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Equation of State for a van der Waals Universe during Reissner–Nordström Expansion

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Abstract

In a previous work [E.M. Prodanov, R.I. Ivanov, and V.G. Gueorguiev, *Reissner–Nordström Expansion*, *Astroparticle Physics*], we proposed a classical model for the expansion of the Universe during the radiation-dominated epoch based on the gravitational repulsion of the Reissner–Nordström geometry — naked singularity description of particles that ”grow” with the drop of the temperature. In this work we model the Universe during the Reissner–Nordström expansion as a van der Waals gas and determine the equation of state.

We recently proposed [1] a classical mechanism for the cosmic expansion. It relies on the assumption that the Universe is a two-component gas. One of the fractions is that of ultra-relativistic "normal" particles of typical mass m and charge q with equation of state of an ideal quantum gas of massless particles. The other component is "unusual" — these are either particles of ultra-high charge or particles of ultra-high mass M and charge Q (up to a few electron charges: $\sim 10^{-34}$ cm), greater than the mass M . The "unusual" particles are modelled as Reissner–Nordström naked singularities and the expansion mechanism is based on their gravitational repulsion. The motion of charged test particles in a Reissner–Nordström field has been studied in [2].

Naked singularities have been subject of significant scrutiny for decades. The general-relativistic description of the gravito-electric field of charged elementary particles is in terms of naked singularities — since the charge-to-mass ratio for elementary particles is greater than one. For the electron, the charge-to-mass ratio is $\sim 10^{21}$. In view of this, in the 1950s, the Reissner–Weyl repulsive solution served as an effective model for the electron. Very recently, a general-relativistic model for the classical electron — a point charge with finite electromagnetic self-energy, described as Reissner–Nordström (spin 0) or Kerr–Newman (spin 1/2) solution of the Einstein–Maxwell equations, — has been studied by Blinder [3]. Naked singularities are disliked — hence the Cosmic Censorship Conjecture [4] — but not ruled out — there is no mathematical proof whatsoever of the Cosmic Censorship. At least one naked singularity is agreed to have existed — the Big Bang — the Universe itself. Of particular importance in the study of naked singularities are the work of Choptuik [5], where numerical analysis of Einstein–Klein–Gordon solutions shows the circumstances under which naked singularities are produced, and the work of Christodoulou [6] who proved that there exist choices of asymptotically flat initial data which evolve to solutions with a naked singularity. The possibility of observing naked singularities at the LHC has been studied in [7] — for example, a proton-proton collision could result in a naked singularity and a set of particles with vanishing total charge or with one net positive charge — an event probably undistinguishable from ordinary particle production. In a cosmological setting, naked singularities have been well studied and classified — see, for example, [8].

As shown in [1], for temperatures below 10^{29} K, quantum effects do not play a role in the interaction between the "unusual" particles and the "normal" particles (see [1] also for the discussion on the proposed model's range of validity, imposed by considering classical gravitational interactions only).

Consider a "normal" particle of specific charge $|q/m| > 1$, and an "unusual" particle of opposite charge ($qQ < 0$). If the "normal" particle approaches the "unusual" particle from infinity, the field of the naked singularity is characterized by three regions [1, 2]:

- (a) *Impenetrable region* — between $r = 0$ and $r = r_0(T)$.

For an incoming test particle, reality of the kinetic energy leads to the existence of a turning radius. This is the radius $r_0(T)$ which can be thought of as radius of the "impenetrable" sphere surrounding the naked singularity. It depends on the energy of the incoming particle (or the temperature T of the "normal" fraction of the Universe): the higher the energy (or the temperature), the deeper the

incoming particle will penetrate into the gravitationally repulsive field of the naked singularity.

- (b) *Repulsive region* — between the turning radius $r_0(T)$ and the critical radius $r_c \geq r_0(T)$.

The critical radius r_c is where the repulsion and attraction interchange. As the temperature drops, the "unusual" particles "grow" (incoming particles have lower and lower energies and turn back farther and farther from the naked singularity). When the temperature gets sufficiently low, the radius of the "unusual" particles $r_0(T)$ grows to r_c (but not beyond r_c , as the region $r > r_c$ is characterized by attraction and an incoming particle cannot turn back while attracted). This means that incoming particles have such low energies that they turn back immediately after they encounter the gravitational repulsion. This does not apply to incoming particles of charge q such that $qQ > 0$ — we shall see that the repulsive region for such particles extends to infinity (the gravitational attraction will not be sufficiently strong to overcome the electrical repulsion).

- (c) *Attractive region* — from the critical radius r_c to infinity. Again, there is no gravitationally attractive region for an incoming particle such that $qQ > 0$.

As shown in [2], when an incoming particle has sufficiently large charge which is also opposite in sign to that of the naked singularity: $\text{sign}(Q)q/m < -1$, the particle will collide with the naked singularity. When the naked singularity "captures" such particle, its charge Q decreases and its mass M increases. If sufficient number of incoming particles are captured, Q will eventually become equal to M — the naked singularity will pick a horizon and turn into a black hole. This black hole will evaporate quickly afterwards. We will assume that our "unusual" particles have survived such annihilation. We will also assume that these superheavy charged particles have survived annihilation through all other different competing mechanisms — for example, they could recombine into neutral particles or decay before or after that (see Ellis et al. [9] on the astrophysical constraints on massive unstable neutral relic particles and Gondolo et al. [10] on the constraints of the relic abundance of a dark matter candidate — a generic particle of mass in the range of $1 - 10^{14}$ TeV, lifetime greater than $10^{14} - 10^{18}$ years, decaying into neutrinos).

Our expansion model assumes that initially, at extremely high energies and pressures, the "normal" particles are within the gravitationally repulsive regions of the "unusual" particles. The particles from the "normal" fraction "roll down" the gravitationally repulsive potentials of the "unusual" particles and in result the Universe expands. The addition of a new class of particles (the "unusual") in the picture of the Universe does not challenge our current understanding of the physical laws governing the Universe. The "unusual" particles interact purely classically with the "normal" component of the Universe and this classical interaction results in the appearance of a repulsive force. Our aim is to offer a possible explanation for the expansion of the Universe while conforming with the well established theoretical models. As shown in [1], during the Reissner–Nordström expansion, the standard relation between the scale factor of the Universe a

and the temperature T holds: $aT = \text{const.}$ Also, during the Reissner–Nordström expansion, the time-dependance of the scale factor is: $a(\tau) \sim \sqrt{\tau}$ (see [1] for details). Such is the behaviour of the scale factor during the expansion of the Universe throughout the radiation-dominated era, obtained by the standard cosmological treatment. These two pictures are not alternatives — they complement each other. This is manifested in the fact that the scale factor behaves in the same way, namely, that the dynamics of the Universe is the same for both pictures. Thus, we believe, the physics of the expanding Universe could be considered as a superposition of these two pictures.

On a large scale, the Universe is isotropic and homogeneous and for a Robertson–Walker Universe (see, for example, [11, 12]), the energy-momentum sources are modeled as a perfect fluid, specified by an energy density and isotropic pressure in its rest frame. This applies for matter known observationally to be very smoothly distributed. On smaller scales, such as stars or even galaxies, this is a poor description. In our picture, the Universe has global Robertson–Walker geometry, but locally it has Reissner–Nordström geometry. On the level of the interaction between the "unusual" particles and the "normal" particles of the Universe, different density and pressure variables should be introduced. We are going to complement the entire radiation-dominated era with Reissner–Nordström expansion and model the interaction between the "unusual" particles and the "normal" particles as interaction between the components of a van der Waals gas. Modeling the Universe as a van der Waals phase is possible in the light of the deep analogies between the physical picture behind the Reissner–Nordström expansion and the classical van der Waals molecular model: atoms are surrounded by imaginary hard spheres and the molecular interaction is strongly repulsive in close proximity, mildly attractive at intermediate range, and negligible at longer distances. The laws of ideal gas must then be corrected to accommodate for such interaction: the pressure should increase due to the additional repulsion and the available volume should decrease as atoms are no longer entities with zero own volumes (see, for example, [13]).

As an interesting development in a similar vein, one should point out the work [14] (see also the references therein) which studies van der Waals quintessence by considering a cosmological model comprising of two fluids: baryons, modelled as dust (large-scale structure fluid) and dark matter with a van der Waals equation of state (background fluid). Van der Waals equation of state for ultra-relativistic matter have been studied by [15].

Returning to the Reissner–Nordström expansion, once the temperature drops sufficiently low so that $r_0(T)$ becomes equal to r_c , the "normal" particle with charge q such that $0 \geq \text{sign}(Q)q/m \geq -1$, will be expelled beyond $r = r_c$ (as $r_0(T) < r$ always) — into the region of gravitational attraction. Due to its ultra-high energy, the "normal" particle will overcome the gravitational attraction and will escape unopposed to infinity. Thus the gravitationally attractive region is of no importance for such particles and for them we can assume that the potential of the naked singularity is infinity from $r = 0$ to $r = r_0(T)$ and zero from $r = r_c$ to infinity.

For "normal" particles such that $qQ > 0$, the potential gradually drops to zero towards infinity. For ultra-high temperatures, the energy E of a "normal" particle is of the order of kT . At temperatures below 10^{10} K, the dominant term in the energy E becomes the

particle's rest energy mc^2 (throughout the paper we use geometrized units $c = 1 = G$) and, as we shall see, the turning radius $r_0(T)$ becomes infinitely large below such temperature. As we model the entire radiation-dominated epoch with Reissner–Nordström repulsion, at recombination (the end of this epoch: $t_{recomb} \sim 300\,000$ years), the free ions and electrons combine to form neutral atoms ($q = 0$) and this naturally ends the Reissner–Nordström expansion — a neutral "normal" particle will now be too far from an "unusual" particle to feel the gravitational repulsion (the density of the Universe will be sufficiently low). During the expansion, the volume V of the Universe is proportional to the number N of "unusual" particles times their volume (one can view the impenetrable spheres of the naked singularities as densely packed spheres filling the entire Universe). At recombination, $V \sim t_{recomb}^3$. Therefore, at recombination, the radius $r_0(T)$ of an "unusual" particle will be of the order of $R_c = N^{-1/3}t_{recomb}$. During the expansion, a "normal" particle is never farther than $r_0(T)$ from an "unusual" particle. We will request that once $r_0(T)$ becomes equal to $R_c = N^{-1/3}t_{recomb}$, then the potential of the interaction between a naked singularity and a particle of charge q , such that $qQ > 0$, becomes zero.

There are many studied examples of such "unusual" particles: these could be either the ultra-heavy charged particles (CHAMPS) of the model of de Rujula, Glashow and Sarid [16], or the collapsed charged objects of very low (\sim Planck) mass studied by Hawking [17], or even Preskill's [18] ultra-heavy magnetic monopoles which were created so copiously in the early Universe that they outweighed everything else in the Universe by a factor of 10^{12} (in the latter case Q will be the magnetic charge of a monopole, all other particles in the Universe will be magnetically neutral: $q = 0$).

In this paper we use a standard treatment [13] to model the van der Waals phase of the Universe as a real gas and, using the virial expansion, we obtain the gas parameters. Combining the van der Waals equation with $aT = \text{const}$, we find the equation of state describing the classical interaction between the ordinary particles in the Universe and the "unusual" particles. The phase after the van der Waals phase is that of an ideal gas of "normal" particles.

Consider now Reissner–Nordström geometry [19, 12] in Boyer–Lindquist coordinates [20] and geometrized units is:

$$ds^2 = -\frac{\Delta}{r^2} dt^2 + \frac{r^2}{\Delta} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \quad (1)$$

where: $\Delta = r^2 - 2Mr + Q^2$, M is the mass of the centre, and Q — the charge of the centre. We will be interested in the case of a naked singularity only, namely: $Q > M$. The radial motion of a test particle of mass m and charge q in Reissner–Nordström geometry can be modeled by an effective one-dimensional motion of a particle in non-relativistic mechanics with the following equation of motion [1, 2] (see also [21] for Schwarzschild geometry) :

$$\frac{\dot{r}^2}{2} + \left[-\left(1 - \frac{q}{m} \frac{Q}{M} \epsilon\right) \frac{M}{r} + \frac{1}{2} \left(1 - \frac{q^2}{m^2}\right) \frac{Q^2}{r^2} \right] = \frac{\epsilon^2 - 1}{2}, \quad (2)$$

where $\epsilon = E/m$ is the specific energy (energy per unit mass) of the three-dimensional motion. The expression in the square brackets is the effective non-relativistic one-dimensional potential and the specific energy of the effective one-dimensional motion is $(1/2)(\epsilon^2 - 1)$. As we will not be interested in the effective one-dimensional motion, we will proceed from equation (2) to derive an expression that will serve as gravitational potential energy $U(r)$ of the three-dimensional motion. In the rest frame of the probe ($\dot{r} = 0$), equation (2) becomes a quadratic equation for the energy ϵ . The bigger root of this equation is exactly the gravitational potential energy $U(r)$ plus the rest energy m (see also [22]). Namely:

$$U(r) = \frac{qQ + m\sqrt{\Delta}}{r} - m = \frac{qQ}{r} + m\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} - m. \quad (3)$$

Since $M \sim Q \sim 10^{-34}$ cm, expression (3) for the potential energy $U(r)$, for distances above 10^{-34} cm, can be approximated by:

$$U(r) = -\frac{Mm}{r} + \frac{qQ}{r} + \frac{m}{2}(-M^2 + Q^2)\frac{1}{r^2}. \quad (4)$$

Motion is allowed only when the kinetic energy is real. Equation (2) determines the region (r_-, r_+) within which motion is impossible. The turning radii are given by [1, 2]:

$$r_{\pm} = \frac{M}{\epsilon^2 - 1} \left[\epsilon \frac{q}{m} \frac{Q}{M} - 1 \pm \sqrt{\left(\epsilon \frac{q}{m} \frac{Q}{M} - 1 \right)^2 - (1 - \epsilon^2) \left(1 - \frac{q^2}{m^2} \right) \frac{Q^2}{M^2}} \right]. \quad (5)$$

We identify the impenetrable radius $r_0(T)$ of an “unusual” particle as the bigger root r_+ . The expansion mechanism is based on the fact that $r_0(T)$ is inversely proportional to the temperature, namely, the naked singularity drives apart all neutral particles and particles of specific charge q/m such that $\text{sign}(Q)q/m \geq -1$.

Note that when $\epsilon \rightarrow 1$ (which happens when the rest energy becomes the dominant term, i.e. when kT drops below m , or below 10^{10} K), then the turning radius $r_0(T)$ tends to infinity.

At the point where gravitational attraction and repulsion interchange, there will be no force acting on the incoming particle. That is, this is the point where the derivative of the potential (4) vanishes:

$$r_c = M \left(\frac{Q^2}{M^2} - 1 \right) \left(1 - \frac{q}{m} \frac{Q}{M} \right)^{-1}. \quad (6)$$

Obviously, the critical radius r_c for an incoming particle charged oppositely to the “unusual” particle ($qQ < 0$) will be smaller than the critical radius for a neutral ($q = 0$) incoming particle — the region of gravitational repulsion will be reduced by the additional electrical attraction. When the incoming probe has charge with the same sign as that of the “unusual” particle (i.e. $qQ > 0$), then r_c does not exist. This means that there will be a region of repulsion only — the gravitational attraction will not be sufficiently strong to overcome the electrical repulsion.

Finally, the potential energy of a charged probe in the field of an “unusual” particle can be written as follows:

$$U(r) = \begin{cases} \infty, & r < r_0(T), \\ -\frac{Mm}{r} + \frac{qQ}{r} + \frac{m}{2}(-M^2 + Q^2)\frac{1}{r^2}, & r_0(T) \leq r \leq R, \\ 0, & r > R, \end{cases} \quad (7)$$

where:

$$R = \begin{cases} r_c, & 0 \geq \text{sign}(Q)q/m \geq -1, \\ R_c, & qQ > 0. \end{cases} \quad (8)$$

Obviously, the expansion beyond r_c will be due to the second type of particles only and we will study them from now on.

Next, we consider the thermodynamics of a real gas.

The virial expansion relates the pressure p to the particle number N , the temperature T and the volume V [13]:

$$p = \frac{NkT}{V} \left[1 + \frac{N}{V}F(T) + \left(\frac{N}{V}\right)^2 G(T) + \dots \right], \quad (9)$$

where the correction term $F(T)$ is due to two-particle interactions, the correction term $G(T)$ is due to three-particle interactions and so forth. We will ignore all interactions involving more than two particles. The correction term $F(T)$ is [13]:

$$F(T) = 2\pi \int_0^\infty \lambda(r) r^2 dr = \beta - \frac{\alpha}{kT}, \quad (10)$$

where $\lambda(r)$ is given by:

$$\lambda(r) = 1 - e^{-\frac{U(r)}{kT}}. \quad (11)$$

Then “van der Waals” equations is [13]:

$$p + \left(\frac{N}{V}\right)^2 \alpha = \frac{NkT}{V} \left(1 + \frac{N}{V}\beta\right). \quad (12)$$

In the limit $N\beta/V \rightarrow 0$, this equation reduces to the usual van der Waals equation [13]:

$$\left[p + \left(\frac{N}{V}\right)^2 \alpha\right] \left(1 - \frac{N}{V}\beta\right) = \frac{NkT}{V}. \quad (13)$$

We now assume that the “unusual” particles leave “voids” in the Universe where “normal” particles cannot enter. Thus, the effective space left for the motion of the “normal” component of the gas is reduced by $N\beta$, where β is the “volume” of an “unusual” particle and N is the number of “unusual” particles. We will also pretend that “unusual”

particles are not present and that the potential in which the “normal” particles move is not due to the “unusual” particles, but rather to the two-particle interactions between the “normal” component of the gas. In essence, we “remove” N “unusual” particles out of all particles and we are dealing with a gas of n “normal” particles such that $qQ > 0$. The “van der Waals” equation (12) then becomes:

$$p + \left(\frac{N}{V}\right)^2 \alpha = \frac{nkT}{V} \left(1 + \frac{N}{V} \beta\right), \quad (14)$$

For the potential determined in (7), we have:

$$\lambda(r) = 1 - e^{-\frac{U(r)}{kT}} = \begin{cases} 1, & r < r_0(T), \\ \frac{U(r)}{kT}, & r_0(T) \leq r \leq R_c, \\ 0, & r > R_c. \end{cases} \quad (15)$$

We then get:

$$\beta = 2\pi \int_0^{r_0(T)} r^2 dr = \frac{2\pi}{3} r_0^3(T) = \frac{1}{2} v_0(T), \quad (16)$$

$$\alpha = 2\pi \int_{r_0(T)}^{R_c} U(r) r^2 dr = \pi m M^2 \left(1 - \frac{Q^2}{M^2}\right) [R_c - r_0(T)] \\ + \pi m M \left(1 - \frac{q}{m} \frac{Q}{M}\right) [R_c^2 - r_0^2(T)], \quad (17)$$

where $v_0(T)$ is the “volume” of an “unusual” particle. Note that both α and β depend on the temperature via the particle’s radius $r_0(T)$.

We have shown [1] that for our expansion model, the standard relation between the scale factor of the Universe a and the temperature T holds: $aT = \text{const}$. Let ρ denote the density of the Universe. Then, as the volume V of the Universe is proportional to the third power of a and as $V \sim 1/\rho$, we have $T \sim \rho^{1/3}$. Therefore, $T/V \sim \rho^{4/3}$.

The volume V of the Universe during the van der Waals phase is proportional to the volume $v_0(T)$ of the “unusual” particles times their number N . Using equation (16), namely: $\beta = \frac{1}{2} v_0(T)$, it immediately follows that $N\beta/V$ is, essentially, constant.

Equation (14) is the equation of state for the van der Waals phase of the expanding Universe and can be written: as:

$$p = \eta \rho^{4/3} - \frac{\alpha}{\beta^2}. \quad (18)$$

Here η is some constant. The second term depends on the temperature via α and β and becomes irrelevant towards the end, as $\alpha \rightarrow 0$ when $r_0(T) \rightarrow R_c$ (we have ideal gas treatment beyond this stage). Note also that the correction term $-\alpha/\beta^2$ is positive as α is negative.

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