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Georgi Grahovski
Dublin Institute of Technology, georgi.grahovski@dit.ie

Rossen Ivanov
Dublin Institute of Technology, rivanov@dit.ie

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The Camassa-Holm Hierarchy and Soliton Perturbations

Georgi G. Grahovski and Rossen I. Ivanov

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Introduction. Integrable equations are widely used as model equations in various problems. Such equations are in some sense exactly solvable, e.g. by the inverse scattering method (ISM) and exhibit global regular solutions.
In hydrodynamic context, even though water waves are expected to be unstable in general, they do exhibit certain stability properties in physical regimes where integrable model equations are accurate approximations for the evolution of the free surface water wave. The model equation is not integrable, but is somehow close to an integrable equation, i.e. can be considered as a perturbation of an integrable equation. In such case it is still possible to obtain approximate analytical solutions.
There two main approaches treating the perturbations of integrable equations: a "Direct Approach" and a "Spectral" Approach.
The direct approach is based on expanding the solutions of the perturbed equation around the unperturbed one, then the corrections due to perturbations are to be determined:

$$\tilde{u}(x,t) = u(x,t) + \Delta u(x,t).$$

Here $\tilde{u}(x,t)$ ($u(x,t)$) is the solution of the perturbed (unperturbed) nonlinear equation and $\Delta u(x,t)$ is a (small) perturbation. The strength of the perturbation is measured by a parameter ϵ: $\Delta u(x,t) = \mathcal{O}(\epsilon)$. By small (weak) perturbation one means $0 < \epsilon \ll 1$. Such perturbations can be studied *directly* in the configuration (coordinate) space, The effect of the perturbations on the scattering data can be studied in the *spectral space* of the associated spectral problem.
Several authors had used various versions of the direct approach in the study of soliton perturbations: D. J. Kaup [3] had used a similar approach for the perturbed sine-Gordon equation. Keener and McLaughlin [4] had proposed a direct approach by obtaining the appropriate Green functions for the nonlinear Schrodinger and sine-Gordon equations. For a comprehensive review of the direct perturbation theory see e.g. [2, 5] and the references therein.
In the spectral space, the study of the soliton perturbations is based on the perturbations of the scattering data, associated to the spectral problem. It is based on the use of the expansions of the "potential" $u(x,t)$ of the associated spectral problem over the complete set of "squared solutions" (the eigenfunctions of the corresponding

recursion operator). Such methods are used by a number of authors, for studying perturbations of various nonlinear evolutionary equations: the sin-Gordon equation – [6]; the nonlinear Schrödinger equation – [7, 8, 9]; etc.
The completeness of the squared eigenfunctions helps to describe the ISM for the corresponding hierarchy as a GFT. The role of the Fourier modes for the GFT is played by the Scattering data of the associated Lax operator. The GFT provides a natural setting for the analysis of small perturbations to an integrable equation: The leading idea is that starting from a purely soliton solution of a certain integrable equation one can 'modify' the soliton parameters such as to incorporate the changes caused by the perturbation. There is a contribution to the equations for the scattering data that comes from the GFT-expansion of the perturbation.

2. The Camassa-Holm Hierarchy: An Overview. Closely related to the KdV hierarchy is the hierarchy of the Camassa-Holm (CH) equation [10]. This equation has the form

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \qquad (1)$$

where ω is a real constant. It is integrable with a Lax pair [10]

$$\Psi_{xx} = \Big(\frac{1}{4} + \lambda(m+\omega)\Big)\Psi, \qquad \Psi_t = \Big(\frac{1}{2\lambda} - u\Big)\Psi_x + \frac{u_x}{2}\Psi + \gamma\Psi, \qquad (2)$$

where $m \equiv u - u_{xx}$ and γ is an arbitrary constant.
Both CH and KdV equations appeared initially as models of the propagation of two-dimensional shallow water waves over a flat bottom. The solitary waves of KdV are smooth solitons, while the solitary waves of CH, which are also solitons, are smooth if $\omega > 0$ and peaked (called "peakons" and representing weak solutions), if $\omega = 0$.
The problem of perturbation of the CH equation arises when one deals with model equations that are in general non-integrable but close to the CH equation. A perturbation could appear for example when one takes into account the viscosity effect.
Another possible scenario comes from the so-called "b-equation" that also is a model of shallow water waves [11] :

$$m_t + b\omega u_x + bmu_x + m_x u = 0.$$

It generalizes the CH equation and is integrable only for: A) $b = 2$ – Camassa-Holm equation; B) $b = 3$ – Degasperis-Procesi equation.
The solutions of the b-equation for values of b close to $b = 2$ can be analyzed in the framework of the CH-perturbation theory. The "b-equation" can be casted into a form of a CH perturbation

$$m_t + 2\omega u_x + 2mu_x + m_x u = (2-b)(\omega u_x + mu_x) \equiv P[u],$$

for a small parameter $\epsilon = b - 2$.

3. Inverse Scattering Method and Generalised Fourier Transform for the Camassa-Holm Hierarchy. For simplicity we consider the case where m

is a Schwartz class function, $m(x) \in \mathcal{S}(\mathbb{R})$. For simplicity we use the notation $q = u - u_{xx} + \omega$; $q(x,t) > 0$ for all t if $q(x,0) > 0$. Let $k^2 = -\frac{1}{4} - \lambda\omega$, i.e. $\lambda(k) = -\frac{1}{\omega}\Big(k^2 + \frac{1}{4}\Big)$. The continuous spectrum in terms of k corresponds to k – real. The discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \dots, N$ where κ_n is real and $0 < \kappa_n < 1/2$. The continuous spectrum vanishes for $\omega = 0$, [12].
The Jost solutions of the spectral problem $f^{\pm}(x,k)$ are fixed by their asymptotic when $x \to \pm\infty$ for all real $k \neq 0$:

$$\lim_{x\to\infty} e^{\mp ikx} f^{\pm}(x,k) = 1,$$

and moreover at the points of the continuous spectrum

$$f^{-}(x,k) = a(k)f^{+}(x,-k) + b(k)f^{+}(x,k), \qquad \text{Im } k = 0.$$

The scattering coefficient $a(k)$ has an analytic continuation in the upper complex plane. At the points $i\kappa_n$ of the discrete spectrum, $a(k)$ has simple zeroes, $a(i\kappa_n) = 0$. Then $f^{-}(x, i\kappa_n) = b_n f^{+}(x, i\kappa_n)$ for some coefficient b_n.
The quantities $\mathcal{R}^{\pm}(k) = b(\pm k)/a(k)$ are known as reflection coefficients. One can define their analogues $R_n^{\pm} = \frac{b_n^{\pm 1}}{i a_n}$ at the points of the discrete spectrum $k = ik_n$ [13] where $\dot{a}_n = [\frac{\partial}{\partial k} a(k)]_{k=k_n}$.
With the asymptotics of the Jost solutions and (2) one can show that

$$L_{\pm} F^{\pm}(x,k) = \frac{1}{\lambda} F^{\pm}(x,k) \qquad L_{\pm} F_n^{\pm}(x) = \frac{1}{\lambda_n} F_n^{\pm}(x), \tag{3}$$

where $\lambda_n = \lambda(i\kappa_n)$; $F^{\pm}(x,k) \equiv (f^{\pm}(x,k))^2$, $F_n^{\pm}(x) \equiv F(x, i\kappa_n)$ are the squares of the Jost solutions and

$$L_{\pm} = (\partial^2 - 1)^{-1}\Big[4q(x) - 2\int_{\pm\infty}^{x} \mathrm{d}y\, m'(y)\Big]\cdot \tag{4}$$

is the recursion operator. The inverse of this operator is also well defined [13]. The completeness relation for the eigenfunctions of the recursion operator is [13]

$$\frac{\omega}{\sqrt{q(x)q(y)}}\theta(x-y) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F^{-}(x,k)F^{+}(y,k)}{k a^2(k)}\mathrm{d}k$$
$$+\sum_{n=1}^{N} \frac{1}{i\kappa_n \dot{a}_n^2}\Big[\dot{F}_n^{-}(x)F_n^{+}(y) + F_n^{-}(x)\dot{F}_n^{+}(y) - \Big(\frac{1}{i\kappa_n} + \frac{\ddot{a}_n}{\dot{a}_n}\Big)F_n^{-}(x)F_n^{+}(y)\Big]. \tag{5}$$

where $\dot{F}_n^{\pm}(x) \equiv [\frac{\partial}{\partial k} F^{\pm}(x,k)]_{k=k_n}$, etc. Therefore $F^{\pm}$, $F_n^{\pm}$ and $\dot{F}_n^{\pm}$ can be considered as 'generalised' exponents. It is possible to expand $m(x)$ and its variation $\delta m(x)$ over the above mentioned basis, or rather the quantities depending on $q(x)$ (which are determined by $m(x)$) [13]:

$$\omega\Big(\sqrt{\frac{\omega}{q(x)}} - 1\Big) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{2k\mathcal{R}^{\pm}(k)}{\lambda(k)} F^{\pm}(x,k)\mathrm{d}k + \sum_{n=1}^{N} \frac{2\kappa_n}{\lambda_n} R_n^{\pm} F_n^{\pm}(x); \tag{6}$$

$$\frac{\omega}{\sqrt{q(x)}}\int_{\pm\infty}^{x} \delta\sqrt{q(y)} \quad \mathrm{d}y = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{\lambda(k)}\delta\mathcal{R}^{\pm}(k)F^{\pm}(x,k)\mathrm{d}k$$
$$\pm \quad \sum_{n=1}^{N}\Big[\frac{1}{\lambda_n}(\delta R_n^{\pm} - R_n^{\pm}\delta\lambda_n)F_n^{\pm}(x) + \frac{R_n^{\pm}}{i\lambda_n}\delta\kappa_n\dot{F}_n^{\pm}(x)\Big] \qquad (7)$$

The expansion coefficients are given by the scattering data and their variations. This makes evident the interpretation of the ISM as a generalized Fourier transform. Now it is straightforward to describe the hierarchy of Camassa-Holm equations. To every choice of the function $\Omega(z)$, known also as the dispersion law we can put into correspondence the nonlinear evolution equation (NLEE) that belongs to the Camassa-Holm hierarchy:

$$\frac{2}{\sqrt{q}}\int_{\pm\infty}^{x}(\sqrt{q})_t\mathrm{d}y + \Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = 0. \qquad (8)$$

An equivalent form of the equation is

$$q_t + 2q\tilde{u}_x + q_x\tilde{u} = 0, \qquad \tilde{u} = \frac{1}{2}\Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}} - 1\Big). \qquad (9)$$

The choice $\Omega(z) = z$ leads to $\tilde{u} = u$ and thus to the CH equation (1). Other choices of the dispersion law and the corresponding equations of the Camassa-Holm hierarchy are discussed in [13]. The CH equation is equivalent to the following linear evolution equations for the scattering data:

$$\mathcal{R}_t^{\pm}(k) \mp ik\Omega(\lambda^{-1})\mathcal{R}^{\pm}(k) = 0, \qquad (10)$$
$$R_{n,t}^{\pm} \pm \kappa_n\Omega(\lambda_n^{-1})R_n^{\pm} = 0, \qquad (11)$$
$$\kappa_{n,t} = 0. \qquad (12)$$

The time-evolution of the scattering data for the CH equation (1) can be computed from the above formulae for $\Omega(z) = z$,

4. Perturbation Theory for for the Camassa-Holm Hierarchy. Let us start with a perturbed equation of the CH hierarchy of the form

$$q_t + 2q\tilde{u}_x + q_x\tilde{u} = P[u], \qquad \tilde{u} = \frac{1}{2}\Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}} - 1\Big), \qquad (13)$$

where again, $P[u]$ is a small perturbation, by assumption in the Schwartz-class. It is useful to write (13) in the form

$$\frac{2}{\sqrt{q}}\int_{\pm\infty}^{x}(\sqrt{q})_t\mathrm{d}y + \Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = \frac{1}{\sqrt{q}}\int_{\pm\infty}^{x}\frac{P(y)}{\sqrt{q(y)}}\mathrm{d}y. \qquad (14)$$

With the completeness relation (5) one can deduce the gereralised Fourier expansion for expressions, like the one on the right-hand side of (13).

Theorem: *Assuming that f^+ and f^- are not linearly dependent at $x=0$ and $g(x) \in \mathcal{S}(\mathbb{R})$, the following expansion formulas hold:*

$$\begin{aligned} \frac{\omega}{\sqrt{q}} \int_{\pm\infty}^{x} \frac{g(y)}{\sqrt{q(y)}} \mathrm{d}y = & \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \tilde{g}^{\pm}(k) F_x^{\pm}(x,k) \mathrm{d}k \\ & \mp \sum_{j=1}^{N} \left(g_{1,j}^{\pm} \dot{F}_{j,x}^{\pm}(x) + g_{2,j}^{\pm} F_{j,x}^{\pm}(x) \right), \end{aligned} \tag{15}$$

and the Fourier coefficients are

$$\begin{aligned} \tilde{g}^{\pm}(k) &= \frac{1}{ka^2(k)} \left(g, F^{\mp} \right), \\ g_{1,j}^{\pm} &= \frac{1}{k_j \dot{a}_j^2} \left(g, F_j^{\mp} \right), \\ g_{2,j}^{\pm} &= \frac{1}{k_j \dot{a}_j^2} \left[\left(g, \dot{F}_j^{\mp} \right) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j} \right) \left(g, F_j^{\mp} \right) \right], \end{aligned}$$

where $(g,F) \equiv \int_{-\infty}^{\infty} g(x)F(x)\mathrm{d}x$.

The substitution of the expansions (15) for $P[u]$, (6) and (7) into the perturbed equation (14) gives the following expressions for the modified scattering data:

$$\mathcal{R}_t^{\pm} \mp ik\Omega(1/\lambda)\mathcal{R}^{\pm} = \mp \frac{i\lambda(P, F^{\mp})}{2ka^2(k)}, \tag{16}$$

$$k_{j,t} = \frac{\lambda_j (P, F_j^{\mp})}{2k_j \dot{a}_j^2 R_j^{\pm}} \tag{17}$$

$$\begin{aligned} R_{j,t}^{\pm} - R_j^{\pm}\lambda_{j,t} & \pm \kappa_j \Omega(1/\lambda_j) R_j^{\pm} \\ & = -\frac{\lambda_j}{2k_j \dot{a}_j^2} \left[(P, \dot{F}_j^{\mp}) - \left(\frac{1}{k_j} + \frac{\ddot{a}_j}{\dot{a}_j} \right) (P, F_j^{\mp}) \right], \end{aligned} \tag{18}$$

From (18) we obtain the following for the coefficient b_j:

$$b_{j,t} + \kappa_j \Omega(1/\lambda_j) b_j = -\frac{\lambda_j}{4\kappa_j \dot{a}_j} \left(P, b_j^2 \dot{F}_j^{+} - \dot{F}_j^{-} \right).$$

The 'perturbed' solution for the hierarchy in the adiabatic approximation can be recovered from the following expansion for $\tilde{u}(x)$ with the 'modified' scattering data keeping the unperturbed 'generalised' exponents:

$$\tilde{u}(x) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{k\Omega(1/\lambda(k))}{\omega\lambda(k)} \mathcal{R}^{\pm}(k) F^{\pm}(x,k) \mathrm{d}k + \sum_{n=1}^{N} \frac{\kappa_n \Omega(1/\lambda_n)}{\omega \lambda_n} R_n^{\pm} F_n^{\pm}(x).$$

This formula follows from the second part of (9) and (6). Note that for the CH equation (1) $\tilde{u} \equiv u$.

5. Conclusions. In our derivations we used completeness relations that are valid only given the assumption that the Jost solutions f^+ and f^- are linearly independent at $x = 0$. The case when this condition is not satisfied is quite exceptional, however this is exactly the case when one has purely soliton solution. Then one has to take into account a nontrivial contribution from the scattering data at $k = 0$ and some of the presented results require modification. This means that no shelf is formed behind the soliton.

The presence of shelf for KdV equation is observed e.g. under the perturbation $P[u] = \epsilon u$ [14]. The evaluation of the perturbation terms for the CH hierarchy could be technically difficult due to the complicated form of the CH multisoliton solutions [15]. However the limit $\omega \to 0$ leads to the relatively simple peakon solutions.

We end up with listing some open problems: 1) Using the presented general formulae to study the perturbations of the peakon parameters; 2) To study the soliton interactions of pure solitons and peakons for CH.

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