + 2b,\ y = 4b$. The first of these possibilities cannot occur because the components of the image of $(a + 2b, a + 2b)$ must each belong to the fully invariant subgroup $\{a \pm 2b, 4b, 0\}$. So it remains only to deal with the case where $y = 4b$. Suppose for a contradiction that there is a matrix $\begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix}$ mapping $(a + 2b, 4b) \mapsto (a, 2b)$. Then $(a + 2b)\psi_1 + 4b\psi_3 = a$. However $(a+2b)\psi_1 \in \{a \pm 2b, 4b, 0\}$. The last two possibilities cannot occur since $4b\psi_3$ cannot have a term in a. However $(a + 2b)\psi_1 = a \pm 2b$ would imply that $a \pm 2b + 4kb = a$ for some integer k. But then $2b + 4kb = 0$ – contradiction. This completes the proof of our Example. $\square$

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