1978-01-01

Essentially-Rigid Families of Abelian p -Groups

Brendan Goldsmith

Technological University Dublin, brendan.goldsmith@tudublin.ie

Follow this and additional works at: <https://arrow.tudublin.ie/scschmatart>

Part of the [Mathematics Commons](#)

Recommended Citation

Goldsmith, B. (1978). Essentially-rigid families of Abelian p -Groups. *Journal of the London Mathematical Society*, vol. 18 pg. 70-74. doi:10.1112/jlms/s2-18.1.70

This Article is brought to you for free and open access by the School of Mathematics at ARROW@TU Dublin. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@TU Dublin. For more information, please contact arrow.admin@tudublin.ie, aisling.coyne@tudublin.ie, gerard.connolly@tudublin.ie.

ESSENTIALLY-RIGID FAMILIES OF ABELIAN p -GROUPS

ESSENTIALLY-RIGID FAMILIES OF ABELIAN p -GROUPS

B. GOLDSMITH

Introduction

In a recent paper Shelah [9] has established the existence of a rigid-like family of 2^λ separable p -groups each of cardinality λ , where λ is a strong limit cardinal of cofinality $> \aleph_0$, that is a family such that (i) the endomorphism ring of each group is the split extension of the p -adic integers by the ideal of small endomorphisms and (ii) every homomorphism between different members of the family is small. (See Pierce [8] or Fuchs [3] for the concept of small homomorphisms.) Then assuming G.C.H. this leaves open the following problem:—

If $\mu = \lambda^{\aleph_0} = 2^\lambda$, is there a rigid-like family of 2^μ separable p -groups, each of cardinality μ ?

This problem seems to be extremely difficult and in this paper we derive a weaker result and in so doing obtain a partial answer to Fuchs [4; Problem 53]. We remark that our technique is mainly group-theoretic and uses a minimal number of notions from set theory.

Finally all groups are additively written abelian groups and we refer to Fuchs [3] and [4] for standard results and notation; for set theoretic concepts we refer to Jech [5].

1. *Essentially-rigid families of p -groups*

Let λ be an infinite cardinal and suppose $\bar{B}$ is the torsion-completion of the group B which is a standard p -group of final rank λ . Recall that if G is a reduced p -group containing B as a basic subgroup then we can regard G as a pure subgroup of $\bar{B}$ and then any endomorphism ϕ of G will have a unique extension $\bar{\phi}$ to $\bar{B}$. Thus we may, and do, regard endomorphisms of G as endomorphisms of $\bar{B}$. Let $E(X)$ denote the endomorphism ring of any group X .

Define the ideal of inessential endomorphisms of G by $I(G) = \{\phi \in E(G) \mid \bar{B}\bar{\phi} \leq G\}$. Clearly $I(G)$ is a 2-sided ideal of $E(G)$ and a left ideal of $E(\bar{B})$.

THEOREM 1.1. *For any infinite cardinal λ , there exists a group G , with basic subgroups of final rank λ , such that $E(G)$ is the split extension of the p -adic integers, $\mathbb{Q}_p^*$, by the ideal $I(G)$, $E(G) = \mathbb{Q}_p^* \oplus I(G)$.*

Proof. Let B be a standard basic group of final rank λ and choose G such that $B \leq G \leq \bar{B}$ and $\bar{B}/G \cong Z(p^\infty)$. Then G has rank $\lambda^{\aleph_0}$ but has a basic subgroup of final rank λ . We show $E(G) = \mathbb{Q}_p^* \oplus I(G)$.

Now we have the exact sequence

$$0 \rightarrow G \rightarrow \bar{B} \rightarrow Z(p^\infty) \rightarrow 0$$

which yields

$$\begin{aligned} 0 \rightarrow \text{Hom}(\bar{B}, G) &\rightarrow \text{Hom}(G, G) \rightarrow P \text{ext}(Z(p^\infty), G) \\ 0 \rightarrow \text{Hom}(Z(p^\infty), Z(p^\infty)) &\rightarrow P \text{ext}(Z(p^\infty), G), \rightarrow 0. \end{aligned}$$

Received 20 December, 1977.

[J. LONDON MATH. SOC. (2), 18 (1978), 70-74]

So $\text{Pext}(Z(p^\infty), G)$ is a cyclic Q_p^* -module and since the image of $\text{Hom}(\bar{B}, G)$ in $\text{Hom}(G, G)$ is just $I(G)$, we have that $E(G) = Q_p^* + I(G)$. Since $Q_p^* \cap I(G) = 0$, the extension is a ring split extension.

LEMMA 1.2 (Leptin [6]). *If G and H are pure subgroups of $\bar{B}$ containing B then $G \cong H$ if and only if there is an automorphism θ of $\bar{B}$ with $G\theta = H$.*

Definition. A group G is said to be a maximal pure subgroup of $\bar{B}$ if G is a pure subgroup of $\bar{B}$ containing B and $\bar{B}/G \cong Z(p^\infty)$.

LEMMA 1.3. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there are 2^μ non-isomorphic maximal pure subgroups of $\bar{B}$, the torsion-completion of the standard basic group B of final rank λ .*

Proof. Since $\bar{B}$ has cardinality $\lambda^{\aleph_0} = \mu$ we have that $\bar{B}/B \cong \bigoplus_\mu Z(p^\infty)$. Now a maximal pure subgroup G of $\bar{B}$ corresponds to a divisible subgroup of corank 1 in this quotient and so there are clearly 2^μ such groups. But by Lemma 1.2 any two maximal pure subgroups will be isomorphic only if there is an automorphism θ of $\bar{B}$ mapping one to the other. However any automorphism of $\bar{B}$ is determined by its action on B , of cardinality λ , and so there are at most μ automorphisms of $\bar{B}$. Hence there are 2^μ non-isomorphic maximal pure subgroups of $\bar{B}$.

For the rest of this section suppose that λ is an infinite cardinal such that $\lambda^{\aleph_0} = 2^\lambda = \mu$ and let $\bar{B}$ denote the torsion-completion of the standard basic group B of final rank λ .

Definition. If G_j and G_k are maximal pure subgroups of $\bar{B}$, let

$$I_j(G_k) = \{\phi \in \text{Hom}(G_j, G_k) \mid \bar{B}\bar{\phi} \leq G_k\}.$$

LEMMA 1.4. *For an arbitrary maximal pure subgroup G of $\bar{B}$, there are at most μ maximal pure subgroups G_k of $\bar{B}$ such that $\text{Hom}(G_k, G) \neq I_k(G)$.*

Proof. Suppose there exists a family $\{G_k\}_{k \in K}$ of more than μ maximal pure subgroups of $\bar{B}$ with $\text{Hom}(G_k, G) \neq I_k(G)$. Then we can construct a family $\{\bar{\phi}_k\}_{k \in K}$ of endomorphisms of $\bar{B}$ such that $\bar{\phi}_k$ does not belong to $I_k(G)$. Since $E(\bar{B})$ has cardinality μ and K has cardinality greater than μ , we must have $\bar{\phi}_k = \bar{\phi}_j$ for some different k, j in K . But then

$$\bar{B}\bar{\phi}_k = (G_j + G_k)\bar{\phi}_k \leq G_j\bar{\phi}_k + G_k\bar{\phi}_k = G$$

which is contrary to our choice of $\bar{\phi}_k$.

LEMMA 1.5. *If G_0 is an arbitrary maximal pure subgroup of $\bar{B}$ then there are at most μ maximal pure subgroups G_k such that $\text{Hom}(G_0, G_k) \neq I_0(G_k)$.*

Proof. Suppose there exists a family $\{G_k\}_{k \in K}$, where $|K| > \mu$, of maximal pure subgroups of $\bar{B}$ such that $\text{Hom}(G_0, G_k) \neq I_0(G_k)$. Then we have a family $\{\bar{\phi}_k\}_{k \in K}$ of endomorphisms of $\bar{B}$ such that $\bar{\phi}_k \notin I_0(G_k)$. But then $\bar{\phi}_k = \bar{\phi}_j$ for $k, j \in K'$ with $|K'| > \mu$. So $G_0\bar{\phi}_k \leq \bigcap_{j \in K'} G_j$ for $k \in K'$. Let $G_k = \langle H_k, x_k \rangle_*$ where $H_k \geq \bigcap_{j \in K'} G_j$

and let π_k denote the projection of $\bar{B}$ onto $\langle x_k \rangle_*$. Set $\bar{\phi}'_k = \bar{\phi}_k + \pi_k$. Then $\phi'_k \in \text{Hom}(G_0, G_k)$ but $\phi'_k \notin I_0(G_k)$. But since there are more than μ such x_k 's we have exhibited more than μ distinct endomorphisms $\bar{\phi}'_k$ of $\bar{B}$ —a contradiction.

Definition. A family $\{G_j\}_{j \in J}$ of separable p -groups is said to be essentially-rigid if

$$\text{Hom}(G_j, G_k) = \begin{cases} Q_p^* \oplus I(G_j) & j = k \\ I_j(G_k) & j \neq k. \end{cases}$$

THEOREM 1.6. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there exists an essentially-rigid family of 2^μ groups, each of cardinality μ .*

Proof. Let ε denote the least ordinal of cardinality 2^μ . We construct a family $\{G_\alpha\}_{\alpha < \varepsilon}$ inductively. For G_0 choose any maximal pure subgroup of $\bar{B}$, the torsion-completion of the standard basic group of final rank λ . Now suppose the family $\{G_\alpha\}_{\alpha < \beta}$ has been constructed where $0 < \beta < \varepsilon$.

For each $\alpha < \beta$ consider the set of maximal pure subgroups G_i of $\bar{B}$ such that $\text{Hom}(G_\alpha, G_i) \neq I_\alpha(G_i)$ or $\text{Hom}(G_i, G_\alpha) \neq I_i(G_\alpha)$. By Lemmas 1.4 and 1.5 we know there are at most μ indices i for which this is true. By the minimality of ε we know that the set of G_i having this property for any α less than β contains less than 2^μ members. So by Lemma 1.3 there exists a maximal pure subgroup G^1 of $\bar{B}$ not in this set. Set $G_\beta = G^1$. Then it follows easily that $\{G_\alpha\}_{\alpha \leq \beta}$ is essentially-rigid. The construction is completed by transfinite induction. The proof is then completed by the observation that each maximal pure subgroup of $\bar{B}$ has cardinality $|\bar{B}| = \mu$.

2. Prescribing the ideal of inessential endomorphisms

The results in §1 imposed very little restriction on the ideal over which the endomorphism ring of a maximal pure subgroup of $\bar{B}$ splits. In this section we show that this ideal can be restricted somewhat. Ideally we would like to show that splitting can occur over the ideal of small endomorphisms; however, this seems to be very difficult. We offer instead a weaker splitting result.

Definition. If G_j and G_k are separable p -groups with a common basic subgroup B , then we define, for an infinite cardinal λ ,

$$I_j^\lambda(G_k) = \{\phi \in \text{Hom}(G_j, G_k) \mid \bar{B}\bar{\phi} \leq G_k \text{ and } |\bar{B}\bar{\phi}| \leq \lambda\}.$$

When $G_j = G_k$ we simply write $I^\lambda(G_k)$.

THEOREM 2.1. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there exists a family $\{G_j\}$ of 2^μ separable p -groups, each of cardinality μ , having a common basic subgroup B of cardinality λ and such that*

$$\text{Hom}(G_j, G_k) = \begin{cases} Q_p^* \oplus I^\lambda(G_k) & j = k \\ I_j^\lambda(G_k) & j \neq k. \end{cases}$$

Proof. Let

$$B = \bigoplus_{n < \omega} B_n \text{ where } B_n = \bigoplus_{\lambda} Z(p^n).$$

Let $\bar{B}$ denote the torsion-completion of B . Then $\bar{B}/B$ is a divisible p -group of rank μ .

Let $\{W_k\}_{k \in K}$ denote the set of endomorphic images of $\bar{B}$ which have rank μ . Since $|E(\bar{B})| = \mu$, there are at most μ such images, i.e. $|K| \leq \mu$. Let W_k^* denote a minimal pure subgroup of $(W_k + B)/B$. Then $\{W_k^*[p]\}_{k \in K}$ is a family of at most μ subspaces of the vector space $(\bar{B}/B)[p]$. Moreover, by choice of W_k , each of these subspaces has dimension μ . Then appealing to Lemma 2.2 below we see that there exist 2^μ maximal subspaces of $(\bar{B}/B)[p]$ which contain no $W_k^*[p]$. Hence we can find 2^μ maximal pure subgroups of $\bar{B}$, each containing B , such that no W_k is contained in any of them. Then using Lemmas 1.4 and 1.5 we can refine this family to a family $\{G_j\}$ of 2^μ maximal pure subgroups such that no W_k is contained in a G_j and the family is essentially rigid.

To obtain the desired result we now observe that the basic subgroup of an image of $\bar{B}$ in a G_j has rank χ less than λ or rank λ . Since the rank of an image of $\bar{B}$ in a G_j cannot be $\lambda^{\aleph_0}$, any image of $\bar{B}$ in a G_j has rank at most $\chi^{\aleph_0}$ or λ . But if χ is less than λ then $\chi^{\aleph_0}$ is less than μ , hence λ has cofinality $\aleph_0$ and then χ less than λ implies that $\chi^{\aleph_0}$ is also less than λ . Thus in either case any image of $\bar{B}$ in a G_j has rank at most λ . The proof is completed by

LEMMA 2.2. *Let V be a vector space of dimension α over a field F , where α is an infinite cardinal. Let $\{W_i\}_{i < \alpha}$ be a family of subspaces each of dimension α . Then there exist 2^α subspaces U_j each of co-dimension 1 such that no W_i is contained in a U_j .*

Proof. This is a standard set-theoretic extension of a well-known result (see [1; Lemma 5.2]).

COROLLARY 2.3. *If λ is an infinite cardinal such that $\mu = \lambda^{\aleph_0} = 2^\lambda$, then there exist 2^μ separable p -groups G_i , each of rank μ , with basic subgroups of final rank λ , such that every homomorphism between different members of the family has image of cardinality at most λ .*

Clearly the homomorphisms between different members of the above family are "small" (but not in the technical sense of Pierce). This gives a partial answer to Fuchs [4; Problem 53]. A similar result has been obtained by Shelah [9] but by entirely different methods.

COROLLARY 2.4. *There exists a family of 2^c , where $c = 2^{\aleph_0}$, separable p -groups $\{G_j\}$ such that $E(G_j) = Q_p^* \oplus E_s(G_j)$ for each j and, moreover, every homomorphism between distinct members of the family is small.*

Proof. Clearly $\lambda = \aleph_0$ satisfies the conditions of Theorem 2.1. But, by a result of Megibben [7], if $\theta: \bar{B} \rightarrow G_j$ is not small then G_j contains an unbounded torsion-complete group. Since such a group must have rank $2^{\aleph_0}$ we deduce from the construction of the G_j 's in Theorem 2.1, that no such group can be contained in any of the G_j . Thus every homomorphism from G_j to G_k ($j \neq k$) is small and the result follows from Theorem 2.1.

We remark that the members of such a family are essentially indecomposable and that this answers a problem raised by Shelah at the end of his paper [9]. We note however that such a family had been previously constructed by Corner [2].

Acknowledgment. This paper was written under the supervision of Dr. A. L. S. Corner; I would like to express my appreciation of his encouragement and help.

References

1. R. A. Beaumont and R. S. Pierce, "Some invariants of p -groups", *Michigan Math. J.*, 11 (1964) 137-149.
2. A. L. S. Corner, "On endomorphism rings of primary abelian groups", *Quart. J. Math.*, 20 (1969), 277-296.
3. L. Fuchs, *Infinite abelian groups, Vol. I* (Academic Press, New York and London, 1970).
4. L. Fuchs, *Infinite abelian groups, Vol. II* (Academic Press, New York and London, 1973).
5. T. J. Jech, *Lectures in set theory*, Lecture Notes in Mathematics, Vol. 217 (Springer-Verlag, Berlin, 1971).
6. H. Leptin, "Zur Theorie der überabzählbaren abelschen p -Gruppen", *Abh. Math. Sem. Univ. Hamburg*, 24 (1960), 79-90.
7. C. Megibben, "Large subgroups and small homomorphisms", *Michigan Math. J.*, 13 (1966), 153-160.
8. R. S. Pierce, "Homomorphisms of primary abelian groups", *Topics in Abelian Groups* (Ed. J. M. Irwin and E. A. Walker), 215-310 (Chicago, Illinois, 1963).
9. S. Shelah, "Existence of rigid-like families of abelian p -groups", *Model Theory and Algebra*, Lecture Notes in Mathematics, Vol. 498 (Ed. D. H. Saracino and V. B. Weispfenning), 384-402 (Springer-Verlag, Berlin, 1975).

College of Technology,
Kevin Street,
Dublin 8, Ireland.

Dublin Institute for Advanced Studies,
10 Burlington Road,
Dublin 4, Ireland.