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On the Peakon and Soliton Solutions of an Integrable PDE with Cubic Nonlinearities

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On the peakon and soliton solutions of an integrable PDE with cubic nonlinearities

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1 Introduction

The interest inspired by the Camassa-Holm (CH) equation and its singular peakon solutions [1] prompted the search for other integrable equations with similar properties. An integrable peakon equation with cubic nonlinearities was first discovered by Qiao [12] and studied further in e.g. [13, 14]. Another equation with cubic nonlinearities was introduced by V. Novikov in [10]. The Lax pair for Novikov's equation is given in [7]. In fact the Qiao equation

$$m_t + (m(u^2 - u_x^2))_x = 0, \qquad m = u - u_{xx} \tag{1}$$

together with the CH equation

$$m_t + 2u_x m + u m_x = 0, \qquad m = u - u_{xx} \tag{2}$$

belong to the bi-Hamiltonian hierarchy of equations described by Fokas and Fuchssteiner [4]. It is known that the Qiao equation has a distinctive W/M-shape travelling wave solution [12, 13]. The peakons of Novikov's equation were studied in [8] while $2+1$ dimensional generalizations of Qiao's hierarchy are studied in [3]. Single and, mutilpeakon dynamics, weak kink, kink-peakon, and stability analysis of the Qiao equation were studied in [15] and [5], while other types of solitons are studied in [9]. Results for the CH and related equations are available in the monographs [6, 2, 11] and the references therein. Equation (1) may also be written as

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0. \tag{3}$$

Qiao introduced a 2×2 Lax pair for this equation given by the linear system $\mathbf{\Psi}_x = \mathbf{U\Psi}$ and $\mathbf{\Psi}_t = \mathbf{V\Psi}$ where

$$\begin{aligned} \mathbf{U} &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}m\lambda \\ -\frac{1}{2}m\lambda & \frac{1}{2} \end{pmatrix}, \\ \mathbf{V} &= \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}m\lambda(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}m\lambda(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}. \end{aligned} \tag{4}$$

Another equation from the same hierarchy is

$$m_t + \left(\frac{1}{m^2}\right)_x - \left(\frac{1}{m^2}\right)_{xxx} = 0. \tag{5}$$

The (white) soliton solutions of (1) and (5) were previously found in [16, 18]. These results rely on the fact that the spectral problem for (1) is gauge-equivalent to the one for the mKdV equation. In this communication we first discuss the peakon solutions of (1). Then we present soliton solutions which approaching a constant value as $|x| \to \infty$ (dark solitons). To this end we are going to reformulate the spectral problem in the form of a Schrödinger operator, which is also the spectral problem for the KdV equation.

2 Peakon solutions

In [7] there is a remark on the peakons of Qiao's equation, stating that their computation is problematic since one encounters a square of a delta-function. This difficulty can be avoided by the following transformation of (1). Assuming peakon solutions which vanish as $x \to \pm\infty$ and writing

$$m(x,t) = \sum_{k=1}^{N} p_k(t)\delta(x - x_k(t))$$

one can integrate (1) to find

$$\partial_t \int_{-\infty}^{x} m(y,t)dy + (u^2 - u_x^2)m(x,t) = 0,$$

giving

$$\partial_t \left(\sum_{k=1}^{N} p_k(t)\theta(x - x_k(t)) \right) + (u^2 - u_x^2)m(x,t) = 0,$$

It follows that

$$\sum_{k=1}^{N} \dot{p}_k(t)\theta(x - x_k(t)) - \sum_{k=1}^{N} p_k(t)\dot{x}_k(t)\delta(x - x_k(t)) + (u^2 - u_x^2) \sum_{k=1}^{N} p_k(t)\delta(x - x_k(t)) = 0,$$

which is only possible if

$$\dot{p}_k(t) = 0, \tag{6}$$
$$\dot{x}_k(t) = (u^2 - u_x^2)_{x = x_k(t)}. \tag{7}$$

The $N = 1$ peakon solution is easily obtained from the above system, $u(x,t) = \pm\sqrt{c}e^{-|x-ct|}$ where $p_1 = 2\sqrt{c}$ =constant. This solution was reported in [7].

To compute the two-peakon solution we notice that

$$H = \frac{1}{2}\int mu\ \mathrm{d}x = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + p_1 p_2 e^{-|x_1 - x_2|}$$

is a conserved quantity. Therefore $\Delta = x_1 - x_2$ is time independent, i.e. the distance between the two peakons is constant and they move together. This explains the M-shape travelling wave solution mentioned earlier, see for example Fig.1. The solution is

$$u(x,t) = \frac{1}{2}p_1 e^{-|x-ct|} + \frac{1}{2}p_2 e^{-|x-ct-\Delta|}, \qquad c = \frac{1}{2}p_1 p_2 e^{-|\Delta|}.$$

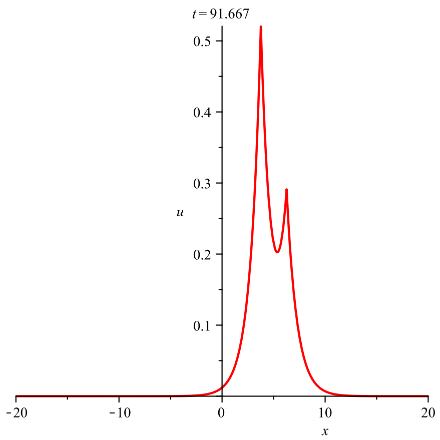

Figure 1: Two peakon profile, $p_1 = 1$, $p_2 = 0.5$, $\Delta = 2.5$.

3 Soliton solutions

3.1 Reformulation of the spectral problem

Let us consider solutions such that

$$m(x,t) > 0, \qquad \lim_{x\to\pm\infty} m(x,t) = m_0, \tag{8}$$

where m_0 is a positive constant. Let us also assume that $m(x,\cdot) - m_0 \in \mathcal{S}(\mathbb{R})$ for any value of t. One can reformulate the spectral problem into a scalar one as follows. Introducing $\mathbf{\Psi} = (\psi, \phi)^T$ the matrix Lax pair written in components becomes

$$2\psi_x = -\psi + m\lambda\phi$$
$$2\phi_x = -m\lambda\psi + \phi.$$

With a change of coordinates

$$\partial_y = \frac{2}{m}\partial_x, \qquad \psi = \frac{1}{\lambda}\left[\frac{\phi}{m} - \phi_y\right] \tag{9}$$

we obtain the following scalar spectral problem for $\phi(y, \lambda)$ (we omit the argument t which acts as an external parameter for the spectral problem being considered)

$$-\phi_{yy} + \left[\left(\frac{1}{m}\right)_y + \frac{1}{m^2}\right]\phi = \lambda^2\phi. \tag{10}$$

Note that this is a Schrödinger's operator with a potential

$$U(y, t) = \left(\frac{1}{m}\right)_y + \frac{1}{m^2}. \tag{11}$$

It is well known how to recover $U(y, t)$ from the scattering data of (10), however the solution is $m(y, t)$ and its recovery from $U(y, t)$ necessitates solving a nonlinear (Riccati) equation. We can express $m(y, t)$ in terms of the eigenfunctions of the Schrödinger's operator. We introduce $\rho(y, \lambda) = \frac{\phi_y}{\phi}$ from which we immediately obtain

$$\rho_y + \rho^2 = \frac{\phi_{yy}}{\phi} = U(y) - \lambda^2.$$

If we define $\rho_0(y) = \rho(y, 0)$ then we have

$$U(y) = \rho_{0,y} + \rho_0^2.$$

However, comparing this with (11) we find a solution $\frac{1}{m} = \rho_0$ or

$$m(y, t) = \frac{1}{\rho_0(y, t)} = \left.\frac{\phi(y, t, \lambda)}{\phi_y(y, t, \lambda)}\right|_{\lambda=0} \tag{12}$$

So far we have worked with y as our variable instead of x. However we can treat y as a parameter, and then (12) represents the solution in parametric form, while the original variable x follows from (9), (12) by:

$$x(y, t) = 2\ln\phi(y, t, 0) + \text{const.} \tag{13}$$

Assuming that $\phi(y, t, 0)$ is positive everywhere, we have a solution in parametric form (12), (13) given entirely in terms of the eigenfunctions $\phi(y, t, 0)$. We can formally write this solution as

$$m(x, t) = 2\int_{-\infty}^{\infty} \delta\left(x - 2\ln\phi(y, t, 0)\right) \mathrm{d}y. \tag{14}$$

where we neglect the constant appearing in (13).

3.2 Inverse scattering and Soliton solutions

It follows from (8) and (11) that $U(y)$ does not decay to 0 as $y \to \pm\infty$. As such we need to introduce the modified potential

$$\tilde{U}(y) = U(y) - \frac{1}{m_0^2}, \qquad (15)$$

which clearly satisfies $\lim_{|y|\to\infty} \tilde{U}(y) = 0$. So we have

$$-\phi_{yy} + \left[U(y) - \frac{1}{m_0^2}\right]\phi = \left(\lambda^2 - \frac{1}{m_0^2}\right)\phi,$$

or, introducing a new spectral parameter

$$k^2 = \lambda^2 - \frac{1}{m_0^2} \qquad (16)$$

we have a standard spectral problem

$$-\phi_{yy}(k,y) + \tilde{U}(y)\phi(k,y) = k^2\phi(k,y), \qquad \tilde{U}(y) \in \mathcal{S}(\mathbb{R}). \qquad (17)$$

However when $\lambda = 0$ we find $k = \pm\frac{i}{m_0}$. This means that if we take an eigenfunction $\phi(k,y)$ of (17) which is analytic in the upper (lower) complex k-plane, we should evaluate it at $k = \frac{i}{m_0}$ $(k = -\frac{i}{m_0})$:

$$m(y,t) = \left.\frac{\phi(y,t,k)}{\phi_y(y,t,k)}\right|_{k=\pm\frac{i}{m_0}} \qquad (18)$$

$$x(y,t) = 2\ln\phi\left(y,t,\pm\frac{i}{m_0}\right). \qquad (19)$$

The spectral theory for the problem (17) is well developed, e.g. [17]. We are going to use these results to construct the soliton solutions of (1), (5). One can introduce scattering data as usual. For the time-dependence of the scattering data one needs the time-evolution of the eigenfunction $\phi(k,x)$. The Lax-pair in x and t variables for (1) has the form

$$\phi_{xx} = \frac{m_x}{m}\phi_x + \left(\frac{1}{4} - \frac{m_x}{2m} - \frac{m^2}{4}\lambda^2\right)\phi, \qquad (20)$$

$$\phi_t = \frac{1}{\lambda^2}\left[\frac{u_x + u_{xx}}{m}\right]\phi - \left[\frac{u + u_x}{\lambda^2 m} + \frac{u^2 - u_x^2}{2}\right]\phi_x + \gamma\phi, \qquad (21)$$

where γ is an arbitrary constant. Asymptotically as $x \to \pm\infty$ equation (21) becomes

$$\phi_t \to -\left[\frac{1}{\lambda^2} + \frac{m_0^2}{2}\right]\phi_x + \gamma\phi.$$

In terms of the (y,k)-variables letting $y \to \pm\infty$ we find,

$$\phi_t \quad \to \quad -\frac{m_0^3}{2}\left[\frac{k^2 m_0^2+3}{k^2 m_0^2+1}\right]\phi_y + \gamma\phi, \tag{22}$$

since $\lim_{|y|\to\infty} m = \lim_{|y|\to\infty} u = m_0$. Defining Jost solutions by

$$\lim_{y\to\pm\infty} \varphi_\pm(y,k)e^{iky} = 1, \tag{23}$$

such that

$$\varphi_-(y,k) = a(k)\varphi_+(y,k) + b(k)\bar{\varphi}_+(y,k), \qquad k \in \mathbb{R} \tag{24}$$

and noting that $\varphi_- \to ae^{-iky} + be^{iky}$ when $y \to \infty$ it follows from (22)

$$a_t = \frac{m_0^3}{4}\left[\frac{k^2 m_0^2+3}{k^2 m_0^2+1}\right](ika) + \gamma a, \qquad b_t = -\frac{m_0^3}{4}\left[\frac{k^2 m_0^2+3}{k^2 m_0^2+1}\right](ikb) + \gamma b.$$

Requiring $a_t = 0$, we find

$$b_t = -ik\frac{m_0^3}{2}\left(\frac{k^2 m_0^2+3}{k^2 m_0^2+1}\right)b(k,t)$$

and thus for the scattering coefficient $r \equiv b/a$ we have

$$r(k,t) = r(k,0)\exp\left[-ik\frac{m_0^3}{2}\left(\frac{k^2 m_0^2+3}{k^2 m_0^2+1}\right)t\right], \tag{25}$$

while the analogue on the discrete spectrum $k = i\kappa_n$, is given by

$$R_n(t) \equiv \frac{b(i\kappa_n)}{ia'(i\kappa_n)} = R_n(0)\exp\left[\frac{\kappa_n m_0^3(3-\kappa_n^2 m_0^2)}{2(1-\kappa_n^2 m_0^2)}t\right]. \tag{26}$$

It is convenient to introduce a dispersion law $f(\kappa) = \frac{\kappa m_0^3(3-\kappa^2 m_0^2)}{2(1-\kappa^2 m_0^2)}$. Then we can write

$$R_n(t) \quad = \quad R_n(0)\exp\left(f(\kappa_n)t\right). \tag{27}$$

For further convenience we introduce

$$\xi_n \equiv y - \frac{f(\kappa_n)}{2\kappa_n}t - \frac{1}{2\kappa_n}\ln\frac{R_n(0)}{2\kappa_n}.$$

The eigenfunctions of the spectral problem (17) are well known, see e.g. [17]. In the purely N-soliton case the eigenfunction analytic in the lower complex k-plane is the Jost solution $\varphi_+(y,k)$ defined in (23) which has the form

$$\varphi_+(y,t,k) = e^{iky}\left(1 + \sum_{n=1}^{N}\frac{\Gamma_n(y,t)}{k-i\kappa_n}\right) \tag{28}$$

with the residues $\Gamma_n(y,t)$ satisfying a linear system

$$\Gamma_n(y,t) = iR_n(t)e^{-2\kappa_n y}\left(1 + i\sum_{m=1}^{N}\frac{\Gamma_m(y,t)}{\kappa_n+\kappa_m}\right).$$

The time-dependence of the scattering data is given by (27). The N- soliton solution then is given in parametric form by (18) and (19) for the eigenfunction (28). The condition $0 < \kappa_n < m_0^{-1}$ is sufficient to ensure smoothness of the solitons.

3.3 Example: One-Soliton Solution

The one-soliton solution corresponds to one discrete eigenvalue $k_1 = i\kappa_1$, where κ_1 is real, positive and $\kappa_1 < m_0^{-1}$. The eigenfunction in this case is (28)

$$\varphi_+(y,t,k) = e^{iky}\left(1 + \frac{1}{k - i\kappa_1}\cdot\frac{iR_1(t)e^{-2\kappa_1 y}}{1+\frac{R_1(t)}{2\kappa_1}e^{-2\kappa_1 y}}\right). \tag{29}$$

Evaluated at $k = \frac{-i}{m_0}$ we find

$$\varphi_+(y,t,\frac{-i}{m_0}) = e^{\frac{y}{m_0}}\left(1 - \frac{1}{\frac{1}{m_0}+\kappa_1}\cdot\frac{R_1(t)e^{-2\kappa_1 y}}{1+\frac{R_1(t)}{2\kappa_1}e^{-2\kappa_1 y}}\right).$$

From (18) and (19) we obtain the one-soliton solutions

$$x(y,t) = \frac{2y}{m_0} + 2\ln\left(1 - \frac{\kappa_1 m_0 e^{-\kappa_1\xi_1}}{(1+\kappa_1 m_0)\cosh\kappa_1\xi_1}\right), \tag{30}$$

$$m(y,t) = \frac{m_0}{1+\frac{\kappa_1^2 m_0^2 \operatorname{sech}^2\kappa_1\xi_1}{1-m_0\kappa_1\tanh\kappa_1\xi_1}}. \tag{31}$$

The extremum (minimum) of m occurs when $\xi_1 = \frac{1}{4\kappa_1}\ln\left(\frac{1-m_0\kappa_1}{1+m_0\kappa_1}\right)$. This is a constant value, e.g. the soliton moves with a velocity $\frac{f(\kappa_1)}{2\kappa_1}$ that depends on the dispersion law (i.e. the chosen equation from the hierarchy). The profile of the dark soliton is given in Fig. 2.

3.4 Example: Two soliton solution

In the case of two discrete eigenvalues we compute

$$\varphi_+(y,t,\frac{-i}{m_0}) = e^{\frac{y}{m_0}}\frac{1+\nu_1 e^{-2\kappa_1\xi_1}+\nu_2 e^{-2\kappa_2\xi_2}+\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2\nu_1\nu_2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}}{1+e^{-2\kappa_1\xi_1}+e^{-2\kappa_2\xi_2}+\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}} \tag{32}$$

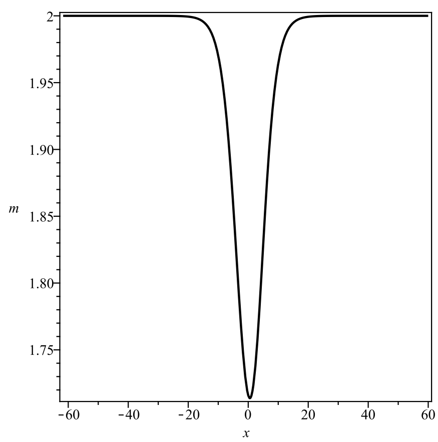

Figure 2: One soliton profile, $m_0 = 2$, $\kappa_1 = 0.2$.

where the following notation is utilized:

$$\nu_j = \frac{\frac{1}{m_0} - \kappa_j}{\frac{1}{m_0} + \kappa_j}, \qquad j = 1, 2.$$

From (18) and (19) we obtain the two-soliton solutions:

$$x(y,t) = \frac{2y}{m_0} + 2\ln\frac{\Delta_1}{\Delta_2} \tag{33}$$

$$m(y,t) = \frac{m_0}{1 + \frac{m_0\Delta_3}{\Delta_1\Delta_2}}. \tag{34}$$

where the following notations are used:

$$\begin{aligned}
\Delta_1(y,t) &= 1 + e^{-2\kappa_1\xi_1} + e^{-2\kappa_2\xi_2} + \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}\\
\Delta_2(y,t) &= 1 + \nu_1 e^{-2\kappa_1\xi_1} + \nu_2 e^{-2\kappa_2\xi_2} + \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 \nu_1\nu_2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}.\\
\Delta_3(y,t) &= \frac{4\kappa_1^2}{m_0^{-1}+\kappa_1}e^{-2\kappa_1\xi_1} + \frac{4\kappa_2^2}{m_0^{-1}+\kappa_2}e^{-2\kappa_2\xi_2}\\
&+ \frac{8(\kappa_1-\kappa_2)^2}{m_0(m_0^{-1}+\kappa_1)(m_0^{-1}+\kappa_2)}e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}\\
&+ \frac{4\kappa_2^2\nu_1}{m_0^{-1}+\kappa_2}\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-4\kappa_1\xi_1-2\kappa_2\xi_2}\\
&+ \frac{4\kappa_1^2\nu_2}{m_0^{-1}+\kappa_1}\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-2\kappa_1\xi_1-4\kappa_2\xi_2}.
\end{aligned} \tag{35}$$

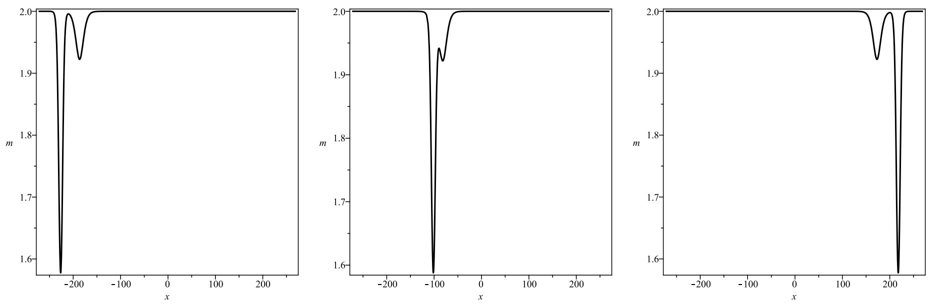

Figure 3: Snapshots of the two (dark) soliton solution of the Qiao equation (1), for three values of t : -30, -12 and 30. The other parameters are $m_0 = 2$, $\kappa_1 = 0.1$, $\kappa_2 = 0.25$.

The interaction of two dark solitons is illustrated in Fig. 3.

4 Conclusions

In this paper we demonstrated firstly, how to obtain peakon solutions of the Qiao equation and secondly how the spectral problem for the Qiao equation can be reduced to the one for the standard Schrödinger operator. Hence the soliton solutions ('dark' solitons) can be obtained in a straightforward manner. This necessitates constant boundary conditions for the solution and also a restriction on the discrete eigenvalues $0 < \kappa_n < m_0^{-1}$. It is interesting what happens to the solutions if this condition is violated. Based on the similarity with Camassa-Holm equation it is likely that there are breaking waves present in this case. Moreover, the equation (1) has a conservation law in the form $X_x(x,t)m(X,t) = m(x,0)$ where X is the solution of

$$X_t(x,t) = u^2(X,t) - u_x^2(X,t), \qquad X(x,0) = x.$$

This last equation is analogous to (6) in the the peakon case. It is likely that this conservation law will play an essential role in the study of the well-posedness, existence and breaking of solutions.

5 Acknowledgments

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