




2014

Examples of G-Strand Equations

Darryl Holm
Imperial College London

Rossen Ivanov
Dublin Institute of Technology, rivanov@dit.ie

Follow this and additional works at: <http://arrow.dit.ie/scschmatcon>

 Part of the [Dynamic Systems Commons](#), [Non-linear Dynamics Commons](#), [Ordinary Differential Equations and Applied Dynamics Commons](#), and the [Partial Differential Equations Commons](#)

Recommended Citation

Holm, D. & R. Ivanov (2014). Examples of G-Strand Equations. In Slavova, A. (ed) *Mathematics in Industry*. Cambridge Scholars Publishing, pp.306-320.

This Conference Paper is brought to you for free and open access by the School of Mathematics at ARROW@DIT. It has been accepted for inclusion in Conference papers by an authorized administrator of ARROW@DIT. For more information, please contact yvonne.desmond@dit.ie, arrow.admin@dit.ie, brian.widdis@dit.ie.



This work is licensed under a [Creative Commons Attribution-NonCommercial-Share Alike 3.0 License](#)



EXAMPLES OF G -STRAND EQUATIONS

DARRYL D. HOLM AND ROSSEN I.
IVANOV

1 Introduction

The G -strand equations for a map $\mathbb{R} \times \mathbb{R}$ into a Lie group G are associated to a G -invariant Lagrangian. The Lie group manifold is also the configuration space for the Lagrangian. The G -strand itself is the map $g(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow G$, where t and s are the independent variables of the G -strand equations. The Euler-Poincaré reduction of the variational principle leads to a formulation where the dependent variables of the G -strand equations take values in the corresponding Lie algebra \mathfrak{g} and its co-algebra, \mathfrak{g}^* with respect to the pairing provided by the variational derivatives of the Lagrangian.

We review examples of two G -strand constructions, including matrix Lie groups and the Diffeomorphism group. In some cases the G -strand equations are completely integrable 1+1 Hamiltonian systems that admit soliton solutions.

Our presentation is based on our previous works [15, 9, 8, 13, 10] and is aimed to illustrate the G -strand construction with two simple but instructive examples:

(i) $SO(3)$ -strand integrable equations for Lax operators, quadratic in the spectral parameter;

(ii) $\text{Diff}(\mathbb{R})$ -strand equations. These equations are in general non-integrable; however they admit solutions in 2 + 1 space-time with singular support (e.g., peakons). The one- and two-peakon equations obtained from the $\text{Diff}(\mathbb{R})$ -strand equations can be solved analytically, and potentially they can be applied in the theory of image registration. Our example is with a system which is a 2 + 1 generalization of the Hunter-Saxton equation.

2 Ingredients of Euler–Poincaré theory for left G -Invariant Lagrangians

Let G be a Lie group. A map $g(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow G$ has two types of tangent vectors, $\dot{g} := g_t \in TG$ and $g' := g_s \in TG$. Assume that the Lagrangian density function $L(g, \dot{g}, g')$ is left G -invariant. The left G -invariance of L permits us to define $l : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ by

$$L(g, \dot{g}, g') = L(g^{-1}g, g^{-1}\dot{g}, g^{-1}g') \equiv l(g^{-1}\dot{g}, g^{-1}g').$$

Conversely, this relation defines for any reduced lagrangian $l = l(\mathbf{u}, \mathbf{v}) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ a left G -invariant function $L : TG \times TG \rightarrow \mathbb{R}$ and a map $g(t, s) : \mathbb{R} \times \mathbb{R} \rightarrow G$ such that

$$\mathbf{u}(t, s) := g^{-1}g_t(t, s) = g^{-1}\dot{g}(t, s) \quad \text{and} \quad \mathbf{v}(t, s) := g^{-1}g_s(t, s) = g^{-1}g'(t, s).$$

Lemma 2.1. *The left-invariant tangent vectors $\mathbf{u}(t, s)$ and $\mathbf{v}(t, s)$ at the identity of G satisfy*

$$\mathbf{v}_t - \mathbf{u}_s = -\text{ad}_{\mathbf{u}}\mathbf{v}. \quad (1)$$

Proof. The proof is standard and follows from equality of cross derivatives $g_{ts} = g_{st}$.

Equation (1) is usually called a *zero-curvature relation*. \square

Theorem 2.2 (Euler-Poincaré theorem for left-invariant Lagrangians).

With the preceding notation, the following two statements are equivalent:

- i *Variational principle on $TG \times TG$ $\delta \int_{t_1}^{t_2} L(g(t, s), \dot{g}(t, s), g'(t, s)) ds dt = 0$ holds, for variations $\delta g(t, s)$ of $g(t, s)$ vanishing at the endpoints in t and s . The function $g(t, s)$ satisfies Euler–Lagrange equations for L on G , given by*

$$\frac{\partial L}{\partial g} - \frac{\partial}{\partial t} \frac{\partial L}{\partial g_t} - \frac{\partial}{\partial s} \frac{\partial L}{\partial g_s} = 0.$$

- ii *The constrained variational principle¹*

$$\delta \int_{t_1}^{t_2} l(\mathbf{u}(t, s), \mathbf{v}(t, s)) ds dt = 0$$

¹As with the basic Euler–Poincaré equations, this is not strictly a variational principle in the same sense as the standard Hamilton’s principle. It is more like the Lagrange d’Alembert principle, because we impose the stated constraints on the variations allowed.

holds on $\mathfrak{g} \times \mathfrak{g}$, using variations of $\mathbf{u} := g^{-1}g_t(t, s)$ and $\mathbf{v} := g^{-1}g_s(t, s)$ of the forms

$$\delta \mathbf{u} = \dot{\mathbf{w}} + \text{ad}_{\mathbf{u}} \mathbf{w} \quad \text{and} \quad \delta \mathbf{v} = \mathbf{w}' + \text{ad}_{\mathbf{v}} \mathbf{w},$$

where $\mathbf{w}(t, s) := g^{-1}\delta g \in \mathfrak{g}$ vanishes at the endpoints. The Euler-Poincaré equations hold on $\mathfrak{g}^* \times \mathfrak{g}^*$ (G -strand equations)

$$\frac{d}{dt} \frac{\delta l}{\delta \mathbf{u}} - \text{ad}_{\mathbf{u}}^* \frac{\delta l}{\delta \mathbf{u}} + \frac{d}{ds} \frac{\delta l}{\delta \mathbf{v}} - \text{ad}_{\mathbf{v}}^* \frac{\delta l}{\delta \mathbf{v}} = 0 \quad \mathcal{E} \quad \partial_s \mathbf{u} - \partial_t \mathbf{v} = [\mathbf{u}, \mathbf{v}] = \text{ad}_{\mathbf{u}} \mathbf{v}$$

where $(\text{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*)$ is defined via $(\text{ad} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g})$ in the dual pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ by,

$$\left\langle \text{ad}_{\mathbf{u}}^* \frac{\delta l}{\delta \mathbf{u}}, \mathbf{v} \right\rangle_{\mathfrak{g}^*} = \left\langle \frac{\delta l}{\delta \mathbf{u}}, \text{ad}_{\mathbf{u}} \mathbf{v} \right\rangle_{\mathfrak{g}}.$$

In 1901 Poincaré in his famous work proves that, when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the well known Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra. These equations are called now in his honor Euler-Poincaré equations. In modern language the contents of the Poincaré's article [14] is presented for example in [7, 5]. English translation of the article [14] can be found as Appendix D in [7].

3 G -strand equations on matrix Lie algebras

Denoting $\mathbf{m} := \delta l / \delta \mathbf{u}$ and $\mathbf{n} := \delta l / \delta \mathbf{v}$ in \mathfrak{g}^* , the G -strand equations become

$$\mathbf{m}_t + \mathbf{n}_s - \text{ad}_{\mathbf{u}}^* \mathbf{m} - \text{ad}_{\mathbf{v}}^* \mathbf{n} = 0 \quad \text{and} \quad \partial_t \mathbf{v} - \partial_s \mathbf{u} + \text{ad}_{\mathbf{u}} \mathbf{v} = 0.$$

For G a semisimple matrix Lie group and \mathfrak{g} its matrix Lie algebra these equations become

$$\begin{aligned} \mathbf{m}_t^T + \mathbf{n}_s^T + \text{ad}_{\mathbf{u}} \mathbf{m}^T + \text{ad}_{\mathbf{v}} \mathbf{n}^T &= 0, \\ \partial_t \mathbf{v} - \partial_s \mathbf{u} + \text{ad}_{\mathbf{u}} \mathbf{v} &= 0 \end{aligned} \tag{2}$$

where the ad-invariant pairing for semisimple matrix Lie algebras is given by

$$\langle \mathfrak{m}, \mathfrak{n} \rangle = \frac{1}{2} \text{tr}(\mathfrak{m}^T \mathfrak{n}),$$

the transpose gives the map between the algebra and its dual $(\cdot)^T : \mathfrak{g} \rightarrow \mathfrak{g}^*$. For semisimple matrix Lie groups, the adjoint operator is the matrix commutator. Examples are studied in [15, 8, 13].

4 Lie-Poisson Hamiltonian formulation

Legendre transformation of the Lagrangian $\ell(\mathfrak{u}, \mathfrak{v}) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ yields the Hamiltonian $h(\mathfrak{m}, \mathfrak{v}) : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$

$$h(\mathfrak{m}, \mathfrak{v}) = \langle \mathfrak{m}, \mathfrak{u} \rangle - \ell(\mathfrak{u}, \mathfrak{v}). \quad (3)$$

Its partial derivatives imply

$$\frac{\delta \ell}{\delta \mathfrak{u}} = \mathfrak{m}, \quad \frac{\delta h}{\delta \mathfrak{m}} = \mathfrak{u} \quad \text{and} \quad \frac{\delta h}{\delta \mathfrak{v}} = -\frac{\delta \ell}{\delta \mathfrak{v}} = \mathfrak{v}.$$

These derivatives allow one to rewrite the Euler-Poincaré equation solely in terms of momentum \mathfrak{m} as

$$\begin{aligned} \partial_t \mathfrak{m} &= \text{ad}_{\delta h / \delta \mathfrak{m}}^* \mathfrak{m} + \partial_s \frac{\delta h}{\delta \mathfrak{v}} - \text{ad}_{\mathfrak{v}}^* \frac{\delta h}{\delta \mathfrak{v}}, \\ \partial_t \mathfrak{v} &= \partial_s \frac{\delta h}{\delta \mathfrak{m}} - \text{ad}_{\delta h / \delta \mathfrak{m}} \mathfrak{v}. \end{aligned} \quad (4)$$

Assembling these equations into Lie-Poisson Hamiltonian form gives,

$$\frac{\partial}{\partial t} \begin{bmatrix} \mathfrak{m} \\ \mathfrak{v} \end{bmatrix} = \begin{bmatrix} \text{ad}^*(\cdot) \mathfrak{m} & \partial_s - \text{ad}_{\mathfrak{v}}^* \\ \partial_s + \text{ad}_{\mathfrak{v}} & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \mathfrak{m} \\ \delta h / \delta \mathfrak{v} \end{bmatrix} \quad (5)$$

The Hamiltonian matrix in equation (5) also appears in the Lie-Poisson brackets for Yang-Mills plasmas, for spin glasses and for perfect complex fluids, such as liquid crystals.

5 Example: Integrable $SO(3)$ G-strands with Lax operator, quadratic in the spectral parameter

Integrable $SO(3)$ G-strand system was studied in [15] by linking it to the integrable P -chiral model of [16, 1, 4]. The Lax operator of the system in

[15] is linear in the spectral parameter. In this example we will provide the zero curvature representation of a $SO(3)$ G-Strand equations and thereby prove its integrability, where the Lax operator is quadratic in the spectral parameter.

5.1 The hat map $\hat{\cdot} : (\mathfrak{so}(3), [\cdot, \cdot]) \rightarrow (\mathbb{R}^3, \times)$

The Lie algebra $(\mathfrak{so}(3), [\cdot, \cdot])$ with matrix commutator bracket $[\cdot, \cdot]$ maps to the Lie algebra (\mathbb{R}^3, \times) with vector product \times , by the linear isomorphism

$$\mathbf{u} := (u^1, u^2, u^3) \in \mathbb{R}^3 \mapsto \hat{\mathbf{u}} := \begin{bmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{bmatrix} \in \mathfrak{so}(3).$$

In matrix and vector components, the linear isomorphism is $\hat{u}_{ij} := -\epsilon_{ijk}u^k$. Equivalently, this isomorphism is given by $\hat{\mathbf{u}}\mathbf{v} = \mathbf{u} \times \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. This is the hat map $\hat{\cdot} : (\mathfrak{so}(3), [\cdot, \cdot]) \rightarrow (\mathbb{R}^3, \times)$, which holds for the skew-symmetric 3×3 matrices in the matrix Lie algebra $\mathfrak{so}(3)$.

One may verify the following useful formulas for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$:

$$\begin{aligned} (\mathbf{u} \times \mathbf{v})^\wedge &= \hat{\mathbf{u}}\hat{\mathbf{v}} - \hat{\mathbf{v}}\hat{\mathbf{u}} =: [\hat{\mathbf{u}}, \hat{\mathbf{v}}], \\ [\hat{\mathbf{u}}, \hat{\mathbf{v}}]\mathbf{w} &= (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}, \\ ((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})^\wedge &= [[\hat{\mathbf{u}}, \hat{\mathbf{v}}], \hat{\mathbf{w}}], \\ \mathbf{u} \cdot \mathbf{v} &= -\frac{1}{2} \text{trace}(\hat{\mathbf{u}}\hat{\mathbf{v}}) =: \langle \hat{\mathbf{u}}, \hat{\mathbf{v}} \rangle, \end{aligned}$$

in which the dot product of vectors is also the natural pairing of 3×3 skew-symmetric matrices.

5.2 The $SO(3)$ G-Strand system in \mathbb{R}^3 vector form

By using the hat map, $\mathfrak{so}(3) \rightarrow \mathbb{R}^3$, the matrix G-Strand system for $SO(3)$ [15] may be written in \mathbb{R}^3 vector form by following the analogy with the Euler rigid body in standard notation,

$$\begin{aligned} \partial_t \Pi + \Omega \times \Pi - \partial_s \Xi - \Gamma \times \Xi &= 0, \\ \partial_t \Gamma - \partial_s \Omega - \Gamma \times \Omega &= 0, \end{aligned} \tag{6}$$

where $\Omega := O^{-1}\partial_t O \in \mathfrak{so}(3)$ and $\Pi := \partial\ell/\partial\Omega \in \mathfrak{so}(3)^*$ are the body angular velocity and momentum, while $\Gamma := O^{-1}\partial_s O \in \mathfrak{so}(3)$ and $\Xi = -\partial\ell/\partial\Gamma \in \mathfrak{so}(3)^*$ are the body angular strain and stress. These G-Strand equations for $\mathfrak{g} = \mathfrak{so}(3)$ equations may be expressed in Lie–Poisson

Hamiltonian form in terms of vector operations in $\mathfrak{so}(3) \times \mathbb{R}^3$ as,

$$\frac{\partial}{\partial t} \begin{bmatrix} \Pi \\ \Gamma \end{bmatrix} = \begin{bmatrix} \Pi \times & \partial_s + \Gamma \times \\ \partial_s + \Gamma \times & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \Pi = \Omega \\ \delta h / \delta \Gamma = \Xi \end{bmatrix}. \quad (7)$$

This Hamiltonian matrix yields a *Lie–Poisson bracket* defined on the dual of the semidirect-product Lie algebra $\mathfrak{so}(3) \ltimes \mathbb{R}^3$ with a two-cocycle given by ∂_s . Namely,

$$\begin{aligned} \{f, h\} &= \int [\delta f / \delta \Pi, \delta f / \delta \Gamma] \cdot \begin{bmatrix} \Pi \times & \partial_s + \Gamma \times \\ \partial_s + \Gamma \times & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \Pi \\ \delta h / \delta \Gamma \end{bmatrix} ds \\ &= \int -\Pi \cdot \frac{\delta f}{\delta \Pi} \times \frac{\delta h}{\delta \Pi} - \Gamma \cdot \left(\frac{\delta f}{\delta \Pi} \times \frac{\delta h}{\delta \Gamma} - \frac{\delta h}{\delta \Pi} \times \frac{\delta f}{\delta \Gamma} \right) \\ &\quad + \frac{\delta f}{\delta \Pi} \partial_s \frac{\delta h}{\delta \Gamma} + \frac{\delta f}{\delta \Gamma} \partial_s \frac{\delta h}{\delta \Pi} ds. \end{aligned} \quad (8)$$

Dual variables are Π dual to $\mathfrak{so}(3)$ and Γ dual to \mathbb{R}^3 . For more information about Lie–Poisson brackets, see [11].

The \mathbb{R}^3 G-Strand equations (6) combine two classic ODEs due separately to Euler and Kirchhoff into a single PDE system. The \mathbb{R}^3 vector representation of $\mathfrak{so}(3)$ implies that $\text{ad}_\Omega^* \Pi = -\Omega \times \Pi = -\text{ad}_\Omega \Pi$, so the corresponding Euler–Poincaré equation has a ZCR. To find its integrability conditions, we set

$$L := \lambda^2 A + \lambda \Pi + \Gamma \quad \text{and} \quad M := \lambda^2 B + \lambda \Xi + \Omega, \quad (9)$$

and compute the conditions in terms of Π , Ω , Ξ , Γ and the constant vectors A and B that are required to write the vector system (6) in zero-curvature form,

$$\partial_t L - \partial_s M - L \times M = 0. \quad (10)$$

By direct substitution of (9) into (10) and equating the coefficient of each power of λ to zero, one finds

$$\begin{aligned} \lambda^4 : A \times B &= 0 \\ \lambda^3 : A \times \Xi - B \times \Pi &= 0 \\ \lambda^2 : A \times \Omega - B \times \Gamma + \Pi \times \Xi &= 0 \\ \lambda^1 : \Pi \times \Omega + \Gamma \times \Xi &= \partial_t \Pi - \partial_s \Xi \quad (\text{EP equation}) \\ \lambda^0 : \Gamma \times \Omega &= \partial_t \Gamma - \partial_s \Omega \quad (\text{compatibility}) \end{aligned} \quad (11)$$

where A and B are taken as constant nonzero vectors. These imply the

following relationships

$$\begin{aligned}\lambda^4 : A &= \alpha B \\ \lambda^3 : A \times (\Xi - \Pi/\alpha) &= 0 \implies \Xi - \Pi/\alpha = \beta A \\ \lambda^2 : A \times (\Omega - \Gamma/\alpha) &= \Xi \times \Pi = \beta A \times \Pi\end{aligned}\quad (12)$$

We solve equations (12) for the diagnostic variables Ξ and Ω , as

$$\Xi - \frac{1}{\alpha}\Pi = \beta A \quad \text{and} \quad \Omega - \frac{\Gamma}{\alpha} - \beta\Pi = \gamma A, \quad (13)$$

where α, β, γ are real scalars.

The conserved quantities can be evaluated from the Lax representation:

$$\begin{aligned}H_{-1} &= \int (A \cdot \Pi) ds \\ H_0 &= \int \left(\frac{(A \times \Pi)^2}{2|A|^2} + A \cdot \Gamma \right) ds \\ H_1 &= \int \left(\Pi \cdot \Gamma - (A \cdot \Pi) \left(\frac{(A \times \Pi)^2}{2|A|^4} + \frac{(A \cdot \Gamma)}{|A|^2} \right) \right) ds.\end{aligned}\quad (14)$$

Let us now try to find a Hamiltonian h as a linear combination of H_{-1} , H_0 and H_1 , i.e. $h = c_{-1}H_{-1} + c_0H_0 + c_1H_1$ for some numerical constants c_k . We need to satisfy the two relations

$$\begin{aligned}\frac{\delta h}{\delta \Pi} &= c_{-1} \frac{\delta H_{-1}}{\delta \Pi} + c_0 \frac{\delta H_0}{\delta \Pi} + c_1 \frac{\delta H_1}{\delta \Pi} = \frac{1}{\alpha} \Gamma + \beta \Pi + \gamma A \equiv \Omega, \\ \frac{\delta h}{\delta \Gamma} &= c_{-1} \frac{\delta H_{-1}}{\delta \Gamma} + c_0 \frac{\delta H_0}{\delta \Gamma} + c_1 \frac{\delta H_1}{\delta \Gamma} = \frac{1}{\alpha} \Pi + \beta A \equiv \Xi\end{aligned}\quad (15)$$

Comparing the scalar coefficients arising in front of the vectors A , Π and Γ from both sides of (15) we obtain

$$\begin{aligned}\alpha &= c_1 = 1, \\ \beta &= c_0 - \frac{A \cdot \Pi}{|A|^2}, \\ \gamma &= c_{-1} - c_0 \frac{A \cdot \Pi}{|A|^2} - \frac{\Pi^2}{2|A|^2} + \frac{3(A \cdot \Pi)^2}{2|A|^4} - \frac{A \cdot \Gamma}{|A|^2},\end{aligned}\quad (16)$$

c_{-1} and c_0 are arbitrary real constants. The most general Hamiltonian of course could contain combinations of all conserved quantities, with coefficients c_k where possibly $k > 1$. In such cases the expressions for α , β and γ of course will contain terms, related to the higher conserved quantities, entering the Hamiltonian.

6 The Diff(\mathbb{R})-strand system

The constructions described briefly in the previous sections can be easily generalized in cases where the Lie group is the group of the Diffeomorphisms. Consider Hamiltonian which is a right-invariant bilinear form $H(u, v)$. The manifold \mathcal{M} where u and v are defined is \mathbb{S}^1 or in the case when the class of smooth functions vanishing rapidly at $\pm\infty$ is considered, we will allow $\mathcal{M} \equiv \mathbb{R}$. Let us introduce the notation $u(g(x)) \equiv u \circ g$. Let us further consider an one-parametric family of diffeomorphisms, $g(x, t) \in \text{Diff}(\mathcal{M})$ by defining the t - evolution as

$$\dot{g} = u(g(x, t), t), \quad g(x, 0) = x, \quad \text{i.e.} \quad \dot{g} = u \circ g \in T_g G; \quad (17)$$

$u = \dot{g} \circ g^{-1} \in \mathfrak{g}$, where \mathfrak{g} , the corresponding Lie-algebra is the algebra of vector fields, $\text{Vect}(\mathcal{M})$. Now we recall the following result:

Theorem 6.1. (A. Kirillov, 1980, [2, 3]) *The dual space of \mathfrak{g} is a space of distributions but the subspace of local functionals, called the regular dual \mathfrak{g}^* is naturally identified with the space of quadratic differentials $m(x)dx^2$ on \mathcal{M} . The pairing is given for any vector field $u\partial_x \in \text{Vect}(\mathcal{M})$ by*

$$\langle m dx^2, u\partial_x \rangle = \int_{\mathcal{M}} m(x)u(x)dx$$

The coadjoint action coincides with the action of a diffeomorphism on the quadratic differential:

$$\text{Ad}_g^* : \quad m dx^2 \mapsto m(g)g_x^2 dx^2$$

and

$$\text{ad}_u^* = 2u_x + u\partial_x$$

Indeed, a simple computation shows that

$$\begin{aligned} \langle \text{ad}_{u\partial_x}^* m dx^2, v\partial_x \rangle &= \langle m dx^2, [u\partial_x, v\partial_x] \rangle = \int_{\mathcal{M}} m(u_x v - v_x u) dx = \\ \int_{\mathcal{M}} v(2mu_x + um_x) dx &= \langle (2mu_x + um_x) dx^2, v\partial_x \rangle, \end{aligned}$$

i.e. $\text{ad}_u^* m = 2u_x m + um_x$.

The Diff(\mathbb{R})-strand system arises when we choose $G = \text{Diff}(\mathbb{R})$. For a two-parametric group we have two tangent vectors

$$\partial_t g = u \circ g \quad \text{and} \quad \partial_s g = v \circ g,$$

where the symbol \circ denotes composition of functions.

In this right-invariant case, the G -strand PDE system with reduced Lagrangian $\ell(u, v)$ takes the form,

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + \frac{\partial}{\partial s} \frac{\delta \ell}{\delta v} &= -\text{ad}_u^* \frac{\delta \ell}{\delta u} - \text{ad}_v^* \frac{\delta \ell}{\delta v}, \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial s} &= \text{ad}_u v. \end{aligned} \quad (18)$$

Of course, the distinction between the maps $(u, v) : \mathbb{R} \times \mathbb{R} \rightarrow \mathfrak{g} \times \mathfrak{g}$ and their pointwise values $(u(t, s), v(t, s)) \in \mathfrak{g} \times \mathfrak{g}$ is clear. Likewise, for the variational derivatives $\delta \ell / \delta u$ and $\delta \ell / \delta v$.

7 The $\text{Diff}(\mathbb{R})$ -strand Hamiltonian structure

Upon setting $m = \delta \ell / \delta u$ and $n = \delta \ell / \delta v$, the right-invariant $\text{Diff}(\mathbb{R})$ -strand equations in (18) for maps $\mathbb{R} \times \mathbb{R} \rightarrow G = \text{Diff}(\mathbb{R})$ in one spatial dimension may be expressed as a system of two 1+2 PDEs in (t, s, x) ,

$$\begin{aligned} m_t + n_s &= -\text{ad}_u^* m - \text{ad}_v^* n = -(um)_x - mu_x - (vn)_x - nv_x, \\ v_t - u_s &= -\text{ad}_v u = -uv_x + vu_x. \end{aligned} \quad (19)$$

The Hamiltonian structure for these $\text{Diff}(\mathbb{R})$ -strand equations is obtained by Legendre transforming to

$$h(m, v) = \langle m, u \rangle - \ell(u, v).$$

One may then write the equations (19) in Lie-Poisson Hamiltonian form as

$$\frac{d}{dt} \begin{bmatrix} m \\ v \end{bmatrix} = \begin{bmatrix} -\text{ad}^*(\cdot)m & \partial_s + \text{ad}_v^* \\ \partial_s - \text{ad}_v & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta m = u \\ \delta h / \delta v = -n \end{bmatrix}. \quad (20)$$

8 Singular solutions of the $\text{Diff}(\mathbb{R})$ -strand equations

For simplicity we continue with the following choice of Lagrangian,

$$\ell(u, v) = \frac{1}{2} \int (u_x^2 + v_x^2) dx, \quad (21)$$

The $\text{Diff}(\mathbb{R})$ -strand equations (19) admit peakon solutions in *both* momenta

$$m = -u_{xx} \quad \text{and} \quad n = -v_{xx},$$

with continuous velocities u and v . This is a two-component generalization of the Hunter-Saxton equation [18, 17].

Theorem 8.1. *The $\text{Diff}(\mathbb{R})$ -strand equations (19) admit singular solutions expressible as linear superpositions summed over $a \in \mathbb{Z}$*

$$\begin{aligned}
m(s, t, x) &= \sum_a M_a(s, t) \delta(x - Q^a(s, t)), \\
n(s, t, x) &= \sum_a N_a(s, t) \delta(x - Q^a(s, t)), \\
u(s, t, x) &= K * m = \sum_a M_a(s, t) K(x, Q^a), \\
v(s, t, x) &= K * n = \sum_a N_a(s, t) K(x, Q^a),
\end{aligned} \tag{22}$$

where $K(x, y) = -\frac{1}{2}|x - y|$ is the Green function of the operator $-\partial_x^2$:

$$-\partial_x^2 K(x, 0) = \delta(x)$$

The solution parameters $\{Q^a(s, t), M_a(s, t), N_a(s, t)\}$ with $a \in \mathbb{Z}$ that specify the singular solutions (22) (which we call 'peakons' for simplicity, although the Green function in this case is unbounded) are determined by the following set of evolutionary PDEs in s and t , in which we denote $K^{ab} := K(Q^a, Q^b)$ with integer summation indices $a, b, c, e \in \mathbb{Z}$:

$$\begin{aligned}
\partial_t Q^a(s, t) &= u(Q^a, s, t) = \sum_b M_b(s, t) K^{ab}, \\
\partial_s Q^a(s, t) &= v(Q^a, s, t) = \sum_b N_b(s, t) K^{ab}, \\
\partial_t M_a(s, t) &= -\partial_s N_a - \sum_c (M_a M_c + N_a N_c) \frac{\partial K^{ac}}{\partial Q^a} \quad (\text{no sum on } a), \\
\partial_t N_a(s, t) &= \partial_s M_a + \sum_{b, c, e} (N_b M_c - M_b N_c) \frac{\partial K^{ec}}{\partial Q^e} (K^{eb} - K^{cb})(K^{-1})_{ae}.
\end{aligned} \tag{23}$$

The last pair of equations in (23) may be solved as a system for the momenta, i.e., Lagrange multipliers (M_a, N_a) , then used in the previous pair to update the support set of positions $Q^a(t, s)$.

9 Example: Two-peakon solution of a Diff(\mathbb{R})-strand

Denote the relative spacing $X(s, t) = Q^1 - Q^2$ for the peakons at positions $Q^1(t, s)$ and $Q^2(t, s)$ on the real line and the Green's function $K = K(X)$. Then the first two equations in (23) imply

$$\begin{aligned}\partial_t X &= -(M_1 - M_2)K(X), \\ \partial_s X &= -(N_1 - N_2)K(X).\end{aligned}\tag{24}$$

The second pair of equations in (23) may then be written as

$$\begin{aligned}\partial_t M_1 &= -\partial_s N_1 - (M_1 M_2 + N_1 N_2)K'(X), \\ \partial_t M_2 &= -\partial_s N_2 + (M_1 M_2 + N_1 N_2)K'(X), \\ \partial_t N_1 &= \partial_s M_1 - (N_1 M_2 - M_1 N_2)K'(X), \\ \partial_t N_2 &= \partial_s M_2 - (N_1 M_2 - M_1 N_2)K'(X).\end{aligned}\tag{25}$$

Assuming $X > 0$, $K'(X) = -\frac{1}{2}\text{sgn}(X) = -\frac{1}{2}$. Introducing for convenience $L_{1,2} = M_{1,2} + iN_{1,2}$ we can rewrite (25) as

$$\begin{aligned}(\partial_t - i\partial_s)L_1 &= \frac{1}{2}L_1\bar{L}_2, \\ (\partial_t - i\partial_s)L_2 &= -\frac{1}{2}\bar{L}_1L_2.\end{aligned}\tag{26}$$

The solution for X can be expressed formally via $L_{1,2}$ from (24) as

$$X = \exp\left(\frac{1}{2}\Delta^{-1}\Re(L_1\bar{L}_2)\right),$$

where $\Delta = \partial_t^2 + \partial_s^2$ and $\Re(z)$ is the real part of z .

From the system (26) we obtain

$$\Delta \ln L_1 = -\frac{1}{4}L_1\bar{L}_2, \quad \Delta \ln L_2 = -\frac{1}{4}\bar{L}_1L_2,\tag{27}$$

thus $\Delta \ln L_1 = \Delta \ln \bar{L}_2$ and $L_1 = \bar{L}_2 e^h$ where $h(s, t)$ is an arbitrary harmonic function: $\Delta h = 0$. Then for the variable $\tilde{Y} = \ln L_1$ we have the equation

$$\Delta \tilde{Y} = -\frac{1}{4}e^{2\tilde{Y}-h},\tag{28}$$

and for $Y = \ln L_1 - \frac{1}{2}h - 2\ln 2 + \pi i$ we arrive at the Liouville's $2D$ equation

$$\Delta Y = e^{2Y}. \quad (29)$$

Solutions of (29) are known in the form

$$Y = \frac{1}{2} \ln \frac{w_s^2 + w_t^2}{f(w)}$$

where the function $f(w)$ could be w^2 , $\cos^2 w$, $\sin^2 w$ or $\sinh^2 w$ with w being an arbitrary harmonic function, $\Delta w = 0$ see e.g. [6, 12, 20]. Thus the solutions $L_{1,2}$ depend on two arbitrary complex harmonic functions h, w . Hence the four peakon parameters $M_{1,2}$ and $N_{1,2}$ can be given in terms of four real arbitrary harmonic functions.

Other examples including complexification of the Camassa-Holm equation [19] are given in [9].

Conclusions

The G -strand equations comprise a system of PDEs obtained from the Euler-Poincaré (EP) variational equations for a G -invariant Lagrangian, coupled to an auxiliary *zero-curvature* equation. Once the G -invariant Lagrangian has been specified, the system of G -strand equations in (2) follows automatically in the EP framework. For matrix Lie groups, some of the the G -strand systems are integrable. The singular solution of the $\text{Diff}(\mathbb{R})$ -strand equations (19) can also be obtain explicitly in some simple situations, and the freedom in the solution is given by several arbitrary harmonic functions of the variables s, t . The *complex* $\text{Diff}(\mathbb{R})$ -strand equations and their peakon collision solutions have also been solved by elementary means [9]. The stability of the single-peakon solution under perturbations into the full solution space of equations (19) would be an interesting problem for future work.

Acknowledgments

We are grateful for enlightening discussions of this material with F. Gay-Balmaz, T. S. Ratiu and C. Tronci. Work by RII was supported by the Science Foundation Ireland (SFI), under Grant No. 09/RFP/MTH2144. Work by DDH was partially supported by Advanced Grant 267382 FCCA from the European Research Council.

Bibliography

- [1] Bordag L. A. and Yanovski A. B., *Polynomial Lax pairs for the chiral $O(3)$ field equations and the Landau-Lifshitz equation*, J. Phys. A: Math. Gen. **28** (1995), 4007–4013.
- [2] Kirillov A., *Orbits of the group of diffeomorphisms of a circle and local Lie superalgebras*, Funct. Anal. Appl. (English transl.) **15** (1981), 135–137.
- [3] ———, *The orbit method. II. Infinite-dimensional Lie groups and Lie algebras*, Contemp. Math. (Providence, RI: Amer. Math. Soc.) **145** (1993), 33–63.
- [4] Yanovski A. B., *Bi-Hamiltonian formulation of the $O(3)$ chiral fields equations hierarchy via a polynomial bundle*, J. Phys. A: Math. Gen. **31** (1998), 8709–8726.
- [5] Marle C.-M., *On Henri Poincaré’s Note “Sur une forme nouvelle des équations de la Mécanique”*, J. Geom. Symm. Phys. **29** (2013), 1–38.
- [6] Crowdy D., *General solutions to the 2D Liouville equation*, Int. J. Engng Sci. **35** (1997), 141–149.
- [7] Holm D. D., *Geometric Mechanics II: Rotating, Translating and Rolling*, World Scientific - Imperial College Press, Singapore, 2011.
- [8] Holm D. D. and Ivanov R. I., *Matrix G -Strands*, Preprint math-ph 1305.4010, available at <http://arxiv.org/pdf/1305.4010v1.pdf>.
- [9] ———, *G -Strands and Peakon Collisions on $\text{Diff}(\mathbb{R})$* , SIGMA **9** (2013), 027 (14 pages).
- [10] Holm D. D. and Lucas A. M., *Toda lattice G -Strands*, Preprint nlin.SI 1306.2984, available at <http://arxiv.org/pdf/1306.2984v1.pdf>.

- [11] Marsden J. E. and Ratiu T. S., *Introduction to Mechanics and Symmetry*, Springer, Berlin, 1999.
- [12] Ibragimov N. H. (ed.), *CRC Handbook of Lie Group to Differential Equations. Vol. II. Applications in engineering and physical sciences*, CRC Press, Boca Raton, Fl., 1995.
- [13] Holm D. D. Gay-Balmaz F. and Ratiu T. S., *Integrable G-Strands on semisimple Lie groups*, Preprint math-ph 1308.3800, available at <http://arxiv.org/pdf/1308.3800v1.pdf>.
- [14] Poincaré H., *Sur une forme nouvelle des équations de la Mécanique*, C. R. Acad. Sci. Paris **CXXXII** (1901), 369.
- [15] Ivanov R. I. Holm D. D. and Percival J. R., *G-Strands*, Journal of Nonlinear Science **22** (2012), 517–551.
- [16] Cherednik I., *On the integrability of the 2-dimensional asymmetric chiral $O(3)$ field equations and their quantum analogue*, J. Nuc. Phys. (in Russian) **33** (1981), 278–282.
- [17] Hunter J. and Zheng Y., *On a Completely Integrable Nonlinear Hyperbolic Variational Equation*, Physica D **79** (1994), 361–386.
- [18] Hunter J. K. and Saxton R. A., *Dynamics of Director Fields*, SIAM J. Appl. Math. **51** (1991), 1498–1521.
- [19] Camassa R. and Holm D., *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.
- [20] Kiselev A. V., *On the geometry of Liouville equation: symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math. **72** (2002), 33–49.