




2009-01-01

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## Recommended Citation

R. Ivanov, Two component integrable systems modelling shallow water waves, Oberwolfach reports, Volume 6, Issue 1, 2009, Wave Motion (pp. pp. 429–462).

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*School of Mathematics*

*Articles*

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*Dublin Institute of Technology*

*Year 2009*

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## Two component integrable systems modelling shallow water waves

ROSSEN I. IVANOV

The aim of this talk is to describe the derivation of shallow water model equations for the *constant vorticity* case and to demonstrate how these equations can be related to two integrable systems: a two component integrable generalization of the Camassa-Holm equation and the Kaup - Boussinesq system.

The motion of inviscid fluid is described by Euler's equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0,$$

where  $\rho$  is a constant density,  $\mathbf{v}(x, y, z, t)$  is the velocity of the fluid at the point  $(x, y, z)$  at the time  $t$ ,  $P$  is the pressure in the fluid,  $\mathbf{g} = (0, 0, -g)$  is the constant Earth's gravity acceleration.

We consider a motion of a shallow water over a flat bottom, which is located at  $z = 0$ . We assume that the motion is in the  $x$ -direction, and that the physical variables do not depend on  $y$ . Let  $h$  be the mean level of the water and let  $\eta(x, t)$  describes the shape of the water surface, i.e. the deviation from the average level. The pressure is  $P = P_A + \rho g(h - z) + p(x, z, t)$ , where  $P_A$  is the constant atmospheric pressure, and  $p$  is a pressure variable, measuring the deviation from the hydrostatic pressure distribution.

On the surface  $z = h + \eta$ ,  $P = P_A$  and therefore  $p = \eta \rho g$ . Taking  $\mathbf{v} \equiv (u, 0, w)$  we can write the kinematic condition on the surface as (e.g. following [1])  $w = \eta_t + u\eta_x$  on  $z = h + \eta$ . Finally, there is no horizontal velocity at the bottom, thus  $w = 0$  on  $z = 0$ .

Let us introduce now dimensionless parameters  $\varepsilon = a/h$  and  $\delta = h/\lambda$ , where  $a$  is the typical amplitude of the wave and  $\lambda$  is the typical wavelength of the wave. Now we can introduce dimensionless quantities, according to the magnitude of the physical quantities, see [1, 2] for details:  $x \rightarrow \lambda x$ ,  $z \rightarrow zh$ ,  $t \rightarrow \frac{\lambda}{\sqrt{gh}} t$ ,  $\eta \rightarrow a\eta$ ,  $u \rightarrow \varepsilon \sqrt{gh} u$ ,  $w \rightarrow \varepsilon \delta \sqrt{gh} w$ ,  $p \rightarrow \varepsilon \rho gh$ .

Now let us notice that there is an exact solution of the governing equations of the form  $u = \tilde{U}(z)$ ,  $0 \leq z \leq h$ ,  $w \equiv 0$ ,  $p \equiv 0$ ,  $\eta \equiv 0$ . This solution represents an arbitrary underlying 'shear' flow. In the presence of a shear flow the horizontal velocity of the fluid will be  $\tilde{U}(z) + u$ . The scaling for such solution is clearly  $u \rightarrow \sqrt{gh}(\tilde{U}(z) + \varepsilon u)$ , and the scaling for the other variables is as before. The system of equations is (the prime denotes derivative with respect to  $z$ ):

$$\begin{aligned} u_t + \tilde{U}u_x + w\tilde{U}' + \varepsilon(uu_x + wu_z) &= -p_x, \\ \delta^2(w_t + \tilde{U}w_x + \varepsilon(uw_x + ww_z)) &= -p_z, \\ u_x + w_z &= 0, \\ w = \eta_t + (\tilde{U} + \varepsilon u)\eta_x, \quad p = \eta, \quad \text{on } z &= 1 + \varepsilon\eta, \\ w = 0 \quad \text{on } z &= 0. \end{aligned}$$

The simplest nontrivial case is a linear shear,  $\tilde{U}(z) = Az$ , where  $A$  is a constant. We choose  $A > 0$ , so that the underlying flow is propagating in the positive direction of the  $x$ -coordinate.

The vorticity is  $\omega = (U + u)_z - w_x$  or in terms of the rescaled variables,  $\omega = A + \varepsilon(u_z - \delta^2 w_x)$ . We are looking for a solution with constant vorticity  $\omega = A$ , and therefore we require that  $u_z - \delta^2 w_x = 0$ . Together with the equation  $u_x + w_z = 0$  it gives

$$u = u_0 - \delta^2 \frac{z^2}{2} u_{0xx} + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon \delta^2), \quad w = -z u_{0x} + \delta^2 \frac{z^3}{6} u_{0xxx} + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon \delta^2),$$

where  $u_0(x, t)$  is the leading order approximation for  $u$ .

With these expressions we obtain the following from the condition on the surface, ignoring terms of order  $\mathcal{O}(\varepsilon^2, \delta^4, \varepsilon \delta^2)$ :

$$(1) \quad \eta_t + A\eta_x + \left[ (1 + \varepsilon\eta)u_0 + \varepsilon \frac{A}{2} \eta^2 \right]_x - \delta^2 \frac{1}{6} u_{0xxx} = 0$$

From the second of the Euler's equations and the condition on the surface we have  $p = \eta - \delta^2 \left[ \frac{1-z^2}{2} u_{0xt} + \frac{1-z^3}{3} A u_{0xx} \right]$ , then the first of the Euler's equations gives (Note that there is no  $z$ -dependence!)

$$(2) \quad \left( u_0 - \delta^2 \frac{1}{2} u_{0xx} \right)_t + \varepsilon u_0 u_{0x} + \eta_x - \delta^2 \frac{A}{3} u_{0xxx} = 0.$$

The linearised equations are

$$(3) \quad u_{0t} + \eta_x = 0, \quad \eta_t + A\eta_x + u_{0x} = 0,$$

giving  $\eta_{tt} + A\eta_{tx} - \eta_{xx} = 0$ . This linear equation has a travelling wave solution  $\eta = \eta(x - ct)$  with a velocity  $c$  satisfying  $c^2 - Ac - 1 = 0$ , or

$$c = \frac{1}{2} \left( A \pm \sqrt{4 + A^2} \right).$$

If there is no shear ( $A = 0$ ), then  $c = \pm 1$ . In general, there is one positive and one negative solution, representing left and right running waves. Suppose that we have only one of these waves, then  $\eta = cu_0 + \mathcal{O}(\varepsilon, \delta^2)$  - e.g. from (3).

By introduction of a new variable  $\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\gamma u_{0xx}$ , where

$$\alpha = \frac{1}{3(1+c^2)} + \frac{2c^2}{3(1+c^2)} \left( 1 + \frac{Ac}{2} \right), \quad \beta = \frac{1 - (3+c^2)(1 + \frac{Ac}{2})}{3(1+c^2)} \alpha, \quad \gamma = \frac{\alpha}{6(c-A)},$$

and a change of variables (rescaling)  $u_0 \rightarrow \frac{1}{\alpha\varepsilon} u_0$ ,  $x \rightarrow \frac{\delta}{\sqrt{B}} x$ ,  $t \rightarrow \frac{\delta}{\sqrt{B}} t$  where  $B = \frac{1}{2} + \frac{1}{6(c-A)} \left( A - \frac{1}{c-A} \right)$  the equations (1), (2) transform into the system

$$(4) \quad m_t + Am_x - Au_{0x} + 2mu_{0x} + u_0 m_x + \rho\rho_x = 0, \quad m = u_0 - u_{0xx}$$

$$(5) \quad \rho_t + A\rho_x + (\rho u_0)_x = 0,$$

Before the rescaling we had  $\alpha\varepsilon\eta = \rho - 1 - \varepsilon^2\beta c^2 u_0^2 - \varepsilon\delta^2\gamma u_{0xx}$ . Since in the leading order  $\eta = cu_0$  the rescaling of  $\eta$  is  $\eta \rightarrow \frac{1}{\alpha\varepsilon}\eta$ . Thus in terms of the rescaled variables  $\eta = \rho - 1 - \frac{\beta c^2}{\alpha^2} u_0^2 - B \frac{\gamma}{\alpha} u_{0xx}$ .

The system (4), (5) is an integrable 2-component Camassa-Holm system that appears in [3], generalizing the famous Camassa-Holm equation [4]. The Lax representation for this system is ( $\zeta$  is a spectral parameter)

$$\begin{aligned}\Psi_{xx} &= \left(-\zeta^2 \rho^2 + \zeta\left(m - \frac{A}{2}\right) + \frac{1}{4}\right)\Psi, \\ \Psi_t &= \left(\frac{1}{2\zeta} - u_0 - A\right)\Psi_x + \frac{1}{2}u_{0x}\Psi.\end{aligned}$$

An alternative derivation for the case of zero vorticity, based on the Green-Naghdi equations is reported in [5].

Another integrable system matching the water waves asymptotic equations to the first order of the small parameters  $\varepsilon, \delta$  is the Kaup - Boussinesq system. We describe briefly its derivation. Introducing  $V = u - \delta^2\left(\frac{1}{2} - \frac{A}{3c}\right)u_{xx}$  the equation (2) can be written as  $V_t + \varepsilon VV_x + \eta_x = 0$ . Equation (1) in the first order in  $\varepsilon, \delta$  is

$$\eta_t + \left[A\eta + (1 + \varepsilon\eta)u_0 + \varepsilon\frac{A}{2}\eta^2\right]_x - \delta^2\frac{1}{6}u_{0xxx} = 0$$

and with a shift  $\eta \rightarrow \eta - \frac{1}{\varepsilon}$  it becomes

$$\eta_t + \varepsilon\left(1 + \frac{Ac}{2}\right)(\eta u_0)_x - \delta^2\frac{1}{6}u_{0xxx} = 0 \quad \text{or} \quad \eta_t + \varepsilon\frac{1+c^2}{2}(\eta V)_x - \delta^2\frac{1}{6}V_{xxx} = 0.$$

Further rescaling leads to the Kaup - Boussinesq system

$$V_t + VV_x + \eta_x = 0, \quad \eta_t - \frac{1}{4}V_{xxx} + \frac{1+c^2}{2}(\eta V)_x = 0,$$

which is integrable iff  $A = 0$  ( $c^2 = 1$ ) with a Lax pair [6]

$$\Psi_{xx} = -\left(\left(\zeta - \frac{1}{2}V\right)^2 - \eta\right)\Psi, \quad \Psi_t = -\left(\zeta + \frac{1}{2}V\right)\Psi_x + \frac{1}{4}V_x\Psi.$$

It is interesting to investigate further which specific properties of the original governing equations are preserved in the 'integrable' approximate models. For example the 2-component Camassa-Holm system for certain initial data admits breaking waves solutions [5].

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