2012

Singular Solutions of Coss-coupled EPDiff Equations: Waltzing Peakons and Compacton Pairs

Colin Cotter
Imperial College London

Darryl Holm
Imperial College London

Rossen Ivanov
Dublin Institute of Technology, rivanov@dit.ie

James Percival
Imperial College London

Singular solutions of cross-coupled EPDiff equations: waltzing peakons and compacton pairs

Colin J. Cotter, Darryl D. Holm, Rossen I. Ivanov and James R. Percival

January 31, 2012

1. EPDiff equations. Let us define an one-parametric group of diffeomorphisms of $\mathbb{R}^n$ with elements that satisfy

$$\frac{\partial \mathbf{X}(\mathbf{x},t)}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{x},t),t), \qquad \mathbf{X}(\mathbf{x},0) = \mathbf{x}, \tag{1}$$

or $\dot{X} = u \circ X$ with $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$, $X \in \mathrm{Diff}(\mathbb{R}^n)$. Let us consider motion in $\mathbb{R}^n$ with a velocity field $u = \dot{X} \circ X^{-1}$; $\mathbf{u}(\mathbf{x},t)$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and define a momentum variable $\mathbf{m} = Q\mathbf{u}$ for some (inertia) operator Q (for example the Helmholtz operator $Q = 1 - \partial_i \partial_i = 1 - \Delta$, where $\partial_i = \frac{\partial}{\partial x^i}$). Let us further define a Lagrangian

$$L[\mathbf{u}] = \frac{1}{2} \int \mathbf{m} \cdot \mathbf{u} \ \mathrm{d}^n \mathbf{x}. \tag{2}$$

Since the velocity $\mathbf{u} = u^i \partial_i \in \mathrm{Vect}(\mathbb{R}^n)$ is a vector field, $\mathbf{m} = m_i \mathrm{d}x^i \otimes \mathrm{d}^n\mathbf{x}$ is a $n+1$-form density, we have a natural right-invariant bilinear form

$$\langle \mathbf{m}, \mathbf{u} \rangle = \int \mathbf{m} \cdot \mathbf{u} \ \mathrm{d}^n \mathbf{x}. \tag{3}$$

The Euler-Poincaré equation for the geodesic motion in this case is [5, 6]

$$\frac{d}{dt}\frac{\delta L}{\delta \mathbf{u}} + \mathrm{ad}^*_{\mathbf{u}} \frac{\delta L}{\delta \mathbf{u}} = 0, \qquad \mathbf{u} = G * \mathbf{m}, \tag{4}$$

where G is the Green function for the operator Q. The corresponding Hamiltonian is

$$H[\mathbf{m}] = \langle \mathbf{m}, \mathbf{u} \rangle - L[\mathbf{u}] = \frac{1}{2} \int \mathbf{m} \cdot G * \mathbf{m} \ \mathrm{d}^n \mathbf{x}, \tag{5}$$

and the equation in Hamiltonian form ($\mathbf{u} = \frac{\delta H}{\delta \mathbf{m}}$) is

$$\frac{\partial \mathbf{m}}{\partial t} = -\mathrm{ad}^*_{\frac{\delta H}{\delta \mathbf{m}}} \mathbf{m}. \tag{6}$$

The left Lie algebra of vector fields is $[\mathbf{u},\mathbf{v}] = -(u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p$. For an arbitrary vector field $\mathbf{v}$ one can write [6]

$$\begin{aligned}
\langle \mathrm{ad}^*_{\mathbf{u}}\mathbf{m},\mathbf{v}\rangle &= \langle \mathbf{m},\mathrm{ad}_{\mathbf{u}}\mathbf{v}\rangle = \langle \mathbf{m},[\mathbf{u},\mathbf{v}]\rangle \\
&= -\langle m_l \mathrm{d}x^l\otimes \mathrm{d}^n\mathbf{x}, (u^k(\partial_k v^p) - v^k(\partial_k u^p))\partial_p\rangle \\
&= -\int m_p(u^k(\partial_k v^p) - v^k(\partial_k u^p))\mathrm{d}^n\mathbf{x} \\
&= \int v^p(u^k(\partial_k m_p) + m_p(\partial_k u^k) + m_k(\partial_p u^k))\mathrm{d}^n\mathbf{x} \\
&= \langle ((\mathbf{u}\cdot\nabla)m_p + \mathbf{m}\cdot\partial_p\mathbf{u} + m_p\mathrm{div}\mathbf{u})\mathrm{d}x^p\otimes \mathrm{d}^n\mathbf{x},\mathbf{v}\rangle,
\end{aligned}$$

and therefore (6) has the form, known as *EPDiff equation*:

$$\frac{\partial m_p}{\partial t} + (\mathbf{u}\cdot\nabla)m_p + \mathbf{m}\cdot\partial_p\mathbf{u} + m_p\mathrm{div}\mathbf{u} = 0. \tag{7}$$

Due to the invariance of the Hamiltonian under the right action of the group Diff($\mathbb{R}^n$) there is a momentum conservation law according to the Noether's Theorem (which can be verified directly with (1)) :

$$m_i(\mathbf{X}(\mathbf{x},t),t)\partial_j X^i(\mathbf{x},t)\det\left(\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right) = m_j(x,0), \tag{8}$$

where $\left(\frac{\partial\mathbf{X}}{\partial\mathbf{x}}\right)_{ij} = \frac{\partial X^i}{\partial x^j}$ is the Jacobian matrix.
The Lie-Poisson bracket is

$$\begin{aligned}
\{A,B\}(\mathbf{m}) &= \langle \mathbf{m},[\frac{\delta A}{\delta\mathbf{m}},\frac{\delta B}{\delta\mathbf{m}}]\rangle \\
&= -\int m_i\Big(\frac{\delta A}{\delta m_k}\partial_k\frac{\delta B}{\delta m_i} - \frac{\delta B}{\delta m_k}\partial_k\frac{\delta A}{\delta m_i}\Big)\mathrm{d}^n\mathbf{x}.
\end{aligned} \tag{9}$$

When $n=1$ the algebra (9), associated with the bracket is the algebra of vector fields on the circle. This algebra admits a generalization with a central extension, which is the famous Virasoro algebra [6, 8]. In two dimensions, $n=2$, the algebra, associated with the bracket is the algebra of vector fields on a torus [1].
2. Singular solutions. The Camassa-Holm (CH) equation [2] can be considered as a member of the family of EPDiff equations in $n=1$ dimension [5]:

$$m_t + 2u_x m + u m_x = 0, \qquad m = u - u_{xx}. \tag{10}$$

The CH equation possesses the so-called N-peakon solution in the form

$$u(x,t) = \frac{1}{2}\sum_{i=1}^{N} p_i(t)\exp(-|x-x_i(t)|), \tag{11}$$

provided p_i and x_i evolve according to the following system of ordinary differential equations:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \tag{12}$$

where the Hamiltonian is $H = \frac{1}{4}\sum_{i,j=1}^{N} p_i p_j \exp(-|x_i - x_j|)$. The momentum is singular,

$$m(x,t) = \sum_{i=1}^{N} p_i(t)\delta(x - x_i(t)), \tag{13}$$

it defines the so-called singular momentum map, [5]. CH is an integrable equation and is very well studied - see e.g. the review article [4].
The singular momentum map (13) suggests the following measure-valued singular momentum solution Ansatz for the $n-$dimensional solutions of the EPDiff equation:

$$\mathbf{m}(\mathbf{x},t) = \sum_{a=1}^{N} \int \mathbf{P}^a(s,t)\, \delta\big(\,\mathbf{x} - \mathbf{Q}^a(s,t)\,\big) \mathrm{d}s.$$

These singular momentum solutions, called “diffeons,” [7] are vector density functions supported in $\mathbb{R}^n$ on a set of N surfaces (or curves) of codimension $(n-k)$ for $s \in \mathbb{R}^k$ with $k < n$. They may, for example, be supported on sets of points (vector peakons, $k = 0$), one-dimensional filaments (strings, $k = 1$), or two-dimensional surfaces (sheets, $k = 2$) in three dimensions.

3. Cross coupled CH equations and waltzing peakons. The Lagrangian for cross coupled CH (CCCH) is [3]

$$l(u,v) = \int_{\mathbb{R}} (uv + u_x v_x)\ \mathrm{d}x.$$

The corresponding two-component EP equations in 1D on $\mathbb{R}$ are

$$\partial_t m = -\,\mathrm{ad}^*_{\delta h/\delta m} m = -\,(vm)_x - m v_x \quad \text{with} \quad v := \frac{\delta h}{\delta m} = K * n,$$

$$\partial_t n = -\,\mathrm{ad}^*_{\delta h/\delta n} n = -\,(un)_x - n u_x \quad \text{with} \quad u := \frac{\delta h}{\delta n} = K * m.$$

with $K(x,y) = \frac{1}{2}e^{-|x-y|}$ being the Green function of the Helmholtz operator. The CCCH Hamiltonian is

$$h(n,m) = \int_{\mathbb{R}} n\, K * m\ \mathrm{d}x = \int_{\mathbb{R}} m\, K * n\ \mathrm{d}x.$$

This Hamiltonian system has **two-**component singular momentum maps

$$m(x,t) = \sum_{a=1}^{M} m_a(t)\,\delta(x - q_a(t)), \qquad n(x,t) = \sum_{b=1}^{N} n_b(t)\,\delta(x - r_b(t)).$$

The total momentum of CCCH is conserved, namely

$$\partial_t\,(u+v) + \partial_x\big(uv + K * (2uv + u_x v_x)\big) = 0.$$

4. Peakon solutions of the cross-flow equations. The CCCH equations are deformations of CH that support two different types of peakons, with velocities

$$u(x,t) = \tfrac{1}{2}\sum_{a=1}^{M} m_a(t)\, e^{-|x-q_a(t)|}\,, \qquad v(x,t) = \tfrac{1}{2}\sum_{b=1}^{N} n_b(t)\, e^{-|x-r_b(t)|}\,, \tag{14}$$

and momenta,

$$m(x,t) = \sum_{a=1}^{M} m_a(t)\,\delta(x-q_a(t))\,, \qquad n(x,t) = \sum_{b=1}^{N} n_b(t)\,\delta(x-r_b(t))\,. \tag{15}$$

The $2M{+}2N$ variables (q_a, m_a), $a = 1,\ldots,M$, and (r_b, n_b), $b = 1,\ldots,N$, are governed by the Hamilton's canonical equations for the Hamiltonian function,

$$H = \tfrac{1}{2}\sum_{a,b=1}^{M,N} m_a(t) n_b(t) e^{-|q_a(t)-r_b(t)|}\,, \tag{16}$$

namely,

$$\dot{q}_a(t) \;=\; \frac{\partial H}{\partial m_a} = \tfrac{1}{2}\sum_{b=1}^{N} n_b(t) e^{-|q_a(t)-r_b(t)|} = v(q_a(t),t)\,, \tag{17}$$

$$\dot{r}_b(t) \;=\; \frac{\partial H}{\partial n_b} = \tfrac{1}{2}\sum_{a=1}^{M} m_a(t) e^{-|q_a(t)-r_b(t)|} = u(r_b(t),t)\,, \tag{18}$$

for the positions of the peakons, and

$$\dot{m}_a(t) \;=\; -\frac{\partial H}{\partial q_a} = \tfrac{1}{2} m_a \sum_{b=1}^{N} n_b \,\mathrm{sgn}\,(q_a - r_b) e^{-|q_a(t)-r_b(t)|} = -\, m_a \frac{\partial v}{\partial x}\Big|_{x=q_a}, \tag{19}$$

$$\dot{n}_b(t) \;=\; -\frac{\partial H}{\partial r_b} = -\tfrac{1}{2} n_b \sum_{a=1}^{M} m_a \,\mathrm{sgn}\,(q_a - r_b) e^{-|q_a(t)-r_b(t)|} = -\, n_b \frac{\partial u}{\partial x}\Big|_{x=r_b} \tag{20}$$

for their canonical momenta. Conserved quantities include the energy H and the total momentum $\sum_a (m_a + n_a)$.

5. The coupled peakon pair. The simplest possible case is $M = N = 1$. Introducing the new variables $X = \frac{q+r}{2}$, $Y = q - r$, respectively the mean position of the peaks and their separation distance. The evolution equations in terms of the new variables are

$$\dot{X} = \frac{(m+n)}{4} e^{-|Y|}, \qquad \dot{Y} = \frac{n-m}{2} e^{-|Y|}.$$

Thus we can define the behavior of the exponential function of the absolute separation of the peaks,

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{|Y|} = \mathrm{sgn}(Y)\frac{n-m}{2}, \tag{21}$$

From (19) - (20)

$$\dot{m} = -\dot{n} = \mathrm{sgn}(Y)\frac{mn}{2}e^{-|Y|} = \mathrm{sgn}(Y)E,$$

where $E = H|_{t=0}$ is the (constant) value of the Hamiltonian, that is to say the total energy of the coupled pair. Differentiating (21) again with respect to time gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(e^{|Y|}\right) = -\mathrm{sgn}^2\left(Y\right)E + 2\delta\left(Y\right)\left(n-m\right)^2.$$

On integrating for a particular signature of $Y|_{t=0} = Y_0 \neq 0$,

$$e^{|Y|} = -\frac{1}{2}m_0 n_0 e^{-|Y_0|}t^2 + \frac{1}{2}\mathrm{sgn}\left(Y_0\right)\left(n_0 - m_0\right)t + e^{|Y_0|},$$

where

$$m_0 = m|_{t=0}\,, \quad n_0 = n|_{t=0} \quad Y_0 = Y|_{t=0}\,.$$

If m_0 and n_0 have the same signature then eventually we will have $|Y| = 0$, regardless of the value of $|Y_0|$. Thus, when m_0 and n_0 share the same signature the half period of their waltzing motion can be found by setting $Y_0 = 0$ and looking for when $e^{|Y|}$ attains unity, namely $t = 2\frac{n_0 - m_0}{n_0 m_0}$. It will be noted that at this time

$$m|_{t=2\frac{m_0-n_0}{m_0 n_0}} = m_0 + m_0 n_0\left(\frac{m_0 - n_0}{m_0 n_0}\right) = n_0,$$

and similarly

$$n|_{t=2\frac{m_0-n_0}{m_0 n_0}} = m_0,$$

so that the two types of peakons do indeed exchange momentum amplitudes over a half cycle, see Fig. 1. The explicit solutions as well as other examples with waltzing peakons and compactons are given in [3].

6. Cross-coupled EPDiff in higher dimensions. The straightforward generalization to higher dimensions is

$$l = \int\left(\mathbf{u}\cdot\mathbf{v} + (\nabla\mathbf{u})\cdot(\nabla\mathbf{v})\right)\mathrm{d}^n x,$$

$$\frac{d\mathbf{m}}{dt} = -\mathrm{ad}^*_{\delta\mathbf{l}/\delta\mathbf{m}}\mathbf{m} = -\mathbf{v}\cdot\nabla\mathbf{m} - (\nabla\mathbf{v})^T\cdot\mathbf{m} - \mathbf{m}\,\mathrm{div}\,\mathbf{v}, \qquad \mathbf{m} = \mathbf{u} - \Delta\mathbf{u},$$

$$\frac{d\mathbf{n}}{dt} = -\mathrm{ad}^*_{\delta\mathbf{l}/\delta\mathbf{n}}\mathbf{n} = -\mathbf{u}\cdot\nabla\mathbf{n} - (\nabla\mathbf{u})^T\cdot\mathbf{n} - \mathbf{n}\,\mathrm{div}\,\mathbf{u}, \qquad \mathbf{n} = \mathbf{v} - \Delta\mathbf{v}.$$

Numerical studies in $n = 2$ dimensions show that the waltzing pairs also appear in higher dimensions. In the case with rotational symmetry the two concentric waves (**u** and **v**) have 'waltzing' fronts and also rotate with respect to each other.

Acknowledgements The work of R.I. is supported by the Science Foundation of Ireland (SFI), under Grant No. 09/RFP/MTH2144.

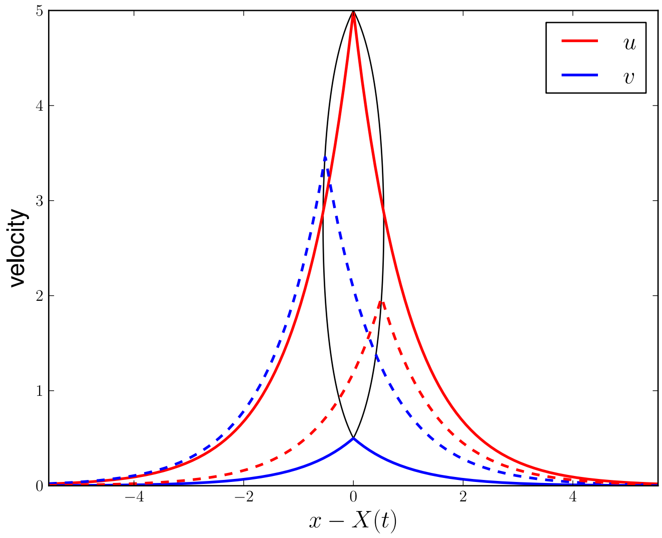

Figure 1: Plot showing velocity fields of a peakon-peakon pair with $m_0 = 5$, $n_0 = 0.5$, $l_0 = 0$ (solid lines). The dotted path indicates the subsequent path of the two peaks in the frame travelling at the particles mean velocity $\dot{X} = \frac{\dot{q}+\dot{r}}{2}$. For these initial conditions the total period for one orbit of the cycle is $T = 3.6$. Also shown is the form of the two peakons at subsequent times $t = 0.45 + 1.8n$, $n \in \mathbb{Z}$

References

[1] V. Arnold and B. Khesin, *Topological Methods in Hydrodynamics* (Springer Verlag, New York, 1998).

[2] R. Camassa and D. Holm, An Integrable Shallow Water Equation with Peaked Solitons, *Phys. Rev. Lett.* **71** (1993) 1661–1664; ArXiv: patt-sol/9305002.

[3] C. Cotter et al., Waltzing peakons and compacton pairs in a cross-coupled Camassa-Holm equation, *J. Phys. A: Math. Theor.* **44** (2011) 265205 (28pp).

[4] D. Holm and R. Ivanov, Smooth and peaked solitons of the Camassa-Holm equation and applications, *J. Geom. Symmetry Phys.* **22** (2011) 13–49.

[5] D. Holm, J. Marsden, in *The breadth of symplectic and Poisson geometry, Progr. Math.* **232**, (Birkhäuser, Boston, MA, 2005).

[6] D. Holm, J. Marsden and T. Ratiu, The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories *Adv. Math.* **137**, 1–81 (1998).

[7] D. Holm and M. Staley, Wave Structure and Nonlinear Balances in a Family of Evolutionary PDEs, *SIAM J. Appl. Dyn. Syst.* **2** (2003) 323–380.

[8] G. Misiołek, Conjugate points in the Bott-Virasoro Group and the KdV equation *Proceedings of the AMS* **125** (1997) 935–940.