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Dark solitons of the Qiao's hierarchy

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Abstract

We obtain a class of soliton solutions of the integrable hierarchy which has been put forward in a series of works by Z. Qiao. The soliton solutions are in the class of real functions approaching constant value fast enough at infinity, the so-called 'dark solitons'.

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1 Introduction

The interest inspired by the Camassa-Holm (CH) equation and its singular peakon solutions [1] prompted search for other integrable equations with similar properties. An integrable peakon equation with cubic nonlinearities has been discovered first by Qiao [11] and studied further e.g. in [12, 13]. Another equation with cubic nonlinearities has been found by V. Novikov [9]. The Lax pair for the Novikov's equation is given in [7], (see also a remark on the peakons of Qiao's equation in [7]). Actually the Qiao's equation

$$m_t + (m(u^2 - u_x^2))_x = 0, \qquad m = u - u_{xx} \tag{1}$$

together with the CH equation

$$m_t + 2u_x m + u m_x = 0, \qquad m = u - u_{xx} \tag{2}$$

belong to the bi-Hamiltonian hierarchy of equations described by Fokas and Fuchssteiner [4]. The Qiao's equation has a distinctive W/M-shape travelling

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wave solutions [11, 12]. The peakons of Novikov's equation have been studied in [8]. $2+1$ dimensional generalizations of Qiao's hierarchy are studied in [3]. Single peakon, mutil-peakon dynamics, weak kink, kink-peakon, and stability analysis of the Qiao's equation were studied in [14] and [5]. For the CH and related equations one can consult the monographs [6, 2, 10] and the references therein.

Equation (1) can also be written as

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0. \tag{3}$$

Qiao presented a 2×2 Lax pair for this equation given by the linear system $\mathbf{\Psi}_x = \mathbf{U}\mathbf{\Psi}$, $\mathbf{\Psi}_t = \mathbf{V}\mathbf{\Psi}$ with

$$\begin{aligned} \mathbf{U} &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}m\lambda \\ -\frac{1}{2}m\lambda & \frac{1}{2} \end{pmatrix}, \\ \mathbf{V} &= \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}m\lambda(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}m\lambda(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}. \end{aligned} \tag{4}$$

There is another equation from the same hierarchy,

$$m_t + \left(\frac{1}{m^2}\right)_x - \left(\frac{1}{m^2}\right)_{xxx} = 0, \tag{5}$$

for which the $\mathbf{V}$-operator is $\mathbf{\Psi}_t = \mathbf{V_2}\mathbf{\Psi}$ where

$$\mathbf{V_2} = \frac{\lambda}{2}\begin{pmatrix} -\frac{\lambda}{m} & \lambda^2 + \frac{m(m_x - m_{xx}) + 3m_x^2}{m^4} \\ -\lambda^2 + \frac{m(m_x + m_{xx}) - 3m_x^2}{m^4} & \frac{\lambda}{m} \end{pmatrix}. \tag{6}$$

The (white) soliton solutions of (1) and (5) have been found previously [15, 17]. These studies rely on the fact that the spectral problem for (1) is gauge-equivalent to the one for the mKdV equation. In this study we will present soliton solutions approaching a constant value for $|x| \to \infty$ (dark solitons). To this end we are going to formulate the spectral problem in the form of a Schrödinger operator, which is the same spectral problem as for the KdV equation.

2 Reformulation of the spectral problem

Let us consider solutions such as

$$m(x,t) > 0, \qquad \lim_{x \to \pm\infty} m(x,t) = m_0, \tag{7}$$

where m_0 is a positive constant. Let us assume also that $m(x,\cdot)-m_0 \in \mathcal{S}(\mathbb{R})$ for any value of t. One can reformulate the spectral problem into a scalar one as follows. Introducing $\mathbf{\Psi} = (\psi, \phi)^T$ the matrix Lax pair written in components is

$$\begin{aligned} 2\psi_x &= -\psi + m\lambda\phi \\ 2\phi_x &= -m\lambda\psi + \phi. \end{aligned}$$

With a change of coordinates

$$\partial_y = \frac{2}{m}\partial_x, \qquad \psi = \frac{1}{\lambda}\left[\frac{\phi}{m} - \phi_y\right] \tag{8}$$

we obtain the following scalar spectral problem for $\phi(y,\lambda)$ (sometimes we do not write the argument t which is an external parameter for the considered spectral problem)

$$-\phi_{yy} + \left[\left(\frac{1}{m}\right)_y + \frac{1}{m^2}\right]\phi = \lambda^2\phi. \tag{9}$$

Note that this is a Schrödinger's operator with a potential

$$U(y,t) = \left(\frac{1}{m}\right)_y + \frac{1}{m^2} \tag{10}$$

It is well known how to recover $U(y,t)$ from the scattering data of (9), however the solution is $m(y,t)$ and its recovery from $U(y,t)$ necessitates solving a nonlinear (Riccati) equation. We can express $m(y,t)$ in terms of the eigenfunctions of the Schrödinger's operator. We introduce $\rho(y,\lambda) = \frac{\phi_y}{\phi}$ from which we immediately obtain

$$\rho_y + \rho^2 = \frac{\phi_{yy}}{\phi} = U(y) - \lambda^2.$$

If we define

$$\rho_0(y) = \rho(y,0)$$

then we have

$$U(y) = \rho_{0,y} + \rho_0^2.$$

However, due to (10) we now have $\frac{1}{m} = \rho_0$ or

$$m(y,t) = \frac{1}{\rho_0(y,t)} = \left.\frac{\phi(y,t,\lambda)}{\phi_y(y,t,\lambda)}\right|_{\lambda=0} \tag{11}$$

So far we treated y as a new variable instead of x. However we can treat y as a parameter, and then (11) represents the solution in parametric form, where the original variable x is given due to (8), (11) by:

$$x(y,t) = 2\ln\phi(y,t,0) + \text{const.} \tag{12}$$

Assuming that $\phi(y,t,0)$ is everywhere positive, we have a solution in parametric form (11), (12) given entirely in terms of the eigenfunctions $\phi(y,t,0)$. One can write formally the solution (neglecting the constant in the last formula) as

$$m(x,t) = 2\int_{-\infty}^{\infty} \delta\left(x - 2\ln\phi(y,t,0)\right)\mathrm{d}y. \tag{13}$$

3 Inverse scattering and Soliton solutions

From (7) and (10) it follows that $U(y)$ does not decay to 0 when $y \to \pm\infty$. To this end we introduce the modified potential

$$\tilde{U}(y) = U(y) - \frac{1}{m_0^2}, \tag{14}$$

for which $\lim_{|y|\to\infty}\tilde{U}(y) = 0$. So we have

$$-\phi_{yy} + \left[U(y) - \frac{1}{m_0^2}\right]\phi = \left(\lambda^2 - \frac{1}{m_0^2}\right)\phi,$$

or, introducing a new spectral parameter

$$k^2 = \lambda^2 - \frac{1}{m_0^2} \tag{15}$$

we have a standard spectral problem

$$-\phi_{yy} + \tilde{U}(y)\phi(k,y) = k^2\phi(k,y), \qquad \tilde{U}(y) \in \mathcal{S}(\mathbb{R}). \tag{16}$$

When $\lambda = 0$ however we find $k = \pm\frac{i}{m_0}$ for k. This means that if one takes an eigenfunction $\phi(k,y)$ of (16) analytic in the upper (lower) half complex k-plane, one should evaluate it at $k = \frac{i}{m_0}$ ($k = -\frac{i}{m_0}$):

$$m(y,t) = \left.\frac{\phi(y,t,k)}{\phi_y(y,t,k)}\right|_{k=\pm\frac{i}{m_0}} \tag{17}$$

$$x(y,t) = 2\ln\phi\left(y,t,\pm\frac{i}{m_0}\right). \tag{18}$$

The spectral theory for the problem (16) is well developed, e.g. [16]. We are going to use these results to construct the soliton solutions of (1), (5). One can introduce scattering data as usual. For the time-dependence of the scattering data one needs the time-evolution of the eigenfunction $\phi(k,x)$. The Lax-pair in x and t variables for (1) has the form

$$\phi_{xx} = \frac{m_x}{m}\phi_x + \left(\frac{1}{4} - \frac{m_x}{2m} - \frac{m^2}{4}\lambda^2\right)\phi, \tag{19}$$

$$\phi_t = \frac{1}{\lambda^2}\left[\frac{u_x + u_{xx}}{m}\right]\phi - \left[\frac{u+u_x}{\lambda^2 m} + \frac{u^2 - u_x^2}{2}\right]\phi_x + \gamma\phi, \tag{20}$$

where γ is an arbitrary constant. The second equation, (20) in asymptotic form $x \to \pm\infty$ is

$$\phi_t \to -\left[\frac{1}{\lambda^2} + \frac{m_0^2}{2}\right]\phi_x + \gamma\phi,$$

or, in terms of k, y-variables when $y \to \pm\infty$,

$$\phi_t \to -\frac{m_0^3}{2}\left[\frac{k^2 m_0^2 + 3}{k^2 m_0^2 + 1}\right]\phi_y + \gamma\phi, \tag{21}$$

since

$$\lim_{|y|\to\infty} m = \lim_{|y|\to\infty} u = m_0.$$

Defining Jost solutions by

$$\lim_{y\to\pm\infty} \varphi_\pm(y,k)e^{iky} = 1, \tag{22}$$

such that

$$\varphi_-(y,k) = a(k)\varphi_+(y,k) + b(k)\bar{\varphi}_+(y,k), \qquad k \in \mathbb{R} \tag{23}$$

and noting that $\varphi_- \to ae^{-iky} + be^{iky}$ when $y \to \infty$ we find from (21)

$$a_t = \frac{m_0^3}{4}\left[\frac{k^2 m_0^2 + 3}{k^2 m_0^2 + 1}\right](ika) + \gamma a \tag{24}$$

$$b_t = -\frac{m_0^3}{4}\left[\frac{k^2 m_0^2 + 3}{k^2 m_0^2 + 1}\right](ikb) + \gamma b. \tag{25}$$

Requiring $a_t = 0$, we find

$$b_t = -ik\frac{m_0^3}{2}\left(\frac{k^2 m_0^2 + 3}{k^2 m_0^2 + 1}\right)b(k,t)$$

and thus for the scattering coefficient $r \equiv b/a$ we have

$$r(k,t) = r(k,0)\exp\left[-ik\frac{m_0^3}{2}\left(\frac{k^2m_0^2+3}{k^2m_0^2+1}\right)t\right], \tag{26}$$

and for the analogue on the discrete spectrum $k = i\kappa_n$,

$$R_n(t) \equiv \frac{b(i\kappa_n)}{ia'(i\kappa_n)} = R_n(0)\exp\left[\frac{\kappa_n m_0^3(3-\kappa_n^2m_0^2)}{2(1-\kappa_n^2m_0^2)}t\right]. \tag{27}$$

For the equation (5) the time evolution of the spectral eigenfunctions is given by

$$\phi_t = \frac{v}{m}\phi + \left(\lambda^2 - v\right)\frac{2}{m}\phi_x + \gamma\phi, \qquad v = \frac{m(m_x+m_{xx})-3m_x^2}{m^4} \tag{28}$$

and analogous considerations give

$$r(k,t) = r(k,0)\exp\left[-2ik\left(\frac{1}{m_0^2}+k^2\right)t\right], \tag{29}$$

$$R_n(t) = R_n(0)\exp\left[2\kappa_n\left(\frac{1}{m_0^2}-\kappa_n^2\right)t\right]. \tag{30}$$

It is convenient to introduce a dispersion law for the hierarchy, which for the considered two members is

$$f(\kappa) = \begin{cases} \frac{\kappa m_0^3(3-\kappa^2m_0^2)}{2(1-\kappa^2m_0^2)} & \text{for eq. (1)}, \\ \frac{2\kappa}{m_0^2}\left(1-m_0^2\kappa^2\right) & \text{for eq. (5)}. \end{cases}$$

Then for the whole hierarchy we can write in general

$$R_n(t) = R_n(0)\exp\left(f(\kappa_n)t\right). \tag{31}$$

For further convenience we introduce

$$\xi_n \equiv y - \frac{f(\kappa_n)}{2\kappa_n}t - \frac{1}{2\kappa_n}\ln\frac{R_n(0)}{2\kappa_n}.$$

The eigenfunctions of the spectral problem (16) are well known, see e.g. [16]. In the purely N-soliton case the eigenfunction, analytic in the lower complex k-plane is the Jost solution $\varphi_+(y,k)$ defined in (22) which has the form

$$\varphi_+(y,t,k) = e^{iky}\left(1+\sum_{n=1}^{N}\frac{\Gamma_n(y,t)}{k-i\kappa_n}\right) \tag{32}$$

with the residues $\Gamma_n(y,t)$ satisfying a linear system

$$\Gamma_n(y,t) = iR_n(t)e^{-2\kappa_n y}\left(1+i\sum_{m=1}^{N}\frac{\Gamma_m(y,t)}{\kappa_n+\kappa_m}\right).$$

The time-dependence of the scattering data is given by (31). The N- soliton solution then is given in parametric form by (17) and (18) for the eigenfunction (32). The condition $0<\kappa_n<m_0^{-1}$ is sufficient to ensure smoothness of the solitons.

4 Example: One-Soliton Solution

The one-soliton solution corresponds to one discrete eigenvalue $k_1 = i\kappa_1$, where κ_1 is real, positive and $\kappa_1 < m_0^{-1}$. The eigenfunction in this case is (32)

$$\varphi_+(y,t,k) = e^{iky}\left(1+\frac{1}{k-i\kappa_1}\cdot\frac{iR_1(t)e^{-2\kappa_1 y}}{1+\frac{R_1(t)}{2\kappa_1}e^{-2\kappa_1 y}}\right). \tag{33}$$

Evaluated at $k=\frac{-i}{m_0}$ we find

$$\varphi_+(y,t,\frac{-i}{m_0}) = e^{\frac{y}{m_0}}\left(1-\frac{1}{\frac{1}{m_0}+\kappa_1}\cdot\frac{R_1(t)e^{-2\kappa_1 y}}{1+\frac{R_1(t)}{2\kappa_1}e^{-2\kappa_1 y}}\right).$$

From (17) and (18) we obtain the one-soliton solutions

$$x(y,t) = \frac{2y}{m_0}+2\ln\left(1-\frac{\kappa_1 m_0 e^{-\kappa_1\xi_1}}{(1+\kappa_1 m_0)\cosh\kappa_1\xi_1}\right), \tag{34}$$

$$m(y,t) = \frac{m_0}{1+\frac{\kappa_1^2 m_0^2 \mathrm{sech}^2\kappa_1\xi_1}{1-m_0\kappa_1\tanh\kappa_1\xi_1}}. \tag{35}$$

The extremum (minimum) of m occurs when

$$\xi_1 = \frac{1}{4\kappa_1}\ln\left(\frac{1-m_0\kappa_1}{1+m_0\kappa_1}\right).$$

This is a constant value, e.g. the soliton moves with a velocity $\frac{f(\kappa_1)}{2\kappa_1}$ that depends on the dispersion law (i.e. the chosen equation from the hierarchy). The profile of the dark soliton is given on Fig. 1.

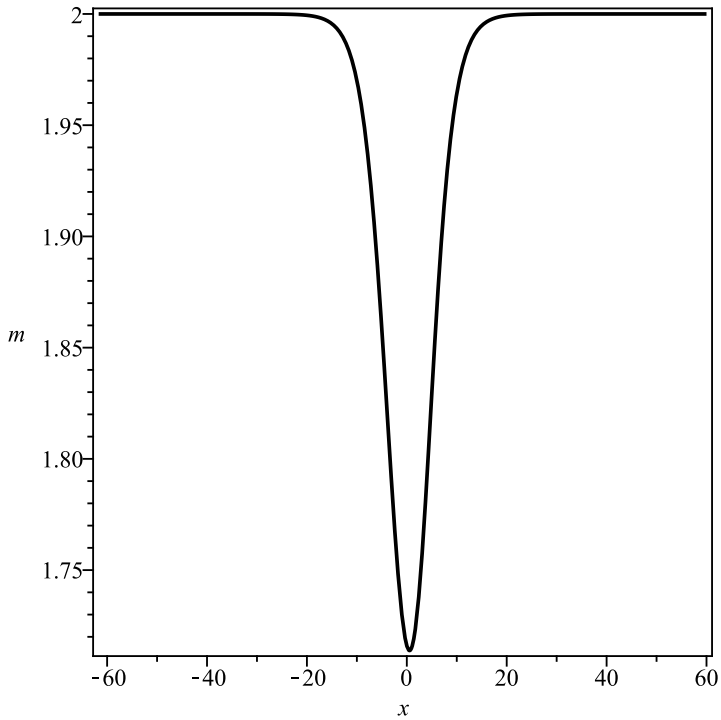

Figure 1: One soliton profile, $m_0 = 2$, $\kappa_1 = 0.2$.

5 Example: Two soliton solution

In the case of two discrete eigenvalues we compute

$$\varphi_+(y,t,\frac{-i}{m_0}) = e^{\frac{y}{m_0}} \frac{1+\nu_1 e^{-2\kappa_1\xi_1}+\nu_2 e^{-2\kappa_2\xi_2}+\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 \nu_1\nu_2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}}{1+e^{-2\kappa_1\xi_1}+e^{-2\kappa_2\xi_2}+\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}} \tag{36}$$

where the following notation is utilized:

$$\nu_j = \frac{\frac{1}{m_0}-\kappa_j}{\frac{1}{m_0}+\kappa_j}, \qquad j=1,2.$$

From (17) and (18) we obtain the two-soliton solutions:

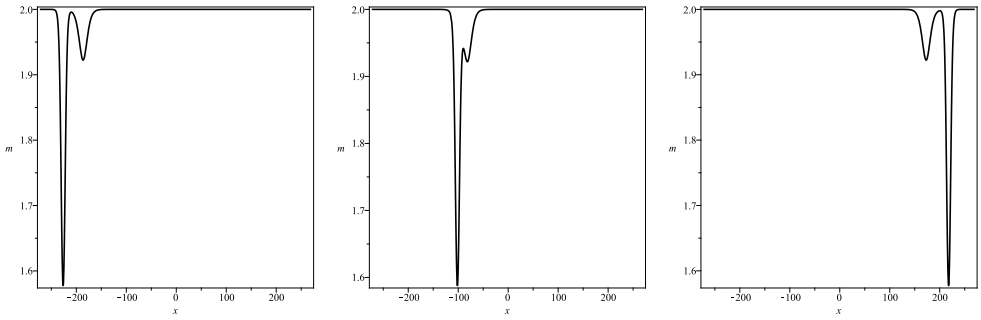

Figure 2: Snapshots of the two (dark) soliton solution of the Qiao equation (1), for three values of t : -30, -12 and 30. The other parameters are $m_0 = 2$, $\kappa_1 = 0.1$, $\kappa_2 = 0.25$.

$$x(y,t) = \frac{2y}{m_0} + 2\ln\frac{\Delta_1}{\Delta_2} \tag{37}$$

$$m(y,t) = \frac{m_0}{1+\frac{m_0\Delta_3}{\Delta_1\Delta_2}}. \tag{38}$$

where the following notations are used:

$$\begin{aligned}
\Delta_1(y,t) &= 1 + e^{-2\kappa_1\xi_1} + e^{-2\kappa_2\xi_2} + \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2} \\
\Delta_2(y,t) &= 1 + \nu_1 e^{-2\kappa_1\xi_1} + \nu_2 e^{-2\kappa_2\xi_2} + \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 \nu_1\nu_2 e^{-2\kappa_1\xi_1-2\kappa_2\xi_2}. \\
\Delta_3(y,t) &= \frac{4\kappa_1^2}{m_0^{-1}+\kappa_1} e^{-2\kappa_1\xi_1} + \frac{4\kappa_2^2}{m_0^{-1}+\kappa_2} e^{-2\kappa_2\xi_2} \\
&+ \frac{8(\kappa_1-\kappa_2)^2}{m_0(m_0^{-1}+\kappa_1)(m_0^{-1}+\kappa_2)} e^{-2\kappa_1\xi_1-2\kappa_2\xi_2} \\
&+ \frac{4\kappa_2^2\nu_1}{m_0^{-1}+\kappa_2}\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-4\kappa_1\xi_1-2\kappa_2\xi_2} \\
&+ \frac{4\kappa_1^2\nu_2}{m_0^{-1}+\kappa_1}\left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2 e^{-2\kappa_1\xi_1-4\kappa_2\xi_2}.
\end{aligned} \tag{39}$$

The interaction of two dark solitons is illustrated on Fig. 2.

6 Conclusions

In this paper we demonstrated how the spectral problem for the Qiao's hierarchy can be reduced to the one for the standard Schrödinger operator and hence the soliton solutions ('dark' solitons) can be obtained in a straightforward manner. This necessitates constant boundary conditions for the solution and also a restriction on the discrete eigenvalues $0 < \kappa_n < m_0^{-1}$. It is interesting what happens to the solutions if this condition is violated. Based on the similarity with Camassa-Holm equation it is likely that there are breaking waves present in this case. Moreover, the equation (1) has a conservation law in the form

$$X_x(x,t)m(X,t) = m(x,0)$$

where X is the solution of

$$X_t(x,t) = u^2(X,t) - u_x^2(X,t), \qquad X(x,0) = x.$$

It is likely that this conservation law will play an essential role in the study of the wellposedness, existence and breaking of the solutions.

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