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# Covariant Relativistic Quantum Mechanics Analysis of a Linearly Accelerated Scalar Particle

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Covariant Relativistic Quantum Mechanics  
Analysis of a Linearly Accelerated Scalar Particle

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Thesis for the award of Ph.D.

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## Abstract

A covariant formalism of Relativistic Quantum Mechanics is demonstrated, through its development and application. The Relativistic Case is shown to follow a similar structure to the established Non-Relativistic formalism. Reasons for preferring the new covariant formalism over the established method are presented.

Solutions to the case of a scalar particle in a one-dimensional field are presented. The Relativistic Energy Eigenfunction is derived. Results are generated from initial Gaussian states via a Green's Function method. A Green's Function for the system is derived and applied. The solution to the Quantum System is shown to follow a scaled version of the classical path.

The equivalent system in the Non-Relativistic case is also analysed. The Energy Eigenfunction for the time-independent Schrödinger Equation is derived. A Green's Function is derived for the time-dependent Schrödinger Equation and solutions to the system are found using initial Gaussian States.

A short demonstration of the established method, that of the Klein-Gordon equation, is provided. Some deficiencies in the Klein-Gordon equation are shown. These deficiencies are shown to not exist in the covariant formalism.

## Declaration

I certify that this thesis which I now submit for examination for the award of PhD is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

This thesis was prepared according to the regulations for postgraduate study by research of the Dublin Institute of Technology and has not been submitted in whole or in part for an award of any other Institute or University.

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## Chapter 1 Introduction

This thesis is concerned with exploring an alternative formulation of Relativistic Quantum Mechanics, as proposed by T. Garavaglia, see [1] and [2], referred to in this work as the Garavaglia Model. The specific case used here is that of a scalar particle undergoing acceleration in a linear potential of the form  $V(z) = kz$  in the Non-Relativistic case and  $e\mathcal{A} = (kz, 0, 0, 0)$  in the Relativistic case. Here  $k$  is some constant. In the conventional theory the Klein-Gordon equation would be the correct method to use. There is, however, a problem with the Klein-Gordon equation, the lack of a positive definite density function. The aim of this new formalism is to establish a Relativistic description of Quantum Mechanics for spin zero – or scalar – particles which is a natural progression from the well established theory of Non-Relativistic Quantum Mechanics. In particular, the Garavaglia Model aims to emulate the success of the Schrödinger equation in describing Non-Relativistic Quantum Mechanics and avoid problems inherent to the established description of Relativistic Quantum Mechanics, the Klein-Gordon equation.

The Schrödinger equation for the case being discussed in this work is:

$$\begin{aligned} \left( \hat{\mathcal{H}} - i \frac{\partial}{\partial t} \right) \Psi(\vec{x}, t) &= 0, \\ \left[ \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 2kz \right) - i \frac{\partial}{\partial t} \right] \Psi(\vec{x}, t) &= 0, \end{aligned} \quad (1.1.1)$$

where  $\Psi(\vec{x}, t)$  is a state which represents some particle (or particles). The Garavaglia Model equivalent, the covariant Schrödinger-type wave equation is:

$$\begin{aligned} \left[ \hat{\mathcal{H}} - i \frac{\partial}{\partial s} \right] \Psi(x^\mu, s) &= 0, \\ \left[ \left( \hat{p} + e\hat{\mathcal{A}} \right)^2 - i \frac{\partial}{\partial s} \right] \Psi(x^\mu, s) &= 0, \\ \left[ \frac{1}{2} \left( -\frac{\partial^2}{\partial t^2} + k^2 z^2 - 2ikz \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - i \frac{\partial}{\partial s} \right] \Psi(x^\mu, s) &= 0. \end{aligned} \quad (1.1.2)$$

where  $x^\mu = (t, \vec{x})$  is the four-position of the particle represented by the state  $\Psi(x^\mu, s)$  and  $s$  is the proper time of the particle. This form of the covariant Schrödinger-type wave equation is derived in §4.4. Proper time is the time as measured in the proper frame, i.e.



the frame of the particle (or particles) being observed. The differences between (1.1.1) and (1.1.2) are due to the treatment of the observer time  $t$  as a variable of the system, it has the same standing as the usual coordinate  $x, y, z$ . The evolution parameter in the Garavaglia Model is the proper time  $s$  which takes the place of the usual time  $t$  in (1.1.1). Also, note that the potential  $e\mathcal{A}$  is a scalar potential, that is the potential function forms the zeroth component of the  $e\mathcal{A}$  four-vector, in the same way that the observer time  $t$  is the zeroth component of the four-position. This leads to the zeroth component of the Hamiltonian operator being expanded as follows:

$$\hat{\mathcal{H}}^0 = (\hat{p}^0 + e\hat{\mathcal{A}}^0)^2 = (\hat{p}^0 + kz)^2 = -\frac{\partial^2}{\partial t^2} + k^2 z^2 - 2ikz \frac{\partial}{\partial t}.$$

Chapter 2 contains a brief outline of the conventional method, i.e. the Klein-Gordon equation. Following this, a discussion of an alternate proposal to generate a proper-time description of Quantum Mechanics is presented. This proposal, the Parameterised Relativistic Dynamics is constructed via the use of a particle independent of the system being considered, the so-called clock-particle. This particle is chosen such that its propagation is well understood, allowing the proper-time for the clock particle to be used as the evolution parameter for the system under consideration. This choice of evolution parameter ensures that it is independent of the space-time coordinates used to describe to system being studied.

In this thesis, the established Non-Relativistic Quantum Mechanics (see Chapter 3) is explored and propagated states which obey the Schrödinger equation are calculated (see §3.4). Similarly, the Garavaglia Model is explored. The derivation of the covariant Schrödinger-type wave equation will be demonstrated (see Chapter 4 and [1] and [2]) and it can be seen that this derivation follows that of the Schrödinger equation in the Non-Relativistic case.

The propagated states are derived using the Green's Function method, (see §3.4 and §4.3, see also §2.4 for a discussion of the Green's function method.). The Green's function method is a very powerful technique – for every system, if a valid Green's function is found, it is possible to find the propagated state for any initial state by integrating the product of the Green's function and the initial state over the initial variables. This property of the Green's function is used in the Path Integral method, a demonstration of which is provided in the Appendix (see §A.1.1).

The properties of the derived propagated states will be studied. It will be demonstrated that these propagated states obey the Schrödinger equation in the Non-Relativistic case (see §3.4 and §A.5) and the covariant Schrödinger-type wave equation in the Relativistic case (see §4.4 and §A.5). A comparison of the quantum system with the classical case, it is shown that the probability density functions associated with the propagated states are maximised on the classical path (see §3.4.2 and §4.5). In the appendix, results important to the Non-Relativistic case are provided. Also, the relevant REDUCE input and output files are provided.

New results presented in this work are:

1. The derivation of the propagated states in the Relativistic case using the Green's function method, with the Green's function as derived in [2]. These states provide the information required to study the propagation of the state undergoing the interaction. From these, probability density functions may be calculated.
2. It is shown §4.4 that these states obey the covariant Schrödinger-type wave equation. So within the Garavaglia Model these propagated states are valid. Also, this means that the method used to derive the states is correct. See also the REDUCE files and their combined output in §A.5.
3. It is demonstrated that these propagated states are maximised on the classical path. See §4.5. This result is important in that it shows that there is a link between the Classical case and the Quantum case.

## Chapter 2 Mathematical Prerequisites

### §2.1 Brief Outline of the Conventional Method: The Klein-Gordon Equation

Here the established method for the treatment of scalar particles, the Klein-Gordon equation, is outlined and a solution to the free particle case is provided. The Klein-Gordon equation may be derived as follows. The total energy of a system is given by the equation:

$$\mathcal{E}^2 = m^2 c^4 + \vec{p}^2 c^2$$

The momentum and energy variables become operators in Relativistic Quantum Mechanics, via the equivalence principle:

$$\mathcal{E} \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \vec{p} \rightarrow -i\hbar \frac{\partial}{\partial \vec{x}} = -i\hbar \hat{\nabla},$$

and letting both sides of the energy-momentum relation act on a wave function we find:

$$\begin{aligned} \left( i\hbar \frac{\partial}{\partial t} \right)^2 \Psi &= \left[ m^2 c^4 + \left( -i\hbar \hat{\nabla} \right)^2 c^2 \right] \Psi, \\ -\hbar^2 \frac{\partial^2}{\partial t^2} \Psi &= \left( m^2 c^4 - \hbar^2 c^2 \hat{\nabla}^2 \right) \Psi. \end{aligned}$$

Using the natural units,  $c = \hbar = 1$  gives:

$$-\frac{\partial^2}{\partial t^2} \Psi = \left( m^2 - \hat{\nabla}^2 \right) \Psi,$$

$$\left( \square^2 - m^2 \right) \Psi(q) = 0.$$

This is the Klein-Gordon equation for the free particle case. A derivation of the Klein-Gordon equation for the accelerated particle being considered in this work is demonstrated in §4.2 and is found to be:

$$\left( \square^2 - m^2 + 2k\hat{p}_0\hat{z} + (k\hat{z})^2 \right) \Psi(q) = 0. \quad (2.1.1)$$

### §2.1.1 Free Particle Case

In the free-particle case the force constant,  $k$ , is set to zero, giving the free-particle Klein-Gordon equation:

$$(\square^2 - m^2) \Psi(q) = 0. \quad (2.1.2)$$

A Green's Function,  $\mathcal{G}(q, q')$  for this case is first derived. By definition, a Green's Function must obey the following:

$$(\square^2 - m^2) \mathcal{G}(q, q') = \delta(q - q'). \quad (2.1.3)$$

In momentum space (2.1.3) becomes:

$$\begin{aligned} (p_\mu p^\mu - m^2) \tilde{\mathcal{G}}(p) &= 1, \\ \tilde{\mathcal{G}}(p) &= \frac{1}{(p_0)^2 - (\vec{p}^2 + m^2)}. \end{aligned}$$

To ensure causality in the system, an imaginary infinitesimal is inserted into the denominator giving:

$$\tilde{\mathcal{G}}(p) = \frac{1}{(p_0)^2 - \mathcal{E}_p^2 + i\epsilon}, \quad (2.1.4)$$

with  $\mathcal{E}_p = +\sqrt{\vec{p}^2 + m^2}$  and  $\epsilon$  is defined to be positive.

The required Green's Function for the coordinate space is given by:

$$\mathcal{G}(q, q') = \int_{-\infty}^{+\infty} \tilde{\mathcal{G}}(p) \frac{e^{ip \cdot (q - q')}}{(2\pi)^4} d^4 p. \quad (2.1.5)$$

This integral has poles at:

$$p_0 = \pm \sqrt{\mathcal{E}_p^2 - i\epsilon},$$

where  $i\epsilon$  raises the negative pole,  $p_0 = -\mathcal{E}_p$ , into the upper half-plane and lowers the positive pole,  $p_0 = +\mathcal{E}_p$ , into the lower half-plane. There are now two contours about the poles. The first contour, about the pole at  $p_0 = +\mathcal{E}_p$  comprises the right half-plane, bound to the left by the line  $p_0 = 0$ , for this contour  $p_0 > 0$ . The second contour, about the pole

at  $p_0 = -\mathcal{E}_p$  comprises the left half-plane, bound above by the line  $p_0 = 0$ , for this contour  $p_0 < 0$ . (2.1.5) can be rewritten as:

$$\mathcal{G}(q, q') = \int_{-\infty}^{+\infty} \frac{e^{i\vec{p}\cdot(\vec{q}-\vec{q}')}}{(2\pi)^3} d^3p \int_{-\infty}^{+\infty} \frac{1}{(p_0)^2 - \mathcal{E}_p^2 + i\epsilon} \frac{e^{-ip_0\cdot(q^0-q'^0)}}{2\pi} dp^0.$$

As stated there are two poles, both are simple. The following result is used below to derive the propagator. If a complex function,  $f$ , has a simple pole at  $z_0$ , i.e. if:

$$f(z) = \frac{A(z)}{B(z)},$$

where  $A$  &  $B$  are analytic at  $z_0$  and  $B$  has a simple zero at  $z_0$  then [3]:

$$\int_C f(z) dz = \int_C \frac{A(z)}{B(z)} dz = 2\pi i \frac{A(z_0)}{B'(z_0)}.$$

The case of interest to this discussion is  $q^0 > q'^0$  with pole at  $p_0 = +\sqrt{\mathcal{E}_p^2 - i\epsilon}$ .

Here  $C$  is a circle<sup>1</sup> which surrounds a single isolated neighbourhood of  $z_0$ . So for this case, using  $z$  to represent  $p_0$  and  $z_0$  the value of  $p_0$  at the pole:

$$\begin{aligned} A(z) &= \frac{e^{-iz\cdot(q^0-q'^0)}}{2\pi}, & A(z_0) &= \frac{\exp[-i\sqrt{\mathcal{E}_p^2 - i\epsilon}\cdot(q^0-q'^0)]}{2\pi}, \\ B(z) &= z^2 - \mathcal{E}_p^2 + i\epsilon, & B'(z) &= 2z, & B'(z_0) &= 2\sqrt{\mathcal{E}_p^2 - i\epsilon}. \end{aligned}$$

Now the limiting case of  $\epsilon$  must be taken:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} A(z_0) &= \frac{e^{-i\mathcal{E}_p(q^0-q'^0)}}{2\pi}, \\ \lim_{\epsilon \rightarrow 0^+} B'(z_0) &= 2\mathcal{E}_p. \end{aligned}$$

So the final form of the integral is:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{1}{(p_0)^2 - \mathcal{E}_p^2 + i\epsilon} \frac{e^{-ip_0\cdot(q^0-q'^0)}}{2\pi} dp^0 &= 2\pi i \frac{e^{-i\mathcal{E}_p(q^0-q'^0)}}{2\pi} \frac{1}{2\mathcal{E}_p}, \\ &= i \frac{e^{-i\mathcal{E}_p(q^0-q'^0)}}{2\mathcal{E}_p} \end{aligned} \tag{2.1.6a}$$

---

<sup>1</sup> Or closed simple curve, i.e. it does not cross itself.

The alternative case, that of  $q^0 < q'^0$  with pole at  $p_0 = -\sqrt{\mathcal{E}_p^2 - i\epsilon}$ , the integral is:

$$\int_{-\infty}^{+\infty} \frac{1}{(p_0)^2 - \mathcal{E}_p^2 + i\epsilon} \frac{e^{-ip_0 \cdot (q^0 - q'^0)}}{2\pi} dp^0 = -i \frac{e^{+i\mathcal{E}_p(q^0 - q'^0)}}{2\mathcal{E}_p} \quad (2.1.6b)$$

Combining the preceding results the  $p^0$  integral is:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{1}{(p_0)^2 - \mathcal{E}_p^2 + i\epsilon} \frac{e^{-ip_0 \cdot (q^0 - q'^0)}}{2\pi} dp^0 &= \theta(q^0 - q'^0) i \frac{e^{-i\mathcal{E}_p(q^0 - q'^0)}}{2\mathcal{E}_p} \\ &\quad - \theta(q'^0 - q^0) i \frac{e^{+i\mathcal{E}_p(q^0 - q'^0)}}{2\mathcal{E}_p} \end{aligned} \quad (2.1.7)$$

This result ensures that causality exists in the system, there is a clear before and after to an event. As can be seen in the preceding equation, past events are separated from future events (if we consider present time to be  $q^0 = 0$ ).

The system can now be restricted to the case of  $q^0 > q'^0$ , in which  $\theta(q^0 - q'^0) = 1$ :

$$\begin{aligned} \Delta_F(q - q') &= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} i \frac{e^{-i\mathcal{E}_p(q^\mu - q'^\mu)}}{2\mathcal{E}_p} d^3p, \\ &= \frac{i}{2(2\pi)^3} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\vec{p}^2 + m^2}} e^{-i\sqrt{\vec{p}^2 + m^2}(q^\mu - q'^\mu)} d^3p \end{aligned} \quad (2.1.8)$$

The Green's function, now represented by  $\Delta_F$ , is known as the Feynman Propagator.

### §2.1.2 Green's Function

By definition, as described later in §2.4, a valid Green's Function,  $\mathcal{G}(q, q')$ , for the Klein-Gordon equation, must satisfy:

$$(\square^2 - m^2) \mathcal{G}(q, q') = \delta(q - q'), \quad (2.1.9)$$

see (2.4.2). The full-form of the Feynman Propagator is used to simplify the calculation:

$$\Delta_F(q - q') = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{1}{p_\mu p^\mu - m^2 + i\epsilon} e^{-ip_\mu \cdot (q^\mu - q'^\mu)} d^4p, \quad (2.1.10)$$

and, after setting  $i\epsilon = 0$ , substituting  $\Delta_F(q - q')$  into (2.1.2) yields:

$$\begin{aligned} (\square^2 - m^2) \Delta_F(q - q') &= \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{p_\mu p^\mu - m^2}{p_\mu p^\mu - m^2} e^{-ip_\mu \cdot (q^\mu - q'^\mu)} d^4p \\ &= \delta(q - q'). \end{aligned}$$

### §2.1.3 Free Particle System

In this section, the Feynman propagator is used to derive a propagated state for the free case using and an initial planewave state:

$$\Psi(q') = \frac{1}{(2\pi)^2} e^{-ip'_\mu q'^\mu}, \quad (2.1.11)$$

where the initial momenta are labelled  $p'$  to differentiate them from the final momenta of the propagated state. To find the propagated state the form of the Feynman propagator to be used is:

$$\Delta_F(q - q') = \int_{-\infty}^{+\infty} \frac{1}{p_\mu p^\mu - m^2} \frac{e^{-ip \cdot (q - q')}}{(2\pi)^4} d^4 p.$$

The propagated state is found as follows:

$$\begin{aligned} \Psi(q) &= \int_{-\infty}^{+\infty} \Delta_F(q - q') \Psi(q') d^4 q', \\ &= \frac{1}{(2\pi)^8} \int_{-\infty}^{+\infty} \frac{1}{p_\mu p^\mu - m^2} e^{-ip_\mu q^\mu} \left[ \int_{-\infty}^{+\infty} e^{i(p-p')q'} d^4 q' \right] d^4 p \\ &= \frac{1}{(2\pi)^6} \int_{-\infty}^{+\infty} \frac{1}{p_\mu p^\mu - m^2} e^{-ip_\mu q^\mu} \left[ (2\pi)^4 \delta^4(p - p') \right] d^4 p \\ &= \frac{1}{(2\pi)^2} \frac{1}{p'_\mu p'^\mu - m^2} e^{-ip'_\mu q^\mu}, \\ \Rightarrow \Psi(q) &= \frac{1}{(2\pi\mathcal{E})^2} e^{-ip'_\mu q^\mu}. \end{aligned}$$

### §2.1.4 Accelerated System

As has been seen, the system for the free particle is quite a simple one. A scalar particle undergoing linear acceleration will now be studied. The Klein-Gordon equation for this case is:

$$\left( \square^2 - m^2 + 2k\hat{p}_0\hat{z} + (k\hat{z})^2 \right) \Psi(q) = 0.$$

The equation governing the propagator for this system is:

$$\left(\square^2 - m^2 + 2k\hat{p}_0\hat{z} + (k\hat{z})^2\right) \mathcal{G}(q, q') = \delta(q - q')$$

which in momentum space is:

$$\left[-k^2 \left(\frac{\partial}{\partial p^3}\right)^2 - 2ikp_0 \frac{\partial}{\partial p^3} + (p_\mu p^\mu - m^2)\right] \tilde{\mathcal{G}}(p) = 1,$$

$$\Rightarrow \left[-k \frac{\partial}{\partial (p^3)} \left(k \frac{\partial}{\partial p^3} + 2ip_0\right) - p_3 p^3 + (p_0)^2 - (p_0)^2 - (p_0)^2 - m^2\right] \tilde{\mathcal{G}}(p) = 1.$$

The method described above for the case of the free particle will not work here as transforming the operators into momentum space will still yield a differential equation. A solution to this case does not exist.

## §2.2 Discussion of the Parameterised Relativistic Dynamics

This section begins with a discussion on the incorporation of proper-time into Relativistic Quantum Mechanics. Parameterized Relativistic Dynamics or the Four Space Formulation has been proposed by Fanchi (see [4] and references therein). Throughout the development of RQM, the aim has been to replicate the established results of Non-Relativistic Quantum Mechanics (NRQM) which have been shown to be consistent with the results of classical mechanics.

The description of a non-relativistic particle is provided by the wave-function,  $\Psi(x, t)$  which obeys the Schrödinger equation:

$$\left(-\frac{\hbar}{2m} \hat{\nabla}^2 + V\right) \Psi(x, t) = -i\hbar \frac{\partial}{\partial t} \Psi(x, t), \quad (2.2.1)$$

where  $V$  is some potential with which the wave-function  $\Psi(x, t)$  interacts. This wave-function is normalisable and has a positive-definite density function:

$$\int_{-\infty}^{+\infty} \Psi^* \Psi d^3x = 1,$$

$$\Psi^* \Psi \geq 0.$$



The Klein-Gordon equation:

$$\left(\frac{\hbar}{i} \frac{\partial}{\partial x_\mu} - \frac{e}{c} A^\mu\right) \cdot \left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{e}{c} A_\mu\right) \Psi = m_0^2 c^2 \Psi, \quad (2.2.2)$$

was proposed as an equivalent description of a relativistic particle: [5], [6] & [7]. The quantity  $A^\mu$  is some four-potential which acts upon the particle – in this case an electromagnetic potential acting upon a particle of charge  $e$ . The Klein-Gordon equation for a free particle is:

$$(\partial_\mu \partial^\mu + m^2) \Psi(x, t) = 0. \quad (2.2.3)$$

Multiplying on the left of (2.2.3) with  $\Psi^*$  and on the left of the complex conjugate of (2.2.3) with  $\Psi(x, t)$  gives the two equations:

$$\Psi^* \partial_\mu \partial^\mu \Psi + m^2 \Psi^* \Psi = 0, \quad (2.2.4a)$$

$$\Psi \partial_\mu \partial^\mu \Psi^* + m^2 \Psi \Psi^* = 0. \quad (2.2.4b)$$

Subtracting (2.2.4b) from (2.2.4a) and noting that  $\Psi \Psi^* = \Psi^* \Psi = |\Psi|^2$  is simply a c-number, the following continuity equation results:

$$\partial_\mu j^\mu = 0, \quad (2.2.5)$$

where the four-vector  $j^\mu$ , defined as  $j^\mu \equiv (\rho, \vec{j})$ , has components:

$$\rho = \Psi^* \left(\frac{\partial}{\partial t} \Psi\right) - \left(\frac{\partial}{\partial t} \Psi^*\right) \Psi, \quad (2.2.6a)$$

$$\vec{j} = \Psi^* (\nabla \Psi) - (\nabla \Psi^*) \Psi. \quad (2.2.6b)$$

The density function,  $\rho$ , arising from this continuity equation is not positive definite – that is, it permits negative probability densities – which “*cast a shadow of doubt on the adequacy*” of the Klein-Gordon equation as a description of relativistic particles, [4]. An attempt was made to side-step this issue and simply accept the emergence of negative probability densities when they arose. This did not work as the resulting description of relativistic particle was incompatible with the established NRQM in the non-relativistic limit.

An alternative description of relativistic particle was provided by Dirac, [8], specifically used to describe the electron. Like the Klein-Gordon equation, the Dirac Equation also

permitted negative probability densities but it was possible to account for these in terms of “holes,” now known to be anti-particles. Dirac’s equation has been used to provide an accurate description of the electron. This theory of “holes” however makes the Dirac theory a many-particle theory and not a single particle theory as desired.

Following the success of the Dirac Equation, attempts were made to modify the Klein-Gordon equation as a field equation, [8]. This attempt did not, however, remove the issue of the non-positive definite density function. For the description of spin-0 relativistic particles, an alternative was needed – the Four Space Formulation (FSF) was proposed, [9], [10], [11] & [12]. In [13], Fock proposed a quantum mechanical proper-time,  $\tau$ , which depended on the four space-time coordinates of the particle via the usual Lorentz transformations from the laboratory frame to the proper frame. This method was an attempt to follow the established methods of Classical Relativistic Mechanics (CRM). This method was unsuccessful and so a modification was made, that the proper-time is actually independent of the space-time coordinates, [14].

In the development of the FSF with independent proper-time, [15], a mass operator arose as part of the formalism. In other work undertaken previously, mass operators were found to be very useful, for example Feynman, [16], defined mass operators to derive the observed spectrum of hadron masses. However, as this mass operator was added in an *ad-hoc* manner a naturally occurring equivalent was desirable.

The conventional theories are deficient as they do not contain proper-time and as such do not provide an adequate description of RQM. An additional issue with the conventional theories is that this lack of proper-time can be linked to the inability of these conventional theories to contain a mass-operator and also to the lack of a density function which depends on time (i.e. to allow particles to decay over time, as is known to happen in nature).

In [9], the advantages of the FSF are demonstrated by applying the formalism to the case of a relativistic scalar particle. In the conventional RQM, only free particle systems may be described using a single-particle theory. The description of an interacting relativistic particle (e.g. a charged particle interacting with an electromagnetic field) can only be provided in terms of a many-body theory. The FSF aims to rectify this deficiency in the conventional theory. The development of the FSF is outlined and some well known problems in RQM are analysed from the point of view of the FSF. A generalised Schrödinger Equation (GSE) is derived with an associated continuity-type equation. These equations

incorporate a fifth component,  $\tau$ , which is identified as the quantum mechanical proper-time, analogous to the well defined classical proper-time.

The FSF contains four important properties: (a) a relativistic covariant proper-time; (b) a positive definite probability density function; (c) a scalar product defined in  $L^2(\vec{x}, ct)$ , where  $L^2(\vec{x}, ct)$  is that space which contains functions which are Lebesgue square-integrable. (A function, real or complex, is square integrable over an interval if the integral of its absolute value squared is finite in the interval – valid wave functions in QM are square integrable.); (d) an eleven parameter symmetry group. To simplify the discussion while still demonstrating the core properties of the FSF, the case being considered has been restricted to that of spin-0 particles – it is stated that the FSF can be used to study relativistic particles with non-zero spin also.

The path to the derivation of the GSE is via a definition of a conditional probability density function  $\rho(\vec{x}, ct|\tau)$  where  $(\vec{x}, ct)$  are the regular space-time coordinates and  $\tau$  is the quantum mechanical proper-time. The density function is normalised over all space-time in which the particle may exist (i.e. the space  $D$  where  $\rho$  may be non-zero):

$$\int_D \rho(\vec{x}, ct|\tau) d^4x = 1; \quad d^4x = dx^0 dx^1 dx^2 dx^3.$$

The density function must disappear as  $|x^\mu| \rightarrow \infty$  and it must obey the continuity equation:

$$\frac{\partial}{\partial \tau} \rho(\vec{x}, ct|\tau) + \frac{\partial}{\partial x^\mu} [\rho(\vec{x}, ct|\tau) V^\mu] = 0, \quad (2.2.7)$$

where  $V^\mu$  is the velocity-field of the particle under consideration. This probability density function is said to be a conditional probability density, where the quantity  $\tau$  provides the conditions. This  $\tau$  is an invariant parameter used as an index for the variation of the dynamical variables  $x^\mu$ .

To determine the meaning of  $V^\mu$ , the expectation value of the space-time coordinates is considered:

$$\langle x^\mu \rangle = \int_D x^\mu \rho d^4x, \quad (2.2.8)$$

$$\begin{aligned}
\frac{d}{d\tau} \langle x^\mu \rangle &= \frac{d}{d\tau} \int_D x^\mu \rho d^4x, \\
&= \int_D \frac{\partial}{\partial \tau} (x^\mu \rho) d^4x, \\
&= \int_D \left[ \left( \frac{\partial}{\partial \tau} x^\mu \right) \rho + x^\mu \left( \frac{\partial}{\partial \tau} \rho \right) \right] d^4x, \tag{2.2.9a}
\end{aligned}$$

$$\begin{aligned}
&= \int_D V^\mu \rho dx - \int_D x^\mu \frac{\partial}{\partial x^\mu} (\rho V^\mu) d^4x, \tag{2.2.9b} \\
&= \langle V^\mu \rangle.
\end{aligned}$$

The continuity equation, (2.2.7), was used in (2.2.9a). The second integral in (2.1.9b) is zero due to boundary conditions (the probability density tends to zero as  $(\vec{x}, ct)$  tends to infinity). This shows that the quantity  $V^\mu$  can be identified as the proper-velocity of the particle. Here  $D$  is the domain in which the particle is localised.

The wave-function for the particle is written as:

$$\psi(\vec{x}, ct, \tau) = \rho(\vec{x}, ct | \tau)^{\frac{1}{2}} \exp [i\phi(\vec{x}, ct, \tau)],$$

where  $\phi$  is some unknown scalar function. Through gauge transformations on both the proper-velocity  $V^\mu$  and potential  $A^\mu$ , the continuity equation written above may be written as:

$$\frac{\partial}{\partial \tau} (\psi^* \psi) + \frac{\partial}{\partial x^\mu} \left[ \psi^* \left( \frac{\hbar}{m} \frac{\partial \phi}{\partial x^\mu} + eA^\mu \right) \psi \right] = 0.$$

From the definition of the wavefunction above, the following equation may be inferred:

$$\frac{\partial \phi}{\partial x^\mu} = -\frac{i}{2\rho} \left( \psi^* \frac{\partial \psi}{\partial} - \psi \frac{\partial \psi^*}{\partial} \right)$$

Substituting this into the preceding equation gives:

$$\frac{\partial}{\partial \tau} (\psi^* \psi) + \frac{\partial}{\partial x^\mu} \left[ -\frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial} - \psi \frac{\partial \psi^*}{\partial} \right) + eA^\mu \psi^* \psi \right] = 0,$$

which can be rewritten as:

$$\psi^* F = F^* = \left( \psi^* F \right)^*,$$

where:

$$F = i\hbar \frac{\partial \psi}{\partial \tau} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_\mu \partial x^\mu} - \frac{i\hbar}{2mc} \left( \frac{\partial A^\mu \psi}{\partial x^\mu} + A^\mu \frac{\partial \psi}{\partial x^\mu} \right).$$

By definition, the product  $\psi^* F$  must be real. To ensure this,  $F$  may be set to have the form  $U\psi$  where  $U$  is an arbitrary, real scalar. Letting  $U = e^2 A^\mu A_\mu / 2\bar{m}c^2$  and setting  $F = U\psi$  gives the following equation, see [9]:

$$i\hbar \frac{\partial \psi}{\partial \tau} = \frac{1}{2\bar{m}} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_\mu} - \frac{e}{c} A^\mu \right)^2 \psi = \hat{p}^\mu \hat{p}_\mu \frac{\psi}{2\bar{m}}. \quad (2.2.10)$$

On identifying  $A^\mu$  as a four-potential,  $e$  as the electric charge,  $\bar{m}$  as a constant with units of mass and  $c$  as the speed of light in vacuum, the above equation becomes a generalised form of the Klein-Gordon equation.

Using this equation, the proper-time derivative of the coordinate expectation value may be calculated:

$$\bar{m} \frac{d\langle x^\mu \rangle}{d\tau} = \int_{-\infty}^{+\infty} \psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{e}{c} A^\mu \right) \psi d^4x = \langle p^\mu \rangle,$$

which defines the expectation value of the four-momentum of the particle.

The method used to construct this equation can be altered to allow the study of non-electromagnetic potentials and particles of non-zero spin. This equation is now similar in form to the Schrödinger equation of the non-relativistic theory.

In the FSF, a relativistic particle may be described in terms of a superposition of mass states. This is analogous to the use of superposition of energy states in the non-relativistic theory. As the operator  $\hat{p}^\mu \hat{p}_\mu$  is hermitian, as  $\hat{p}^\mu$ , it follows that the above wave equation can be written as:

$$i\hbar \frac{\partial \psi}{\partial \tau} = \hat{\mathcal{H}}\psi; \quad \hat{\mathcal{H}} = \frac{1}{2\bar{m}} \hat{p}^\mu \hat{p}_\mu.$$

There exist some set of wave-functions  $\psi_q$  such that:

$$\int_{-\infty}^{+\infty} \psi_{q'}^* \psi_q d^4x = \delta_{q'q}; \quad \hat{p}^\mu \hat{p}_\mu \psi_q = q^2 \psi_q \quad (2.2.11)$$

The quantity  $q$  is the square of momentum with magnitude defined as  $m_0^2 c^2$ , where  $m_0$  is the rest mass of the particle and  $c$  the speed of light in a vacuum. This definition of the magnitude of  $q$  and hence  $\hat{p}^\mu \hat{p}_\mu$  builds on work of Cooke [14] and Feynman *et.al.* [17]. The point is made that the Hamiltonian above remains hermitian no matter the strength of the potential field. This overcomes a problem with other theories, for instance in the Two-Component Formalism [18] the Hamiltonian of the Klein-Gordon equation is hermitian only for slowly varying, weak potential fields. The above equations are very similar to

the Schrödinger equation in the non-relativistic theory allowing an analogous method to be used. The general solution to the wave equation is given by a superposition of the eigenfunctions [9]:

$$\psi(\vec{x} ct, \tau) = \sum_q A(q) \psi_q(\vec{x}, ct) \exp\left(\frac{iq\tau}{2m\hbar}\right) \quad (2.2.12)$$

where the sum over the eigenvalues  $q$  would become an integral where appropriate. As there have been no restrictions placed on the value of  $q$ , both positive and negative values are permitted. Negative values of  $q$  correspond to tachyons - particles which travel faster than light - whereas positive values of  $q$  correspond to tardyons - particles which travel at less than the speed of light. This sum over eigenfunctions may be identified as a sum over mass states, which is a similar concept to a sum of energy states used in non-relativistic quantum mechanics.

A notable problem of conventional relativistic quantum theories, such as the Klein-Gordon equation, is the Klein Paradox. Consider a particle with energy  $\omega_1$ , incident from the left (i.e. originates at  $x = -\infty$ ) onto a step potential at  $x = 0$ . The step potential is  $A = (A^0(x), 0, 0, 0)$  where:

$$A^0(x) = \begin{cases} 0, & x < 0, \\ \alpha, & x > 0. \end{cases}$$

Should this potential be sufficiently strong - i.e. the energy of the barrier is greater than the energy of the incident particle,  $\alpha > \omega_1$  - then the particle is reflected from the barrier with a reflection coefficient greater than one (in such a case, there are three coefficients or magnitudes - connected to probability density, they may not exceed one - these are the incident, reflection and transmission coefficients). That is, more particles are reflected by the barrier than were actually incident. This creation of extra particles has not been detected experimentally and therefore must be considered false. The FSF avoids this issue. The system is set up in the usual way, with the wave-function and its first derivative are continuous at the barrier. Four solutions are found, all of which have a reflection constant which is not larger than one. In addition, the sum of the reflection and transmission coefficients is one in each of the four solutions (see Table 1 in [9]). One case of the four works only for a value of the barriers potential energy to be of the order  $2mc^2$ . An experiment to explore this case would serve as a test of the FSF to determine its accuracy. In the case of particles being transmitted through the barrier, these are found to be particles which are propagating backwards in time. These particles are identified to be the anti-particle of the incident particle.

As discussed in [4], the definition of the expectation value in conventional RQM is incompatible with that of the expectation value of NRQM. This point is expanded upon here. In the FSF, a joint density function is written as:

$$\rho(\vec{x}, ct | \tau) = \bar{\rho}(t | \tau) \rho_c(\vec{x} | ct, \tau), \quad (2.2.13)$$

that is the product of a marginal density function  $\bar{\rho}(t | \tau)$  and a conditional density function  $\rho_c(\vec{x}, ct | \tau)$ . The expectation of an arbitrary function  $g(\vec{x}, t)$  is:

$$\langle g \rangle_{FSF} = \int \bar{\rho}(t | \tau) \left[ \int g(\vec{x}, t) \rho_c(\vec{x}, ct, | \tau) d^3x \right] d(ct).$$

To demonstrate this point, we consider the specific case of a function,  $f(\vec{x})$ , independent of time and a wave-function  $\psi_q$  which has an oscillatory time dependence  $\psi_q \sim \exp[i\omega t] / \sqrt{(2T)}$ . The conventional expectation value of  $f$ , using the definition of Feshbach and Villars, [18], is:

$$\langle f \rangle_{con} = \frac{\int_{-\infty}^{+\infty} \psi_q^* \psi_q (\hbar\omega - eA^0) f(\vec{x}) d^3x}{\int_{-\infty}^{+\infty} \psi_q^* \psi_q d^3x},$$

and the analogous expectation value in the FSF is:

$$\langle f \rangle_{FSF} = \frac{\lim_{T \rightarrow \infty} \int_{-T}^T \int_{-\infty}^{+\infty} f(\vec{x}) \psi_q^* \psi_q d^3x d(ct)}{\lim_{T \rightarrow \infty} \int_{-T}^T \int_{-\infty}^{+\infty} \psi_q^* \psi_q d^3x d(ct)}.$$

Here  $\psi_q$  is the solution to the differential equation (2.2.10). These definitions coincide in cases where  $A^0$  or  $f$  are independent of the spatial coordinates but will differ otherwise. In the non-relativistic limit, the FSF definition becomes the non-relativistic expectation value, however this is not always true of the conventional definition. That the conventional theory has this property is undesired and one which FSF can be used to rectify.

As a demonstration of the emergence of a mass distribution, an experiment is described. The experiment considered is that of a wave-packet in a particle accelerator. The system is constructed in such a way that the type of particle is restricted to a specific value of mass. This is referred to as “*preparing a pure state*”. The method used to prepare this state exploits the equation of the particles moving along an circular path in a magnetic field:

$$m_0^2 = \epsilon_c^2 - e^2 B^2 a^2,$$

where  $m_0$  is the rest mass of the particle,  $\epsilon_c$  is the classical energy of the particle,  $e$  is the charge on the particle,  $B$  is the magnitude of the magnetic field and  $a$  is the radius of the circular path. By placing collectors at all radii except  $a$  removes all particles with a rest mass not equal to  $m_0$ . The particle (or wave-packet) obeys the generalised form of the Klein-Gordon equation:

$$i\hbar \frac{\partial \psi(x, \tau)}{\partial \tau} = \frac{1}{2\bar{m}} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_\mu} - \frac{e}{c} A^\mu \right) \cdot \left( \frac{\hbar}{i} \frac{\partial}{\partial x^\mu} - \frac{e}{c} A^\mu \right) \psi(x, \tau)$$

where the four-potential is defined as  $A = (0, -yB, 0, 0)$ . Using the substitution:

$$\psi(x, \tau) = \phi(x) \exp \left( -\frac{iq\tau}{2\bar{m}\hbar} \right),$$

reduces the equation to one of form:

$$p^\mu p_\mu \phi(x) = q\phi(x).$$

The solutions are:

$$\phi(x) = \exp [i(k_x x + k_z z - \epsilon t)] P(y), \quad (2.2.14)$$

where:

$$P(\xi) = \exp \left( \frac{i\xi^2}{2} \right) g_j(i\alpha, -i\xi^2), \quad j = 1, 2$$

$$\xi^2 = i|eB| \left( y + \frac{k_x}{|eB|} \right)^2,$$

$$\alpha = -i \frac{(m^2 - \epsilon^2)}{|eB|}.$$

Note here that the variable  $\xi$  is a complex number, which can be seen from the definition of  $\xi^2$  as an imaginary number. Where  $\xi$  appears in the definition of  $P(\xi)$  it is always in the form  $i\xi^2$  which is a real number.  $P(\xi)$  maps to real numbers. The  $g_j$  are confluent hypergeometric functions, see [19]:

$$g_1(b, \tau') = F \left( \frac{1}{4}(1+b), \frac{1}{2}, \tau' \right),$$

$$g_2(b, \tau') = t^{\frac{1}{2}} F \left( \frac{1}{4}(1+b) + \frac{1}{2}, \frac{3}{2}, \tau' \right).$$

The eigenvalue relation for this  $q$  is:

$$q = m_0^2 = \epsilon^2 - (2n+1)|e|B \quad n = 0, 1, 2, \dots$$



This equation gives a mass spectrum, for a fixed energy  $\epsilon$  there is a discrete set of value of  $m_0$  depending upon  $n$ . It is then shown, via perturbation theory and the completeness relation for quantum mechanics, that such analysis is valid for arbitrary potential fields. This includes potentials for which the Klein-Gordon equation cannot be solved.

We now follow the procedure in [20] of building a Generalised Quantum Field Theory (GQFT) upon the FSF discussed above, with the aim of establishing the FSF as a viable alternative to the Klein-Gordon equation.

It will be shown that the proposed GQFT contains within it the Lagrangian Quantum Field Theory (LQFT) proposed by Schwinger, [21], and that the GQFT closely follows the equivalent work in the classical case, Classical Field Theory (CFT).

In LQFT, the action operator is defined as an integral of a function of field operators  $\psi_\alpha(x)$  and their first derivatives over an infinite four-volume bounded by space-like surfaces:

$$W_s = \int_{R_s} \mathcal{L}_s(\psi_\alpha(x), \partial_\mu \psi_\alpha(x)) dx,$$

with  $dx = dx^1 dx^2 dx^3 dx^4$  and  $\partial_\mu = \partial/\partial x^\mu$ . This action,  $W_s$  is varied and set equal to the difference between covariant transformations at each surface bounding the four-volume.

In the classical theory, the action is defined as:

$$I_{cl} = \int_{\tau_{cl1}}^{\tau_{cl2}} L_{cl} d\tau,$$

where  $\tau$  is the proper time and the system being studied starts at  $\tau = \tau_{cl1}$  and ends at  $\tau = \tau_{cl2}$ . This action is varied and, according to Hamilton's Principle,  $\delta I_{cl} = 0$ , where  $\delta I$  denotes that a variation was applied to the action  $I$ . A variation to the action is actually the manipulation of the path by which a particle moves from the initial point to the final point, end points of the paths are fixed but intermediate points upon the path may change.

Another motivation given for the development of a GQFT is the lack of a mass operator in the LQFT. As mentioned above in the discussion, mass operators have been demonstrated to be very useful in modern theories but these tend to be defined rather than derived in established theories in order to obtain a desired result. It is much more desirable to have a theory in which a mass operator occurs naturally.

The development of this system begins by considering a system evolving through proper-time,  $\tau$ , where this proper-time is independent of the four usual space-time coordinates. This follows on from work begun by Cooke in the late 1960s, [14], and expanded

upon by other authors in [22], [23] and [24]. This choice of proper-time provides the theory with an invariant scalar parameter which can be used to index the evolution of a system.

The development of the system, described in [20], is as follows. A physical system – a particle moving in some potential – evolves through proper-time, running from  $\tau = \tau_1$  to  $\tau = \tau_2$ . A set of commuting operators at the beginning and end points,  $\xi_1$  and  $\xi_2$  respectively, with their associated state vectors,  $\phi(\xi_1)$  and  $\phi(\xi_2)$ , are used to describe the particle. The transformation matrix from the start to the end of the evolution can now be written as:

$$\langle \xi_1 | \xi_2 \rangle = \langle \phi(\xi_1) | \phi(\xi_2) \rangle.$$

This transformation matrix defines one path along which the system may evolve. Other paths may be found by canonically varying  $\langle \xi_1 | \xi_2 \rangle$ , through the application of an infinitesimal canonical transformation, defined as:

$$\delta\psi_\alpha = i[G, \psi_\alpha],$$

where  $\psi$  is some operator and  $G$  is the generator of the transformation (see [20] Appendix A). The use of canonical transformations ensure that the eigenvalues of the original operators are the same as the transformed operators. The transformed operator at  $\tau_2$  is  $\xi_2 + \delta\xi_2$ , where  $\delta\xi_2 = i[G, \xi_2]$  and the transformed states become  $\phi(\xi_2) \rightarrow \phi(\xi_2) + \delta\phi(\xi_2)$ , where  $\delta\phi(\xi_2) = iG(\tau_2)\phi(\xi_2)$ . The corresponding transformation applied at  $\tau_1$  yields the transformed operator  $\xi_1 + \delta\xi_1$  where  $\delta\xi_1 = i[G, \xi_1]$  and transformed states are  $\phi(\xi_1) + \delta\phi(\xi_1)$ , where  $\delta\phi(\xi_1) = iG(\tau_1)\phi(\xi_1)$ . The new transformation matrix corresponding to the canonically varied system is  $\langle \xi_1 | \xi_2 \rangle + \delta\langle \xi_1 | \xi_2 \rangle$  where  $\delta\langle \xi_1 | \xi_2 \rangle = i\langle \xi_1 | \delta A | \xi_2 \rangle$ , and  $\delta A = G(\tau_2) - G(\tau_1)$ .

The next step taken is to assume that the quantity  $A$  is the action of the system, i.e.  $A = \int_1^2 L d\tau$ , where  $L$  is the Lagrangian corresponding to the system. So the variation in the system can be expressed as the difference in the generator of the variation at the start and end points of the system's evolution. This is similar in structure to the LQFT developed originally by Schwinger, [21]. This property, written as::

$$\delta A = G(\tau_2) - G(\tau_1)$$

is a postulate of the system.

This is now applied to find the generalised field equation. For this case, the action is defined as:

$$A = \int_{s_1}^{s_2} \int_D \mathcal{L} dx ds,$$

where  $\mathcal{L}$  is the Lagrangian Density which has the form  $\mathcal{L} = \mathcal{L}(q, \partial_\mu q, \dot{q}, s)$ . From Hamilton's Principle:

$$\delta A = \int_{s_1}^{s_2} \int_R \delta \mathcal{L} dx ds = 0. \quad (2.2.15)$$

This is varied as follows:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \delta(\partial_\mu q) + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathcal{L}}{\partial s} \delta s. \quad (2.2.16)$$

By assuming that the parameter end points are fixed, integrating by parts over  $\delta s$  gives:

$$\int_{s_1}^{s_2} \int_R \frac{\partial \mathcal{L}}{\partial s} \delta s dx ds = 0. \quad (2.2.17)$$

Using integrating over parts on the term  $\delta \dot{q}$  gives:

$$\begin{aligned} \int_{s_1}^{s_2} \int_R \frac{\partial}{\partial \dot{q}} \delta \left( \frac{\partial q}{\partial s} \right) dx ds &= \int_{s_1}^{s_2} \int_R \left( \frac{d}{ds} (\delta q) \right) dx ds, \\ &= - \int_{s_1}^{s_2} \int_R \left[ \frac{d}{ds} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right] \delta q dx ds, \end{aligned} \quad (2.2.18)$$

where  $\delta q$  vanishes due to the assumption of fixed parameter end points. The  $\partial_\mu q$  term is simplified:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \delta(\partial_\mu q) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \frac{\partial(\delta q)}{\partial x^\mu}, \\ &= \frac{\partial}{\partial x^\mu} [\pi_\mu \delta q] - \left( \frac{\partial \pi^\mu}{\partial x^\mu} \right) \delta q, \end{aligned} \quad (2.2.19)$$

where:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu q)}. \quad (2.2.20)$$

Using (2.2.16), (2.2.17), (2.2.18), (2.2.19) and (2.2.20), the variation of the action becomes:

$$\begin{aligned} \delta A &= \int_{s_1}^{s_2} \int_R \left\{ \left[ \frac{\partial \mathcal{L}}{\partial q} - \partial_\mu \pi^\mu - \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right] \delta q \right\} dx d\tau \\ &\quad + \int_{s_1}^{s_2} \int_R \frac{\partial}{\partial x^\mu} (\pi^\mu \delta q) dx d\tau \end{aligned} \quad (2.2.21)$$

The second integral in (2.2.21) vanishes after the application of Gauss's Theorem and integration over the whole space. The first integral must then be zero for the postulate to hold. In the derivation, the region of integration  $R$  and the quantity  $\delta_0\psi_\alpha$  are arbitrary. This means that for the integrand to be zero the quantity within the square bracket must be zero, that is:

$$\frac{\partial\mathcal{L}}{\partial q} - \frac{d}{d\tau} \frac{\partial\mathcal{L}}{\partial\dot{q}} = \partial_\mu\pi^\mu.$$

This equations is the parameterised Euler-Lagrange equation. Note the above derivation is taken from [25], Chapter 3. It should be noted here that the generalised Euler-Lagrange equation in the PRD is not the same as the typical Euler-Lagrange, there is an extra term. In the case of field theory, the Euler-Lagrange of the standard theory is:

$$\frac{\partial\mathcal{L}}{\partial\psi_\mu} - \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \right) = 0,$$

whereas the equivalent in the PRD [20] is:

$$\frac{\partial\mathcal{L}}{\partial\psi_\mu} - \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \right) - \frac{d}{d\tau} \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = 0.$$

This extra term is due to there being five mathematically independent variables, rather than the four in the standard theories. This means that when calculating the differential of a function that depends on the five independent variables, there will be a term for each of the independent variables, including a term that is dependent on the derivative with respect to the evolution parameter. The mathematical independence of variables is the source of additional terms [26].

The LQFT can be demonstrated to be a special case of GQFT by replicating the results of LQFT or, as presented in [20], to show that Schwinger's foundation of LQFT is contained within GQFT. Taking the property of LQFT, that the field operators are independent of the proper-time,  $\tau$ , the variational principle may be rewritten as:

$$\delta\mathcal{L}_S = \int_{\sigma_2} \mathcal{J}^\mu d\sigma_\mu - \int_{\sigma_1} \mathcal{J}^\mu d\sigma_\mu.$$

This in turn can be written as:

$$\delta\mathcal{L}_S = F_S[\sigma_2] - F_S[\sigma_1].$$

where the functions  $F$  are the generators of the observables, on the surfaces  $\sigma_i$ , defined as:

$$F_S \equiv \int_{\sigma} \mathcal{J}^{\mu} d\sigma_{\mu}.$$

Here  $\sigma_1$  is the surface on which the initial position of the particle exists and  $\sigma_2$  is the final. This corresponds to the relations established by Schwinger in [21]. The quantity  $\mathcal{J}^{\mu}$  is defined in [20] as:

$$\mathcal{J}^{\mu} = \pi^{\alpha\mu} \delta\psi_{\alpha} + \partial^{\mu}\psi_{\alpha}(x, \tau)\delta x_{\nu} + \dot{\psi}_{\alpha}(x, \tau)\delta\tau.$$

FSF transformations are generated by three functions:  $J^{\mu\rho}$  – dependent upon the angular momentum tensor  $M^{\mu\nu\rho}$ ;  $P^{\nu}$  – dependent upon the energy-momentum tensor  $t^{\mu\nu}$ ; and  $H_{op}$  – dependent upon the Hamiltonian  $H^{\mu}$ . The quantities  $J$  and  $P$  are understood by restricting the system to the LQFT (doing so removes any dependence upon  $\tau$ ), they relate to a rotation and spatial transformation respectively. The quantity  $H_{op}$  corresponds to a transformation in proper-time. According to [20],  $H_{op}$  has eigenvalues the inner product of the canonical momentum four-vector for spinless particles. In the case of a non-interacting spinless particle,  $H_{op}$  has the same eigenvalues of  $\hat{p}^{\mu}\hat{p}_{\mu}$  which identifies  $H_{op}$  as a mass operator which means that  $H^{\mu}$  is a mass flux density due to the definition:

$$\begin{aligned} H_{op} &\equiv \int_1^2 \oint_S H^{\mu} dS_{\mu} d\tau, \\ &= \int_1^2 \int_R \partial_{\mu} H^{\mu} dx d\tau. \end{aligned}$$

The function  $H^{\mu}$  is defined as:

$$H^{\mu} \equiv -\pi^{\alpha\mu} \dot{\psi}_{\alpha}.$$

Note the second form of the equation is a result of Stoke's Theorem. This emergence of the mass operator is a desired result.  $H_{op}$  is shown to be proportional to the rest mass of the lowest-energy state  $|1\rangle$  and may be used to develop a multi-particle state through repeated application.

Requiring that the variation of action is zero, i.e.  $\delta A = 0$ , three conservation laws are evident. The first two are the conservation of energy-momentum:

$$\partial_{\mu} \bar{t}^{\mu\nu} = 0,$$

and the conservation of angular momentum:

$$\partial_\mu M^{\mu\nu\rho} = 0.$$

The third law of conservation details the conservation of mass flux density:

$$\partial_\mu H^\mu = 0,$$

and this is a new result of the FSF. This property may be employed to show that the rest mass of a free scalar particle is conserved.

The discussion so far has concerned the development of the formalism and some key properties. The rest of this discussion will focus upon the methodology of the formalism. The method explored in [27] to provide a correction to Relativistic Quantum Mechanics (RQM) is the Parameterized Relativistic Dynamics (PRD) which is itself a further refinement of the Four Space Formulation discussed above. In the accepted theory of Special and General Relativity, time is considered to be a coordinate of the system, much like – though with some special properties – position in space. This allows us to study the mechanics of relativistic systems using four-vectors, where the position of a particle/object in this four-space given by a combination of the typical three dimensional position vector and a temporal component. In Special Relativity, there is no restriction, on a mathematical basis, on time-travel. There is no such property as an Arrow of Time. This idea is at odds with the Newtonian view which is more in keeping with our common-sense view of the world.

The central idea of PRD is to provide an invariant scalar quantity of time,  $s$ , which is used to measure movement through all other space-time coordinates. Essentially, we would have a Newtonian-type time plus the relativistic space-time together in one theory. This quantity  $s$  is independent of the four space-time coordinates, i.e. it does not depend upon the system under consideration. This means that this time,  $s$ , may be used to study two or more reference frames at once. The choice between a proper time or laboratory time is no longer required. Before discussing in more detail the concept of time in the PRD, the classical equivalent to the PRD will be discussed and then the probabilistic form of the PRD and what this means for the concept of mass.

In classical mechanics there are two equivalent formalisms, the Lagrangian formalism and the Hamiltonian formalism, Quantum Mechanics for the most part closely follows the Hamiltonian formalism. The equivalent of the Lagrangian method in QM is the Path

Integral method as originally proposed by Feynman. Hamilton's principle states that on the path which a classical particle follows, the action of the system is an extremum, that is it is maximised or minimised. The action is defined as the integral of the Lagrangian taken over the path followed by the particle. The integral is a time integral running from the initial time to the final time. For the case of PRD, which incorporates an independent evolution parameter  $s$  which runs from  $s = s_1$  to  $s = s_2$  this takes the form:

$$A = \int_{s_1}^{s_2} \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, s) ds,$$

where there are  $n$  generalised coordinates and  $\dot{q} = dq/ds$  (for this discussion,  $n = \mu = 4$ ). Hamilton's principle as written mathematically as  $\delta A = 0$  or  $\int_{s_1}^{s_2} \delta \mathcal{L}(q^\mu, \dot{q}^\mu, s) = 0$ . Under variation the change in the Lagrangian is:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^\mu} \delta q^\mu + \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \delta \dot{q}^\mu.$$

Substituting this into the integral above and integrating by parts yields:

$$\delta A = \int_{s_1}^{s_2} \left( \frac{\partial \mathcal{L}}{\partial q^\mu} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} \right) \delta q^\mu ds = 0.$$

As the actual variation (the quantity  $\delta q^\mu$ ) is arbitrary, the term in the square brackets must be zero for all  $\mu = 0, 1, 2, 3$ . The resulting equations:

$$\frac{\partial \mathcal{L}}{\partial q^\mu} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} = 0$$

are the Euler-Lagrange equations of motion. The Hamiltonian is closely related to the Lagrangian and is defined as follows:

$$H(q^\mu, p_\mu, s) = \dot{q}^\mu p_\mu - \mathcal{L}(q^\mu, \dot{q}^\mu, s).$$

Note that the Einstein summation convention applies here. The following differential relations exist between the Hamiltonian and the Lagrangian:

$$\begin{aligned} \frac{\partial H}{\partial s} &= -\frac{\partial \mathcal{L}}{\partial s}, \\ p_\mu &= \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}, \\ \dot{q}^\mu &= \frac{\partial H}{\partial p_\mu}, \\ \frac{\partial H}{\partial q^\mu} &= -\frac{\partial \mathcal{L}}{\partial q^\mu}. \end{aligned}$$

These relations lead to Hamilton's equations:

$$\begin{aligned}\dot{p}_\mu &= -\frac{\partial H}{\partial q^\mu}, \\ \dot{q}^\mu &= \frac{\partial H}{\partial p_\mu}.\end{aligned}$$

Before moving on, an important function is the Poisson bracket. Let  $F$  be some function of  $q$ ,  $p$  and  $s$ . Its time differential is:

$$\frac{dF}{ds} = \frac{\partial F}{\partial s} + \frac{\partial F}{\partial q^\mu} \dot{q}^\mu + \frac{\partial F}{\partial p_\mu} \dot{p}_\mu.$$

Using Hamilton's equations, this can be rewritten as:

$$\frac{dF}{ds} = \frac{\partial F}{\partial s} + \{F, H\},$$

where the Poisson bracket,  $\{F, H\}$ , is defined as:

$$\{F, H\} = \frac{\partial F}{\partial q^\mu} \frac{\partial H}{\partial p_\mu} - \frac{\partial H}{\partial q^\mu} \frac{\partial F}{\partial p_\mu}.$$

If the function  $F$  has no dependence upon  $s$ , i.e.  $dF/ds = 0$ , and  $\{F, H\} = 0$ , then the variable which  $F$  represents is conserved with respect to  $s$ . The Poisson bracket is used to evaluate properties of conservation of physical variables.

In Relativity, the mass of a particle is related to its physical condition (momentum, presence of an external potential field, etc.). This is shown using the example of a free particle of mass  $m$ . The Hamiltonian is defined as:

$$\mathcal{H} = \frac{g^{\mu\nu}}{2m} p_\mu p_\nu,$$

where  $g^{\mu\nu}$  is the metric for the space in which the particle exists. The simplest example of such a metric is that of the Minkowski space  $g^{00} = -1$ ,  $g^{11} = g^{22} = g^{33} = 1$  and  $g^{\mu\nu} = 0$ ,  $\mu \neq \nu$ . The coordinate and momenta variables are:

$$\begin{aligned}q^\mu &= (q^0, q^1, q^2, q^3) = (ct, \vec{q}), \\ p_\mu &= (p_0, p_1, p_2, p_3) = \left(\frac{E}{c}, -\vec{p}\right),\end{aligned}$$



Using Hamilton's equations above, the following is found:

$$\begin{aligned}\dot{q}^\mu &= \frac{p^\mu}{m}, \\ \dot{p}_\mu &= 0.\end{aligned}$$

These equations show that the particle moves with constant velocity/momentum, i.e. it is a free particle. The Lagrangian is found to be, using the definition used for the Hamiltonian above:

$$\mathcal{L} = \frac{m}{2} \dot{q}^\nu \dot{q}_\nu.$$

As the particle is not accelerating, the equation of motion is quite simple:

$$q^\mu = q_0^\mu + \frac{p_0^\mu s}{m},$$

where  $q_0^\mu$  and  $p_0^\mu$  are the initial position and momentum of the particle respectively. Varying the equation of motion and taking the product gives of these variations gives:

$$\delta q^\mu \delta q_\mu = \frac{p_0^\mu p_{0\mu}}{m^2} (\delta s)^2,$$

and using the initial condition on the particle that:

$$M^2 = \frac{p_0^\mu p_{0\mu}}{c^2},$$

gives:

$$\delta q^\mu \delta q_\mu = c^2 (\delta s)^2.$$

This is the typical relation between four-position of the particle and the proper-time where this proper time is that used to define the action earlier. Note that the equation above for  $M^2$  shows that the mass is a constant of the system.

An equivalent analysis of the motion of a charged particle in an electromagnetic field is also included. The Hamiltonian for a particle of charge  $e$  and mass  $M$  in an e.m. field  $A^\mu$  is:

$$\mathcal{H} = \frac{g^{\mu\nu}}{2M} \left( p_\mu - \frac{e}{c} A_\mu \right) \cdot \left( p_\nu - \frac{e}{c} A_\nu \right),$$

where  $A^\mu$  may depend only on four-position  $q$  and not on the proper-time  $s$ . The equations of motion are, derived using Hamilton's equations above:

$$\dot{q}_\mu = \frac{\pi_\mu}{M}, \tag{2.2.22a}$$

$$\dot{p}_\mu = \frac{e}{Mc} \pi^\nu \partial_\mu A_\nu \tag{2.2.22b},$$

where the momentum is:

$$\pi_\mu = p^\mu - \frac{e}{c}A^\mu.$$

The Lorentz force, the force on a charged particle in an electromagnetic field, is:

$$\dot{p}_\mu = \frac{e}{c}(\partial_\mu A_\nu)\dot{q}^\nu.$$

The derivative of the parameterised momentum  $\pi_\mu$  above yields:

$$\dot{\pi}_\mu = \dot{p}_\mu - \frac{e}{c}(\partial_\nu A_\mu)\dot{q}^\nu.$$

Substituting the relation for the Lorentz force above gives:

$$\dot{\pi}_\mu = \frac{e}{c}\mathcal{F}_{\mu\nu}\dot{q}^\nu,$$

where  $\mathcal{F}_{\mu\nu}$  is the electromagnetic field tensor defined as:

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Taking the inner-product of the Lorentz force with the momentum  $\pi^\mu$  gives:

$$\dot{p}_\mu\pi^\mu = \frac{e}{c}\mathcal{F}_{\mu\nu}\dot{q}^\nu\pi^\mu,$$

which may be rewritten as:

$$\frac{1}{2}\frac{d}{ds}(\pi_\mu\pi^\mu) = \frac{e}{Mc}\mathcal{F}_{\mu\nu}\dot{q}^\nu\dot{q}^\mu,$$

The right hand side is zero as  $\dot{q}^\nu\dot{q}^\mu$  is symmetric while  $\mathcal{F}_{\mu\nu}$  is anti-symmetric. Thus  $\frac{d}{ds}(\pi_\mu\pi^\mu) = 0$  or  $\pi_\mu\pi^\mu \equiv \kappa^2$ . Here  $\kappa$  does not depend on  $s$ . To find the form of  $\kappa$ , note that the potential  $A^\mu$  has not been defined – the above analysis must work for all possible electromagnetic potential fields including  $A^\mu = 0$ . Therefore:

$$\begin{aligned}\kappa^2 &= \pi_\mu\pi^\mu, \\ &= \left(p_\mu - \frac{e}{c}A_\mu\right) \cdot \left(p^\mu - \frac{e}{c}A^\mu\right) \\ &= p_\mu p^\mu = M^2 c^2.\end{aligned}$$

So again, the mass of the particle within PRD is a constant of the motion and does not depend on the physical condition of the particle (energy or momentum).

The PRD may be formulated as a probabilistic theory. It is assumed that there exists a conditional probability density function,  $\rho(x|s)$  which represents the physical properties of the system. The four-position of the particle is represented by  $x \equiv x^\mu$  and here  $s$  is the evolution parameter discussed above. This parameter places a “*condition*” on the density function. For the case of a single particle, the density function may be rewritten as:

$$\rho(x|s) = \rho(\vec{x}|x^0, s)\rho(x^0|s),$$

where  $\rho(x^0|s)$  is a marginal probability density function in time. The “*condition*” the evolution parameter places on the density function is that this function means that when  $\rho(x^0|s) = 0$  the probability of finding the particle (at any point  $\vec{x}$ ) at the time  $x^0$  is zero. This allows, therefore, the appearance/disappearance of particles in PRD. To be a probabilistic theory, the density function must be positive definite, i.e.:

$$\rho(x|s) = \psi^*(x, s)\psi(x, s) \geq 0.$$

and be normalisable:

$$\int_D \rho(x|s) d^4x = 1.$$

Note the similarity of these conditions to those present in NRQM. Through the positive-definite property of the probability density function, the continuity equation of the following form holds:

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x_\mu} (\rho v^\mu) = 0, \quad (2.2.23)$$

where  $v^\mu = \dot{q}^\mu$  is defined in (2.2.22a). The continuity equation (2.2.23) and the expansion of the four-velocity may be used to derive the following differential equation for the state  $\psi$ :

$$i\hbar \frac{\partial \psi}{\partial s} = \mathcal{K}\psi,$$

where  $\mathcal{K}$  is a mass operator defined as:

$$\mathcal{K} = \frac{\pi^\mu \pi_\mu}{2m} + V,$$

with  $V$  being the potential energy and  $\pi^\mu$  the momentum operator. The differential equation (2.2.23) is the *Stueckelberg equation* for a single particle, see [27]. By considering a free, scalar particle, the Stueckelberg equation becomes:

$$i\hbar \frac{\partial \psi_f}{\partial s} = -\hbar^2 \partial_\mu \partial^\mu \psi_f,$$

which has the general solution:

$$\psi_f = \int \eta_{f\kappa} \exp [\kappa(k_f)s + ik_{f\mu}x^\mu] dk_f,$$

with:

$$\kappa(k_f) = -\frac{\hbar}{2m} k_{f\mu} k_f^\mu.$$

The integral is over energy-momentum ( $k_f$  being the four-vector with  $k_f^0$  being the energy of the particle and  $\vec{k}_f$  the momentum of the particle), see [27]. The quantities  $\eta_{f\kappa}$  are normalisation constants. The four-velocity has the expectation value:

$$\langle v_f^\mu \rangle = \frac{d\langle x^\mu \rangle}{ds} = \frac{\langle p_f^\mu \rangle}{m}.$$

By integrating over the infinitesimal interval  $s \rightarrow s + \delta s$  gives:

$$\delta\langle x_f^\mu \rangle = \frac{\langle p_f^\mu \rangle}{m} \delta s.$$

The observable world-line is given by the inner-product of  $\delta\langle x_f^\mu \rangle$  with itself:

$$\begin{aligned} \delta\langle x_f^\mu \rangle \delta\langle x_{f\mu} \rangle &= \frac{\langle p_f^\mu \rangle \langle p_{f\mu} \rangle}{m^2} (\delta s)^2, \\ \Rightarrow m^2 &= \frac{\langle p_f^\mu \rangle \langle p_{f\mu} \rangle}{\delta\langle x_f^\mu \rangle \delta\langle x_{f\mu} \rangle} (\delta s)^2. \end{aligned}$$

Notice now that  $m^2 > 0$ . By definition, the change in the proper time – the quantity  $\delta s$  – is positive and monotonically increasing function. Therefore two cases are permitted: 1)  $\langle p_f^\mu \rangle \langle p_{f\mu} \rangle > 0$  and  $\delta\langle x_f^\mu \rangle \delta\langle x_{f\mu} \rangle > 0$ , or 2)  $\langle p_f^\mu \rangle \langle p_{f\mu} \rangle < 0$  and  $\delta\langle x_f^\mu \rangle \delta\langle x_{f\mu} \rangle < 0$ . Case 1 is that of a time-like motion or path whereas case 2 is that of a space-like motion or path. A time-like path is the typical type encountered in nature, the velocity of the particle is always less than the speed of light in a vacuum,  $c$ . A particle of this type is referred to as a tardyon. In case 2, however, the particle must always have a velocity greater than  $c$  – this type of theoretical particle is called a tachyon.

Taking now the condition on the energy of the particle from the conventional theory:

$$m_s c^2 = p_f^\mu p_{f\mu},$$

where the  $s$  subscript denotes that this mass is that of the PRD. The momentum dispersion is given by:

$$\Delta^2 p_f = \langle p_f^\mu p_{f\mu} \rangle - \langle p_f^\mu \rangle \langle p_{f\mu} \rangle,$$

see [27], equation (9.20). Using the equations above, this may be rewritten as:

$$\begin{aligned} \Delta^2 p_f &= \langle p_f^\mu p_{f\mu} \rangle - \frac{m^2}{(\delta s)^2} \delta \langle x_f^\mu \rangle \delta \langle x_{f\mu} \rangle, \\ \Rightarrow m^2 &= \left( \langle p_f^\mu p_{f\mu} \rangle - \Delta^2 p_f \right) \frac{(\delta s)^2}{\delta \langle x_f^\mu \rangle \delta \langle x_{f\mu} \rangle} \end{aligned}$$

Inserting  $c^2$  on both sides and noting that in the classical limit the dispersion becomes zero, this equation becomes:

$$m^2 c^2 = \left[ \frac{c^2 (\delta s)^2}{\delta \langle x_f^\mu \rangle \delta \langle x_{f\mu} \rangle} \right] \langle p_f^\mu p_{f\mu} \rangle.$$

The addition of  $c^2$  renders the fraction in the square brackets dimensionless. Taking the classical limit, the term in the square bracket becomes one, which gives:

$$m^2 c^2 \approx \text{sgn} \left( \delta \langle x_f^\mu \rangle \delta \langle x_{f\mu} \rangle \right) \langle p_f^\mu p_{f\mu} \rangle$$

Again, like above, this holds for both time-like and space-like paths. The sign of the quantity  $\delta \langle x_f^\mu \rangle \delta \langle x_{f\mu} \rangle$  must not be discarded as it may be negative. As discussed above  $\delta s$  is always positive.

In everyday experience, “*time*” moves in one set direction, it is always increasing. From the point of view of thermodynamics, time seems intimately connected to the idea of entropy – entropy is the measure of disorder of a system (e.g. the universe). From the second law of thermodynamics, we know that the entropy of the universe is steadily increasing, that is the universe is developing from a highly-ordered state (that at or shortly after the big bang) to a highly disordered state. This property, therefore, provides us with an arrow of time. As stated earlier, Special & General Relativity do not take this property into account and so there is no arrow of time present in these theories. This places the Relativistic theory at odds with known observations in the universe and implies that improvements to Relativity are required.

The discussion returns now to the concept of time in the PRD. It was stated above that the time  $s$  used in PRD is independent of the system(s) under consideration. A

mechanism is required that will allow the measurement of this quantity  $s$ . To achieve this, an observer particle (or particles) – referred to as a clock-particle in the terminology of the PRD – which is independent of the system under consideration is introduced. For simplicity, the particle undergoing the interaction is referred to as particle 1 and there is one clock particle referred to as particle 2. Particle 2 is used to track the invariant scalar quantity  $s$ . To enable this, particle 2 is chosen to be a free scalar particle for which the world line is known. This allows the experimenter to measure the propagation of particle 2 relative to his/her own reference frame, i.e. measure  $s$ . In turn this  $s$  is used to parameterise the dynamics of particle 1 as it undergoes the interaction under study. For this method to be valid, three conditions must be met [27]:

- 1) that the rest frames are calibrated to one another;
- 2) Particle 1 does not interact with Particle 2 but may interact with other particles through a potential  $\mathcal{V}$ ;
- 3) Particle 2 is a free, scalar particle.

Condition 1) requires that some initial condition of the reference frames is known. Let's say that the experiment starts at some value of  $s = s_0$  (this may be thought of as the initial proper-time). Then the state of particle 1 should be known at this point and the Lorentz transformation from the reference frame of Particle 1 to Particle 2 is known.

Condition 2) is necessary to ensure that the evolution parameter is independent of the experiment being undertaken. This property is lacking from the coordinate time in Special & General Relativity as any motion in four-space influences the time measured. By keeping particle 2 wholly independent of the experiment being studied an independent evolution parameter is achieved.

Condition 3) is required to ensure that the clock-particle obeys the PRD and may be understood on the same basis as the particle being studied.

The Stueckelberg equation for this case of particles 1 and 2 is [27]:

$$i\hbar \frac{\partial \psi(1, 2, s)}{\partial s} - \left( \frac{p_1^\mu p_{1\mu}}{2m_1} + \frac{p_2^\mu p_{2\mu}}{2m_2} + \mathcal{V} \right) \psi(1, 2, s) = 0.$$

Here  $\psi(1, 2, s)$  is an  $s$ -dependent wavefunction of the two-particle system with four-position of the particle given by  $y_{1\mu}$  and  $y_{2\mu}$ . It is assumed that a separable solution of the form

$\psi(1, 2, s) = \psi(1, s)\psi(2, s)$  is valid. This yields two differential equations where the separation constant may be taken to be zero as the particles are independent of each other:

$$\begin{aligned} i\hbar \frac{\partial \psi(1, s)}{\partial s} - \left( \frac{p_1^\mu p_{1\mu}}{2m_1} + \mathcal{V} \right) \psi(1, s) &= 0, \\ i\hbar \frac{\partial \psi(2, s)}{\partial s} - \frac{p_2^\mu p_{2\mu}}{2m_2} \psi(2, s) &= 0. \end{aligned} \quad (2.2.24)$$

As particle 2 is – by definition – a free particle, there is no potential term in (2.2.24). This equation is further reduced to:

$$i\hbar \frac{\partial}{\partial s} \psi_f(2, s) = -\frac{\hbar^2}{2m_2} \frac{\partial^2}{\partial y_2^\mu \partial y_{2\mu}} \psi_f(2, s),$$

which is the free Stueckelberg equation as above. This equation has solution:

$$\psi_f(2, s) = \sqrt{\eta_f} \exp \left( -\frac{i\hbar}{2m_2} (k_2^\mu k_{2\mu} s + ik_{2\mu} y_2^\mu) \right),$$

see [27], equation (11.6). This provides the condition on the evolution parameter  $s$ . In the classical limit the most probable path of the particle is given by:

$$(\delta s)^2 = (s - s_0)^2 = \frac{1}{c^2} \delta \langle y_2^\mu \rangle \delta \langle y_{2\mu} \rangle,$$

where  $s_0$  is the initial point at which the rest frames are calibrated (condition 1) and may be defined to be zero. This gives:

$$s^2 = \frac{1}{c^2} \delta \langle y_2^\mu \rangle \delta \langle y_{2\mu} \rangle.$$

Ignoring statistical variations in  $y$  (restricting to this path of highest probability) this becomes:

$$\begin{aligned} s^2 &= \frac{1}{c^2} (c^2 \delta t^2 - \delta x^2), \\ &= \delta t^2 - \frac{\delta x^2}{c^2}, \\ &= \delta t^2 (1 - \beta^2), \end{aligned}$$

where  $\beta = v^2/c^2 = (\delta x^2/\delta t^2)/c^2$ .

### §2.2.1 Conclusion

The methods described above provide a viable alternative to the established theories of RQM (the Klein-Gordon equation and the Dirac Equation – the latter is not discussed here as it applies to spin- $\frac{1}{2}$  particles). The core issue, which prompts the necessity of a new theory, is the treatment of time. As described above, the concept of time in relativistic theories is fundamentally at odds with that of everyday experience, so-called Newtonian-time. There is no fundamental problem, mathematically, in Relativity with time-travel, the idea of a before and after is just convention. Two events ( $E_1$  and  $E_2$ ) which appear to occur simultaneously for one observer may appear to occur one after another (first  $E_1$ , then  $E_2$  or first  $E_2$ , then  $E_1$ ) for a different observer depending upon the relative motions of the observers to each other and the position of the events. However, in Newtonian-time there is a definite before and after to an event.

A consequence of this ambiguity of time may be seen in the Klein-Gordon equation. The Klein-Gordon equation is an attempt to combine Special Relativity and Quantum Mechanics. In doing so, it carries over the concept of a four-dimensional space-time. The time part of this four-vector is treated in a similar fashion to the coordinate part, which leads to a differential equation which is second order in both time and coordinate variable; the Klein-Gordon equation. This leads to the continuity equation associated with the Klein-Gordon equation having probability density function which is not positive definite. This is incompatible with the equivalent description of NRQM, the Schrödinger equation, which is a first order differential equation in time and has a positive definite probability density function. Typically, the probability density is defined to be value in the range zero to one. A probability outside this range would be meaningless. This issue with the Klein-Gordon equation is resolved in the PRD through the use of proper-time.

It is a fundamental requirement of the PRD that the proper-time  $s$  be independent of the system being studied – that is, it does not depend on the energy or momentum or configuration of the system being studied in any way. This allows the use of  $s$  to parameterise two or more evolving systems leading to an acceptable historical time – different observers are able to use the same historical time and there will be a before and



after to events. To achieve this the different observers must agree on the clock being used. This agrees with the Newtonian concept of time.

The concepts developed in the FSF are similar in aim to those in the Garavaglia Model [1] & [2], however they differ in implementation. In the Garavaglia Model, the “*proper-time*” used is the proper-time  $\tau$  of the particle undergoing interaction, which is not independent of the system. It is then a requirement of the Garavaglia Model that  $\tau$  is actually assumed independent of the system, which allows the derivation of the covariant Schrödinger-type wave equation. This covariant Schrödinger-type wave equation has a positive-definite density function.

### §2.3 *Garavaglia Model*

The following quoted sections are taken verbatim from [1] and outline the Garavaglia Model as proposed by Dr. T. Garavaglia. The quoted text is within inverted commas and in a different font, notes are added in similar typesetting to rest of this discussion and preceded by “**Note:**” All reference numbers in the following quoted text are those in the bibliography of original paper. Note also that only sections **2** and **4** are copied from [1].

#### “2. COVARIANT ACTION AND COVARIANT CLASSICAL DYNAMICS

The covariant classical methods, which are necessary for the development of covariant quantum theory, are described here. The covariant action is defined as

$$\mathcal{S} = \int \mathcal{L}(q, u) ds, \quad (2.1)$$

where  $\mathcal{L}(q, u)$  is a Lorentz invariant. The requirement that the action be an extremum under a variation leads to the covariant Euler-Lagrange equation

$$\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial u^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0. \quad (2.2)$$

The covariant Lagrangian for a particle of mass  $m$  and charge  $e$  interacting with a four-vector field  $A^\mu(q)$  is

$$\mathcal{L} = \frac{m}{2} u^2 - eA \cdot u, \quad (2.3)$$

and the generalized four-momentum is

$$p_\mu = \frac{\partial \mathcal{L}}{\partial u^\mu} = mu_\mu - eA_\mu. \quad (2.4)$$

The equation of motion is found from Eq. (2.2) to be

$$\frac{dp_\mu}{ds} + \partial_\mu eA \cdot u = 0, \quad (2.5)$$

where  $\partial_\mu = \partial/\partial q^\mu$ . Since

$$\frac{dA_\mu(q)}{ds} = u^\nu \partial_\nu A_\mu(q), \quad (2.6)$$

the equation of motion becomes

$$ma_\mu = eu^\nu F_{\nu\mu}, \quad (2.7)$$

with

$$a_\mu = \frac{d^2 q_\mu}{ds^2}, \quad F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu. \quad (2.8)$$

From the covariant Lagrangian Eq. (2.3), the covariant Hamiltonian is defined as

$$\mathcal{H} = p \cdot u - \mathcal{L}. \quad (2.9)$$

Since  $u_\mu = (p_\mu + eA_\mu)/m$ , a simple calculation gives

$$\mathcal{H} = \frac{(p + eA)^2}{2m} = \frac{(mu)^2}{2m} = \frac{m}{2}. \quad (2.10)$$

From Eq. (2.9) and

$$\frac{d\mathcal{L}}{ds} = \frac{\partial \mathcal{L}}{\partial q} \cdot u + \frac{\partial \mathcal{L}}{\partial u} \cdot a, \quad (2.11)$$

it follows that

$$\frac{d\mathcal{H}}{ds} = \left( \frac{dp}{ds} - \frac{\partial \mathcal{L}}{\partial q} \right) \cdot u - \left( p - \frac{\partial \mathcal{L}}{\partial u} \right) \cdot a = 0, \quad (2.12)$$

which is a consequence of Eq (2.2). In addition

$$\frac{d\mathcal{H}}{ds} = \frac{\partial \mathcal{H}}{\partial q} \cdot u + \frac{\partial \mathcal{H}}{\partial p} \cdot f = 0, \quad (2.13)$$

with  $f^\mu = dp^\mu/ds$ , and this implies the Hamiltonian equations

$$u_\mu = \frac{\partial \mathcal{H}}{\partial p^\mu} \quad f_\mu = -\frac{\partial \mathcal{H}}{\partial q^\mu} \quad (2.14)''$$

**Note:** An alternative route to (2.13) is to use the fact that the Hamiltonian is constant only on-shell:

$$\frac{d\mathcal{H}}{ds} = 0.$$

This is expanded to become:

$$\frac{d\mathcal{H}}{ds} = \frac{\partial p^\mu}{\partial s} \cdot u - \frac{\partial q^\mu}{\partial s} \cdot f = 0,$$

with  $u$  and  $f$  defined in (2.14) in [1].

“These equations and the covariant Hamiltonian Eq (2.10) give

$$u_\mu = \frac{\partial \mathcal{H}}{\partial p^\mu} = (p + eA)_\mu / m, \quad (2.15)$$

and

$$f_\mu = -\frac{\partial \mathcal{H}}{\partial q^\mu} = -u^\nu \partial_\mu eA_\nu. \quad (2.16)$$

From Eq. (2.15) it follows that

$$f_\mu = ma_\mu - e \frac{dA_\mu}{ds}, \quad (2.17)$$

which with Eq. (2.6) and Eq. (2.16) results in the equation of motion Eq. (2.7).

For real function  $F(q, p, s)$  and  $G(q, p, s)$  of  $q_\nu$ ,  $p_\nu$ , and  $s$ , the Poisson bracket is defined as

$$\{F, G\} = \partial F \cdot \bar{\partial} G - \partial G \cdot \bar{\partial} F, \quad (2.18)$$

with  $\partial_\mu = \partial/\partial q^\mu$  and  $\bar{\partial}_\mu = \partial/\partial p^\mu$ . Since

$$\partial_\mu q^\nu = \bar{\partial}_\mu p^\nu = \delta_\mu^\nu, \quad (2.19)$$

one finds

$$\{q^\mu, p^\nu\} = g^{\mu\nu}. \quad (2.20)$$

From

$$\frac{dF}{ds} = \frac{\partial F}{\partial q} \cdot u + \frac{\partial F}{\partial p} \cdot f + \frac{\partial F}{\partial s}, \quad (2.21)$$

and Eq. (2.14), it follows that

$$\frac{dF}{ds} = \{F, \mathcal{H}\} + \frac{\partial F}{\partial s} \quad (2.22)$$

The dynamical equations that are equivalent to Hamiltonians equations Eq. (2.14) are

$$u_\mu = \{q_\mu, \mathcal{H}\} \quad f_\mu = \{p_\mu, \mathcal{H}\} \quad (2.23)$$

The Hamiltonian is the generator for a one parameter transformation that leaves the Poisson bracket invariant. This is seen from

$$\frac{d}{ds}\{q(s)^\mu, p(s)^\nu\} = \left\{\frac{dq(s)^\mu}{ds}, p(s)^\nu\right\} + \left\{q(s)^\mu, \frac{dp(s)^\nu}{ds}\right\} = 0, \quad (2.24)$$

which follows from the dynamical equations Eq. (2.23) and the Jacobi identity for the poisson bracket. This leads to the conclusion that

$$\{q(s)^\mu, p(s)^\nu\} = \{q(0)^\mu, p(0)^\nu\} = g^{\mu\nu}. \quad (2.25)$$

To illustrate the use of the covariant classical dynamical equation, examples are given that are to be compared with the corresponding covariant quantum solutions. These are associated with a vector potential of the form  $A^\mu = (A^0, \vec{A}) = (-V(\vec{q}), 0)$ . From the covariant Hamiltonian Eq. (2.10) and the dynamical equations Eq. (2.14) or Eq. (2.23), it follows that

$$u^0 = (p^0 + A^0)/m \quad f^0 = \frac{(p + A)^0}{m} \partial_0 A_0 = 0; \quad (2.26)$$

hence,  $p^0 = E$ , a constant, and

$$E = m\gamma + V(\vec{q}), \quad (2.27)$$

since  $u^0 = dq^0/ds = \gamma$ . Also

$$f^i = \frac{(p + A)^0}{m} \partial_i A_0 = -u^0 \partial_i V(\vec{q}) \quad (2.28)$$

and

$$\frac{d(m\gamma\vec{\beta})}{dq^0} = -\nabla V(\vec{q}). \quad (2.29)$$

For a free particle with  $V(\vec{q}) = 0$ , one finds  $E = \gamma m$ , and  $\vec{p} = \gamma m \vec{\beta}$ ."

**Note:** (2.15) is the equation of motion in the observer's reference frame, where  $d\vec{\beta}/dq^0$  and  $\nabla V(\vec{q})$  are the conventional acceleration and force.

“For a particle in a uniform field with potential  $V(z) = -Kz$ , the dynamical equations give

$$\frac{dq^0}{ds} = (p^0 - V(z))/m, \quad \frac{dp^0}{ds} = 0, \quad p^0 = E, \quad (2.30)$$

and

$$\frac{dq^3}{ds} = \frac{p^3}{m}, \quad \frac{dp^3}{ds} = (E - V(z)) \frac{\partial V(z)}{\partial z}. \quad (2.31)$$

Since  $p^3 = m\dot{z}(dq^0/ds) = m\dot{z}(E - V(z))/m$ , the invariant Hamiltonian becomes

$$(E - V(z))^2(1 - \dot{z}^2) = m^2. \quad (2.32)$$

For  $z(0) = \dot{z}(0) = 0$ , the solution is

$$z(t) = \frac{m}{K} \left( \sqrt{1 + \left(\frac{Kt}{m}\right)^2} - 1 \right). \quad (2.33)$$

Additional related relativistic particle dynamics is found in [4], and applications of covariant dynamics in curvilinear coordinates to betatron physics may be found in [5] and [6]. Related concepts of covariant dynamics have been used also in the development of relativistic wave equations [7] and covariant field theory [8].”

#### “4. COVARIANT SCHRÖDINGER TYPE WAVE EQUATION

“The Schrödinger type equation in covariant form for the scalar case with wave function

$$\Psi(s, q) = \langle q | \Psi(s) \rangle, \quad (4.1)$$

is

$$i \frac{\partial \Psi(s, q)}{\partial s} = \hat{\mathcal{H}} \Psi(s, q). \quad (4.2)$$

Using a separable solution  $\Psi(s, q) = \psi(s)\Phi(q)$ , this equation becomes

$$\frac{i \frac{\partial \psi(s)}{\partial s}}{\psi(s)} = \lambda = \frac{\hat{\mathcal{H}} \Phi(q)}{\Phi(q)}. \quad (4.3)$$

For  $\psi(s) \propto \exp(-ims/2)$ , one finds

$$\hat{\mathcal{H}}\Phi(q) = \frac{(\hat{p} + eA)^2}{2m}\Phi(q) = \frac{m}{2}\Phi(q), \quad (4.4)$$

with  $\hat{p}^\mu = i\partial^\mu$ ."

**Note:** The aim here is to have an oscillatory solution for  $\psi(s)$ , i.e.  $\psi(s) \approx e^{-(ims/2)}$ , see [28]. Thus from :

$$\frac{i\frac{\partial\psi(s)}{\partial s}}{\psi(s)} = \lambda,$$

we get  $\lambda = m/2$ . Thus:

$$\hat{\mathcal{H}}\phi(q) = \frac{m}{2}\phi(q).$$

"For the free particle case,  $A^\mu = 0$ ,

$$\hat{\mathcal{H}} = \frac{(\hat{p})^2}{2m} = -\frac{1}{2m}\square^2, \quad (4.5)$$

with

$$\square^2 = \frac{\partial^2}{\partial^2 q^0} - \nabla^2 = \frac{\partial^2}{\partial^2 q^0} - \frac{\partial^2}{\partial^2 q^1} - \frac{\partial^2}{\partial^2 q^2} - \frac{\partial^2}{\partial^2 q^3}. \quad (4.6)$$

Potential problems can be included using the vector potential  $A^\mu = (-V(q), 0, 0, 0)$ , and the equation for  $\Phi(q)$  becomes

$$\left( (\hat{p}^0 - V(q))^2 + \nabla^2 \right) \Phi(q) = m^2 \Phi(q). \quad (4.7)$$

The time dependence in this equation is separable, and the solution  $\Phi(q)$  takes this form

$$\Phi(q) \propto e^{-iEt} \phi(x, y, z). \quad (4.8)$$

This gives

$$\left( (E - V(x, y, z))^2 + \nabla^2 \right) \phi(x, y, z) = m^2 \phi(x, y, z). \quad (4.9)$$

In the one dimensional case, this becomes

$$(E - V(z))^2 \phi(z) = m^2 \phi(z) - \frac{\partial^2 \phi(z)}{\partial^2 z}. \quad (4.10)$$

The non-relativistic limit is found from

$$(E - V(z))\sqrt{\phi(z)} = m\sqrt{\phi(z)}\sqrt{1 - \frac{1}{m^2\phi(z)}\frac{d^2\phi(z)}{dz^2}}, \quad (4.11)$$

and this becomes

$$(E - V(z))\sqrt{\phi(z)} \approx m\sqrt{\phi(z)}\left(1 - \frac{1}{2m^2\phi(z)}\frac{d^2\phi(z)}{dz^2} + \dots\right), \quad (4.12)$$

which is equivalent to

$$\frac{d^2\phi(z)}{dz^2} + 2m(E - m - V(z))\phi(z). \quad (4.13)$$

This is the usual time-independent Schrödinger equation; however, the eigenvalues  $E - m$  contain the rest mass of the particle. The rest mass should be retained in the non-relativistic limit because in the free particle case the phase of the wave function is a Lorentz invariant,  $Et - \vec{p} \cdot \vec{q} = ms$ . Furthermore, the operator  $\hat{p}^0$  is a generator for the displacement of the time coordinate  $q^0$ ; however, in this theory, it is associated with a phase space Hamiltonian operator through a time dependent Schrödinger equation. In this way, time and energy are not conjugate coordinates expressible in terms of phase space operators, and an uncertainty relation of the type described in Appendix A is not possible. A related comment can be found in [12].”

**Note:** (4.8) is the time-independent Schrödinger equation, an alternative solution to which is discussed in §3.2. Also, (4.12) is incorrect and should read:

$$\frac{d^2\phi(z)}{dz^2} + 2m(E - m - V(z))\phi(z) = 0.$$

“The equation of continuity for a free scalar particle follows from the covariant Schrödinger type equation. From Eq. (4.2) and the fact that  $\hat{\mathcal{H}}$  is Hermitian, one can form

$$\psi^*(q, s)i\frac{\partial\psi(q, s)}{\partial s} + \psi(q, s)i\frac{\partial\psi^*(q, s)}{\partial s} = \psi^*(q, s)\mathcal{H}\psi(q, s) - \psi(q, s)\mathcal{H}\psi^*(q, s), \quad (4.14)$$

and this can be written as

$$i\frac{\partial\rho(q, s)}{\partial s} - \hat{p}_\mu j^\mu(q, s) = 0, \quad (4.15)$$

with

$$\rho(q, s) = \psi^*(q, s)\psi(q, s), \quad (4.16)$$

and

$$j^\mu(q, s) = \frac{1}{2m}(\phi^*(q, s)\hat{p}^\mu\phi(q, s) - \phi(q, s)\hat{p}^\mu\phi^*(q, s)). \quad (4.17)$$

Here  $\rho(q, s)$  represents the probability density in the four-vector  $q^\mu$  representation, and  $j^\mu(q, s)$  is associated with the charge density current [13]. The normalization condition  $\langle\Psi(s)|\Psi(s)\rangle = 1$  requires

$$\int d^4q \rho(q, s) = 1. \quad (4.18)$$

For this theory, the probability density is positive definite, and this removes the historical problem associated with finding a suitable probability density for the Klein-Gordon equation.”

**Note:** The probability density function for the covariant Schrödinger-type wave equation will be compared to that of the Klein-Gordon equation in §2.5.1 and §2.5.2.



## §2.4 Green's Function Theory

Green's Functions are used to find a propagated state which will be a solution to the covariant Schrödinger-type wave equation. This use of Green's functions will be shown later in both the non-relativistic and relativistic cases.

Green's Functions are a convenient way of finding the solution to differential equations, particularly inhomogeneous differential equations. The Green's function was first developed by George Green, in an 1828 essay entitled an "*Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism.*" Green developed this method when trying to find a solution to the differential equation:

$$\nabla^2 f(\vec{r}, \vec{r}_0) = -4\pi\delta(\vec{r} - \vec{r}_0).$$

This is not the exact form of the equation in question, modern mathematical notation is vastly different than that used in Green's time, see [29].

The function  $\delta(\vec{r} - \vec{r}_0)$  is the Dirac Delta Function. This is not a function in the true sense of the word and as such was only accepted as an unfortunate result of the calculations required to solve certain differential equations, see [29]. It was not until Dirac put it on a stronger mathematical foundation that its use became fully accepted and further applied, see [29]. Take the Dirac delta function as a function of  $t$  for example. Then,  $\delta(t - t')$  has the following properties:

$$\int_{\mathcal{I}} f(t')\delta(t - t') dt' = \begin{cases} f(t), & t \in \mathcal{I}, \\ 0, & t \notin \mathcal{I}, \end{cases}$$

where  $f(t')$  is some arbitrary function defined in the interval  $\mathcal{I}$ . The Dirac delta function is often represented by the following integral:

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(x-x')t} dt.$$

Another function is the Heaviside Unit-Step Function (see [30], p273), defined as:

$$\theta(s) = \begin{cases} 1 & s > 0 \\ 0 & s < 0 \end{cases}$$

and has the property:

$$\delta(s) = \frac{\partial}{\partial s}\theta(s).$$

The Green's function method has been applied to various problems since it was developed, for example the Heat Equation, Laplace's Equation and the Wave Equation. Here the technique is outlined in the one dimensional case. Take an inhomogeneous differential equation:

$$\tilde{\mathcal{L}}u(t) = f(t),$$

where  $\tilde{\mathcal{L}}$  is a differential operator depending only on  $t$ . The function  $f(t)$  is an evolution function for the system in question.

To solve this using the Green's function method, there is a function  $g(t)$  such that:

$$u(t) = \int_{t'}^t g(t-t')f(t') dt'$$

since  $u(t)$  is a convolution of the functions  $g(t)$  and  $f(t)$ . Setting  $f(t) = \delta(t)$  gives:

$$\begin{aligned} u(t) &= \int_{t'}^t g(t-t')\delta(t') dt' = g(t) \\ \tilde{\mathcal{L}}g(t) &= \delta(t) \end{aligned} \tag{2.4.1}$$

So for any differential equation of the form above, the Green's function is defined as in (2.4.1). For higher dimensional cases write:

$$\tilde{\mathcal{L}}g(\vec{r}, \vec{r}_0) = \delta(\vec{r}, \vec{r}_0). \tag{2.4.2}$$

The system under consideration is that of an accelerated scalar particle moving in a one-dimensional potential field, specifically an electric field acting on a charged particle along the  $z$ -axis. In the non-relativistic case, the Schrödinger equation for the accelerated scalar particle is:

$$\left[ -\frac{1}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + z - i\frac{\partial}{\partial t} \right] \Psi(x, y, z, t) = 0.$$

The Green's function for this case is defined by the equation:

$$\left[ -\frac{1}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + z - i\frac{\partial}{\partial t} \right] g(x, y, z, t) = \delta(x-x')\delta(y-y')\delta(z-z')\delta(t).$$

That the Green's function is a solution to the Schrödinger equation is due to the fact that a Dirac delta function may be used as an initial state. Using the Green's function method, integrating over the product of the Green's function and the initial state returns the Green's function as the propagated wave function.

In the covariant case, the Schrödinger-type Equation for the accelerated scalar particle is:

$$\left[ -\frac{\partial^2}{\partial t^2} + z^2 - 2iz\frac{\partial}{\partial t} + \hat{\nabla}^2 + i\frac{\partial}{\partial s} \right] \Psi(t, x, y, z, s) = 0,$$

with the Green's function defined by:

$$\left[ -\frac{\partial^2}{\partial t^2} + z^2 - 2iz\frac{\partial}{\partial t} + \hat{\nabla}^2 + i\frac{\partial}{\partial s} \right] g(t, x, y, z, s) = \delta(t - t')\delta(x - x')\delta(y - y')\delta(z - z')\delta(s).$$

As another initial condition for this project, the Green's function is defined as:

$$g(q, q', s) = \theta(s)\mathcal{G}(q, q', s).$$

The inclusion of  $\theta(s)$  restricts the proper time of the system to  $s \geq 0$ . Note that the four-position  $q$  is  $q = t, x, y, z$ .

The function  $\mathcal{G}(q, q', s)$  is the evolution function or propagator for the system. This function can be rewritten as:

$$\mathcal{G}(q, q', s) = \langle q | \hat{U}(s) | q' \rangle,$$

where  $\hat{U}$  is a unitary operator, defined in this case to be  $\hat{U} = e^{-is\hat{\mathcal{H}}}$ . A unitary operator is one where its inverse is equal to its Hermitian conjugate, i.e. if  $\hat{X}$  is unitary then  $\hat{X}^{-1} = \hat{X}^\dagger$ .

This can also be expressed in terms of the action of the system, which will be elaborated upon in §3.3.

The Path Integral method, which was first explored by Dirac and then further developed by Feynman, is based on the above use of the Green's function as a propagator for the initial state. Instead of having one integral of the product of the Green's function and the initial state, there are an infinite number of integrals over all possible paths, see [31].

The Green's function method provides a mechanism to find a propagated state which is a solution to the differential equation under study when given any initial state:

$$\Psi(q, s) = \int_{-\infty}^{+\infty} g(q, q', s)\Psi(q', 0)d^4q'.$$

## §2.5 Covariant vs. Established Formalism

The Klein-Gordon equation, for the case of a potential field

$e\mathcal{A}_\mu = (-V(z), \vec{0})$  is:

$$\left[ \left( i \frac{\partial}{\partial t} + V(z) \right)^2 - \nabla^2 - m^2 \right] \Psi(q, t) = 0. \quad (2.5.1)$$

Below the continuity equations for both cases will be derived but first a quick note is needed on why the Klein-Gordon equation and covariant Schrödinger-type Equation are different.

This difference between the Klein-Gordon equation and the covariant Schrödinger-type wave equation results from the basic principle of the two approaches. In the development of the Klein-Gordon equation, the mechanics of the particle in the reference frame of the observer was under consideration. The time as measured in this theory is itself a coordinate of the system and not an evolution parameter. The covariant Schrödinger-type wave equation is founded on the use of the proper-time  $s$  of the particle as the evolution parameter for the system. For the covariant Schrödinger-type wave equation to work,  $s$  is assumed to be independent of the coordinates of the particle being studied.

The construction of the covariant Schrödinger-type wave equation is shown here. An initial state (i.e. at  $s=0$ ) is  $|\Psi(0)\rangle$  and a propagated state is found via:

$$|\Psi(s)\rangle = \hat{U}(s)|\Psi(0)\rangle.$$

$\hat{U}(s) = e^{-is\hat{\mathcal{H}}}$  is a unitary transformation:

$$\hat{U}^\dagger(s)\hat{U}(s) = \hat{\mathcal{I}},$$

with  $\hat{\mathcal{I}}$  the identity operator. For small values of proper-time,  $s \ll 1$ ,  $\hat{U}(s)$  can be expanded via a Taylor expansion about the point  $s = 0$ :

$$\hat{U}(s) \approx \hat{\mathcal{I}} - is\hat{\mathcal{H}}.$$

Using this the propagated state becomes:

$$\begin{aligned} |\Psi(s)\rangle &= \hat{U}(s)|\Psi(0)\rangle \\ &= |\Psi(0)\rangle - is\hat{\mathcal{H}}|\Psi(0)\rangle \end{aligned}$$

Altering this equation to find the differential of  $|\Psi(0)\rangle$  at  $s = 0$  gives:

$$\begin{aligned} \frac{|\Psi(\delta s)\rangle - |\Psi(0)\rangle}{\delta s} &= -i\hat{\mathcal{H}}|\Psi(0)\rangle \\ \Rightarrow \quad i\frac{\partial}{\partial s}|\Psi(0)\rangle &= \hat{\mathcal{H}}|\Psi(0)\rangle \end{aligned}$$

For the approach of the Garvaglia Model, use was made of the proper frame of the particle along with that of the observer. In the proper frame, the particle is at rest, so the total energy is given by  $\mathcal{E} = mc^2$ , which becomes with  $c = 1$ ,  $\mathcal{E} = m$  or  $\mathcal{E}^2 = m^2$ , with  $m$  the rest mass of the particle. The invariant of the system is  $\mathcal{E}^2 - (c\vec{p})^2$ , which is used to construct the Schrödinger-type equation. This leads to a density function which is positive definite, as is shown below.

The Klein-Gordon equation is also shown to be a special case of the covariant Schrödinger-type equation using a separable solution. It is the *time-independent* covariant Schrödinger-type wave equation and can be rewritten as:

$$\hat{\mathcal{H}}\Psi(q, s) = m^2\Psi(q, s).$$

### §2.5.1 Continuity Equation: Klein-Gordon Equation

As stated in Chapter 1, the problem with the Klein-Gordon equation is that the associated density function,  $\rho(q, t)$ , is not positive-definite. Along with the probability density function,  $\rho$ , there is the probability current density  $\vec{j}$ , and together these satisfy the continuity equation:<sup>2</sup>

$$\frac{\partial}{\partial t}\rho - \nabla \cdot \vec{j} = 0, \quad (2.5.2)$$

derived as follows. Taking the case of a zero field, i.e.  $e\mathcal{A}_\mu = 0$ , the Klein-Gordon equation becomes:<sup>3</sup>

$$(\partial_\mu \partial^\mu + m^2) \Psi(q, t) = 0. \quad (2.5.3)$$

Next act on the left of (2.5.3) with  $\Psi^*(q, t)$ <sup>4</sup> and on the left of the complex conjugate of (2.5.3) with  $\Psi(q, t)$ :

$$\begin{aligned} \Psi^* \frac{\partial^2}{\partial t^2} \Psi - \Psi^* (\nabla^2 \Psi) + m^2 |\Psi|^2 &= 0, \\ \Psi \frac{\partial^2}{\partial t^2} \Psi^* - \Psi (\nabla^2 \Psi^*) + m^2 |\Psi|^2 &= 0, \end{aligned}$$

Subtract the resulting equations gives:

$$\begin{aligned} & \left[ \Psi^* \left( \frac{\partial^2}{\partial t^2} \Psi \right) - \left( \frac{\partial^2}{\partial t^2} \Psi^* \right) \Psi \right] - [\Psi^* (\nabla^2 \Psi) - (\nabla^2 \Psi^*) \Psi] = 0 \\ \Rightarrow \frac{\partial}{\partial t} & \left[ \Psi^* \left( \frac{\partial}{\partial t} \Psi \right) - \left( \frac{\partial}{\partial t} \Psi^* \right) \Psi \right] - \nabla \cdot [\Psi^* (\nabla \Psi) - (\nabla \Psi^*) \Psi] = 0. \end{aligned}$$

So the usual continuity equation has been constructed:

$$\frac{\partial}{\partial t}\rho - \nabla \cdot \vec{j} = 0,$$

---

<sup>2</sup> Where functions are written without variables these are taken as understood from the context of the discussion. In this subsection,  $\rho \equiv \rho(q, t)$ ,  $\Psi \equiv \Psi(q, t)$ , etc.

<sup>3</sup>  $\partial_\mu$  and  $\partial^\mu$  are as follows:

$$\partial_\mu = i\hat{p}_\mu = \partial/\partial x^\mu, \quad \partial^\mu = i\hat{p}^\mu = \partial/\partial x_\mu.$$

<sup>4</sup> \* denotes complex conjugate, i.e.  $\Psi^*(q, t)$  is the complex conjugate of  $\Psi(q, t)$ .

with:

$$\begin{aligned}\rho &= \Psi^* \left( \frac{\partial}{\partial t} \Psi \right) - \left( \frac{\partial}{\partial t} \Psi^* \right) \Psi, \\ \vec{j} &= \Psi^* (\nabla \Psi) - (\nabla \Psi^*) \Psi.\end{aligned}$$

For simplicity the continuity equation, in this contravariant notation, can be rewritten as:

$$\partial_\mu j^\mu = 0,$$

where:

$$j^\mu = (\rho, \vec{j}).$$

That the Klein-Gordon equation fails is due to the fact that it is second order in the time derivative, which leaves a time derivative in the equation for  $\rho(q, t)$  above.

### §2.5.2 Continuity Equation: Covariant Schrödinger-type Equation

This section is a restatement of the discussion in [1] encompassing equations (4.14) to (4.18) in that paper. The covariant Schrödinger-type equation is:

$$\hat{\mathcal{H}}\Psi(q, s) = i \frac{\partial}{\partial s} \Psi(q, s), \quad (2.5.4)$$

where  $s$  is the so-called proper distance,  $c\tau$ , the product of the speed of light  $c$  with the proper time of the particle  $\tau$ . The Hamiltonian operator,  $\hat{\mathcal{H}}$ , can be expanded as:

$$\hat{\mathcal{H}} = \frac{(\hat{p}_\mu - e\mathcal{A}_\mu)^2}{2m}.$$

So for the case of a free particle, the Hamiltonian operator can be rewritten:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m}. \quad (2.5.5)$$

Using (2.5.5), the Schrödinger-type equation becomes for the free particle case:

$$\frac{1}{2m} \hat{p}_\mu \hat{p}^\mu \Psi - i \frac{\partial}{\partial s} \Psi = 0. \quad (2.5.6)$$

The same method is used as before — act on the left of (2.5.6) with  $\Psi^*$  and act on the left of the complex conjugate of (2.5.6) with  $\Psi$ :

$$\begin{aligned} -\frac{1}{2m}\Psi^*\hat{p}_\mu\hat{p}^\mu\Psi - i\Psi^*\frac{\partial}{\partial s}\Psi &= 0, \\ -\frac{1}{2m}\Psi\hat{p}_\mu\hat{p}^\mu\Psi^* + i\Psi\frac{\partial}{\partial s}\Psi^* &= 0. \end{aligned}$$

Subtracting these two equations gives:

$$\hat{p}_\mu \left( \frac{1}{2m} [\Psi\hat{p}^\mu\Psi^* - \Psi^*\hat{p}^\mu\Psi] \right) - i\frac{\partial}{\partial s}(\Psi^*\Psi) = 0.$$

The continuity equation is now:

$$i\frac{\partial}{\partial s}\rho - \hat{p}_\mu j^\mu = 0, \tag{2.5.7}$$

with:

$$\begin{aligned} \rho &= \Psi^*\Psi = |\Psi|^2 \geq 0 \\ j^\mu &= \frac{1}{2m} (\Psi\hat{p}^\mu\Psi^* - \Psi^*\hat{p}^\mu\Psi). \end{aligned}$$

The density function associated with the covariant Schrödinger-type equation can be seen to be positive-definite by definition. So the covariant approach removes some inconsistencies when dealing with spin-0 particles.



## Chapter 3 Non-Relativistic Theory

### §3.1 Introduction

This project is concerned with applying a covariant formalism for Relativistic Quantum Mechanics to the system consisting of a charged scalar particle in a one-dimensional potential field,  $V(z) = -kz$ . Before discussing the Relativistic case in Chapter 4, the system will be studied here in the Non-Relativistic case to explore the methods.

As a first step to providing a complete description of the system, the time-independent Schrödinger equation will be solved. The time-independent Schrödinger equation is:

$$\hat{\mathcal{H}}|\Psi\rangle = \mathcal{E}|\Psi\rangle, \quad (3.1.1)$$

where  $\hat{\mathcal{H}}$  is the Hamiltonian operator and  $\mathcal{E}$  is the total energy of the system. This is similar in form to the case of a time-independent Hamiltonian (i.e. a conserved system, one that has constant energy,  $\mathcal{E}$ ) in classical mechanics simply being the total energy of the system:

$$\mathcal{H} = \mathcal{E},$$

where  $\mathcal{H}$  is the classical Hamiltonian which for the system described has the form:

$$\mathcal{H} = \frac{p^2}{2m} - kz.$$

The solution to (3.1.1) will be shown in §3.2 to be the Airy Function,  $\Phi(-\xi)$ , defined as:

$$\Phi(\xi) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(u\xi + \frac{u^3}{3}\right) du.$$

See §3.2.2 for a further discussion of the Airy function.

Following this, the method of solution to the time-dependent Schrödinger equation will be demonstrated. The time-dependent Schrödinger equation is:

$$\hat{\mathcal{H}}|\Psi\rangle = i\frac{\partial}{\partial t}|\Psi\rangle.$$

The method used to find a solution to this equation is via the Green's Function Method, as discussed in some detail in §2.4. The method is derived in such a way that all that is needed is an initial state to find the final solution.

Two examples of initial state are given, a plane-wave state and a Gaussian state. Once the final states – i.e. the propagated wave functions – are found, the corresponding density functions are derived. As will be seen in §3.4.2, the density function for the Gaussian state is centered on the classical path of the particle while the density function for the plane-wave state is a constant number - which shows that there is a constant probability of finding the planewave state at all points.

### §3.2 Time-independent Schrödinger Equation

The time-independent Schrödinger equation for a non-relativistic scalar particle of mass  $m$  in an electric potential  $V(z) = -kz$  is:

$$\hat{\mathcal{H}}|\Psi\rangle = \mathcal{E}|\Psi\rangle, \quad (3.2.1)$$

with:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} - k\hat{z}.$$

To simplify the equations involved, the operators are transformed into a new, dimensionless, form:

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{p_s^2}{2m} \left( \frac{\hat{p}}{p_s} \right)^2 - kz_s \left( \frac{\hat{z}}{z_s} \right), \\ &= \left( \frac{p_s^2}{m} \right) \left[ \frac{\hat{p}_d^2}{2} - \frac{kz_s m}{p_s^2} \hat{z}_d \right], \end{aligned} \quad (3.2.2)$$

where the momentum and position operators are rewritten as  $\hat{p}_d = \hat{p}/p_s$  and  $\hat{z}_d = \hat{z}/z_s$  respectively and  $p_s$  and  $z_s$  are scaling factors. These new operators are dimensionless, with subscript  $d$  to denote this fact. The commutation relation for the dimensionless operators is:

$$[\hat{z}_d, \hat{p}_d] = \left[ \frac{\hat{z}}{z_s}, \frac{\hat{p}}{p_s} \right] = i \frac{\hbar}{z_s p_s},$$

Defining the following:

$$p_s z_s = \hbar, \quad (3.2.3)$$

the commutation relation of the new dimensionless operators is now:

$$[\hat{z}, \hat{p}] = i, \quad (3.2.4)$$

where the subscript  $d$  has dropped. To further simplify the calculations, it is required that the coefficient of  $\hat{z}$  in (3.2.2) is unity, ie:

$$\frac{kz_s m}{p_s^2} = 1 \quad (3.2.5)$$

Finally, the dimensionless Hamiltonian for the system can be written:

$$\begin{aligned} \hat{\mathcal{H}}_d &= \frac{\hat{\mathcal{H}}}{\left(\frac{p_s^2}{m}\right)}, \\ &= \frac{\hat{p}^2}{2} - \hat{z}. \end{aligned}$$

The new form of the time-independent Schrödinger equation, (3.2.1), therefore is:

$$\begin{aligned} \left(\frac{\hat{p}^2}{2} - \hat{z}\right) |\Psi\rangle &= \alpha |\Psi\rangle, \\ \left(\frac{\hat{p}^2}{2} - (\hat{z} + \alpha)\right) |\Psi_\alpha\rangle &= 0, \end{aligned} \quad (3.2.4)$$

where  $\alpha = \frac{\mathcal{E}m}{p_s^2}$ . From here on, all variables are in this new dimensionless form. To simplify the discussion, write  $|\Psi_\alpha\rangle = |\alpha\rangle$ . It is known that, in coordinate space:

$$\begin{aligned} \langle q|\hat{p}|\Psi\rangle &= \int_{-\infty}^{+\infty} \langle q|\hat{p}|q'\rangle \langle q'|\Psi\rangle dq' \\ &= \int_{-\infty}^{+\infty} \left(-i\frac{\partial}{\partial q}\right) \delta(q-q') \Psi(q') dq' = -i\frac{\partial}{\partial q} \Psi(q). \\ \langle q|\hat{p}^2|\Psi\rangle &= \int_{-\infty}^{+\infty} \langle q|\hat{p}|q'\rangle \int_{-\infty}^{+\infty} \langle q'|\hat{p}|q''\rangle \langle q''|\Psi\rangle dq' dq'' \\ &= \int_{-\infty}^{+\infty} \langle q|\hat{p}|q'\rangle \left(-i\frac{\partial}{\partial q'}\right) \Psi(q') dq', \\ &= \int_{-\infty}^{+\infty} \left(-i\frac{\partial}{\partial q}\right) \delta(q-q') \left(-i\frac{\partial}{\partial q'}\right) \Psi(q') dq', \\ &= \left(-i\frac{\partial}{\partial q}\right)^2 \Psi(q) = -\frac{\partial^2}{\partial q^2} \Psi(q). \end{aligned}$$

The coordinate space solution to the energy-eigenfunction equation is found by acting on the left of (3.2.4) with the bra  $\langle z|$ :

$$\begin{aligned} \langle z| \left( \frac{\hat{p}^2}{2} - (z + \alpha) \right) |\alpha\rangle &= 0, \\ - \left( \frac{1}{2} \frac{\partial^2}{\partial z^2} + (z + \alpha) \right) \langle z|\alpha\rangle &= 0. \end{aligned} \quad (3.2.5)$$

(3.2.5) is a second order differential equation but this can be simplified by transforming the problem into momentum space. The method is as follows.

The following relation exists in Quantum Mechanics:

$$\begin{aligned} e^{i\alpha\hat{p}}\hat{z}e^{-i\alpha\hat{p}} &= \hat{z} + i\alpha[\hat{p}, \hat{z}], \\ &= \hat{z} + \alpha. \end{aligned}$$

See [32], pg. 127. Acting on the left of (3.2.4) with  $e^{-i\alpha\hat{p}}$  and using this relation:

$$\begin{aligned} e^{-i\alpha\hat{p}} \left( \frac{\hat{p}^2}{2} - e^{i\alpha\hat{p}}\hat{z}e^{-i\alpha\hat{p}} \right) |\alpha\rangle &= 0, \\ \left( \frac{\hat{p}^2}{2} - \hat{z} \right) e^{-i\alpha\hat{p}} |\alpha\rangle &= 0. \end{aligned}$$

Using the identity operator,  $\hat{I} = \int |p'\rangle\langle p'| dp'$ , and acting on the left with the bra  $\langle p|$ , (3.2.5) becomes:

$$\int_{-\infty}^{+\infty} \langle p| \left( \frac{\hat{p}^2}{2} - \hat{z} \right) |p'\rangle\langle p'| e^{-i\alpha\hat{p}} |\alpha\rangle dp' = 0. \quad (3.2.6)$$

As this equation is now in momentum space, the following relations result:

$$\langle p|\hat{p}^2|p'\rangle = p^2\langle p|p'\rangle = p^2\delta(p - p'), \quad (3.2.7a)$$

$$\langle p|\hat{z}|p'\rangle = i\frac{\partial}{\partial p}\langle p|p'\rangle = i\frac{\partial}{\partial p}\delta(p - p'). \quad (3.2.7b)$$

(3.2.6) now becomes:

$$\left( \frac{p^2}{2} - i\frac{\partial}{\partial p} \right) \int_{-\infty}^{+\infty} \delta(p - p')\langle p'|e^{-i\alpha\hat{p}}|\alpha\rangle dp' = 0,$$

which simplifies further to:

$$\left( \frac{p^2}{2} - i\frac{\partial}{\partial p} \right) \langle p|e^{-i\alpha\hat{p}}|\alpha\rangle = 0,$$

which is a first order differential equation. Solving this:

$$\langle p|e^{-i\alpha\hat{p}}|\alpha\rangle = e^{-i\alpha p}\langle p|\alpha\rangle = A \exp\left(-i\frac{p^3}{6}\right).$$

Thus

$$\langle p|\alpha\rangle = A \exp\left[i\left(\alpha p - \frac{p^3}{6}\right)\right],$$

where  $A$  is the normalisation constant, determined as follows. As stated earlier, the system is normalised to unity. That is written, in momentum space, as:

$$\int_{-\infty}^{+\infty} \Psi^*(p, t)\Psi(p, t) dp = 1. \quad (3.2.8)$$

Recalling the identity operator from above and the labelling condition on the states, the condition (3.2.8) can be rewritten as:

$$\langle \alpha|\alpha'\rangle = \delta(\alpha - \alpha'), \quad (3.2.9)$$

which can be expanded as follows:

$$\begin{aligned} \langle \alpha|\alpha'\rangle &= \int_{-\infty}^{+\infty} \langle \alpha|p\rangle\langle p|\alpha'\rangle dp, \\ &= \int_{-\infty}^{+\infty} \langle p|\alpha\rangle^*\langle p|\alpha'\rangle dp, \\ &= \int_{-\infty}^{+\infty} A^* \exp\left[-i\left(\alpha p - \frac{p^3}{6}\right)\right] A \exp\left[i\left(\alpha' p - \frac{p^3}{6}\right)\right] dp, \\ &= |A|^2 \int_{-\infty}^{+\infty} e^{ip(\alpha' - \alpha)} dp, \\ &= |A|^2 (2\pi\delta(\alpha - \alpha')). \end{aligned} \quad (3.2.10)$$

By comparing (3.2.9) and (3.2.10), the following can be seen:

$$\begin{aligned} 2\pi|A|^2\delta(\alpha - \alpha') &= \delta(\alpha - \alpha'), \\ |A|^2 &= \frac{1}{2\pi}, \\ \Rightarrow A &= \frac{1}{\sqrt{2\pi}}. \end{aligned}$$

An energy eigenfunction in coordinate space is  $\langle z|\alpha\rangle$ . To determine the form of this wave function, the Fourier Transformation of the momentum space wave function is found – note that  $\langle z|p\rangle = \frac{e^{ipz}}{\sqrt{2\pi}}$ :

$$\begin{aligned}\langle z|\alpha\rangle &= \int_{-\infty}^{+\infty} \langle z|p\rangle \langle p|\alpha\rangle dp, \\ &= \int_{-\infty}^{+\infty} \left( \frac{e^{ipz}}{\sqrt{2\pi}} \right) \left( \frac{\exp\left[i\left(\alpha p - \frac{p^3}{6}\right)\right]}{\sqrt{2\pi}} \right) dp, \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[i\left(p(z + \alpha) - \frac{p^3}{6}\right)\right] dp, \tag{3.2.11a}\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left[i\left(\xi u - \frac{u^3}{3}\right)\right] \cdot 2^{\frac{1}{3}} du, \tag{3.2.11b}$$

where  $\xi = 2^{\frac{1}{3}}(z + \alpha)$  and  $u = p/2^{\frac{1}{3}}$ . Notice that (3.2.11b) can be written as:

$$\langle z|\alpha\rangle = \frac{2^{\frac{1}{3}}}{\sqrt{\pi}} \Phi\left(-2^{\frac{1}{3}}(z + \alpha)\right), \tag{3.2.12}$$

where  $\Phi(\xi)$  is the Airy Function and is defined as:

$$\Phi(\xi) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \cos\left(u\xi + \frac{u^3}{3}\right) du, \tag{3.2.13}$$

where the relation  $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$  was used. This function,  $\Phi(\xi)$ , is the Airy  $Ai(\xi)$  function. The alternative solution to this equation is the Airy  $Bi(\xi)$  function. This is not used as it diverges as  $\xi \rightarrow \infty$  (see [33] and [34] for further information on this specific Airy Function).

Consider now a potential  $V = kz$ . Via the method outlined above, the dimensionless Schrödinger equation is:

$$\hat{\mathcal{H}}|\Psi\rangle = \left(\frac{\hat{p}^2}{2} + \hat{z}\right)|\Psi\rangle = \beta|\Psi\rangle,$$

where  $\beta$  is used to denote the energy of the particle and distinguish this case from that studied above. Note that  $\beta = \alpha$ . Following the same method as above, the momentum space wave function is:

$$\langle p|\beta\rangle = \frac{1}{\sqrt{2\pi}} \exp\left[i\left(\beta p + \frac{p^3}{6}\right)\right],$$

and the coordinate space wave function is:

$$\begin{aligned}\langle z|\beta\rangle &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left[ i \left( p(z + \beta) + \frac{p^3}{6} \right) \right] dp, \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left[ i \left( \zeta v + \frac{v^3}{3} \right) \right] \cdot 2^{\frac{1}{3}} dv,\end{aligned}$$

where  $\zeta = 2^{\frac{1}{3}}(z + \beta)$  and  $v = p/2^{\frac{1}{3}}$ .

$$\langle z|\beta\rangle = \frac{2^{\frac{1}{3}}}{\sqrt{\pi}} \Phi \left( +2^{\frac{1}{3}}(z + \beta) \right), \quad (3.2.14)$$

with  $\Phi(\xi)$  defined in (3.2.13).

### §3.2.1 Asymptotic Expansion

Recall from (3.2.11a) the wave function in coordinate space:

$$\Psi_\alpha(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left[ i \left( p(z + \alpha) - \frac{p^3}{6} \right) \right] dp, \quad (3.2.15)$$

where  $\alpha = \mathcal{E}m/p_s^2 = \mathcal{E}m/(kz_s m) = \mathcal{E}/k$ . Here the scaling factor  $z_s$  is arbitrary and so may be set to one. (3.2.15) is rewritten:

$$\Psi_\mathcal{E}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left\{ i \left[ p \left( z + \frac{\mathcal{E}}{k} \right) - \frac{p^3}{6} \right] \right\} dp,$$

Consider the transformation  $z \rightarrow z + a$ . Under this transformation, the wave function becomes:

$$\begin{aligned} \Psi_\mathcal{E}(z + a) &= \int_{-\infty}^{+\infty} \exp \left\{ i \left[ p \left( z + a + \frac{\mathcal{E}}{k} \right) - \frac{p^3}{6} \right] \right\} dp, \\ &= \int_{-\infty}^{+\infty} \exp \left\{ i \left[ p \left( z + \frac{\mathcal{E} + ka}{k} \right) - \frac{p^3}{6} \right] \right\} dp = \Psi_{\mathcal{E} + ka}(z), \\ \Psi_\mathcal{E}(z + a) &= \Psi_{\mathcal{E} + ka}(z) \end{aligned} \quad (3.2.16)$$

It is only necessary to study one specific eigenfunction as all others may be determined by this relation.

The asymptotic expansion of the Airy function as  $|z| \rightarrow \infty$  is found using the method described in [35], §5.6. Choose  $\mathcal{E} = 0$ :

$$\Psi_0(z) = \int_{-\infty}^{+\infty} \exp \left[ i \left( pz - \frac{p^3}{6} \right) \right] dp.$$

Note that this is not a ground state as  $V(z)$  may have any value between  $\infty$  and  $-\infty$ . Rewrite the exponent as  $u = pz - \frac{p^3}{6}$ . This is an analytic function of  $p$ . That is at every point on the complex plane,  $p_0$ , the function has a Taylor Expansion which converges. The



function  $e^{iu(p)}$  oscillates rapidly over  $p$  as  $z$  becomes infinitely large. These oscillations will cancel out except at points where  $\partial u/\partial p$  is close to or equal to zero:

$$\begin{aligned}\frac{\partial u}{\partial p} &= z - \frac{p^2}{2} = 0, \\ \Rightarrow p_0 &= \pm\sqrt{2z}.\end{aligned}$$

where the solutions to the above equation are denoted  $p_0$ .

Take now the Taylor expansion of  $u(p)$  in the neighbourhood of  $p_0$ , stopping at the second order term:

$$u(p) = p_0 z - \frac{p_0^3}{6} - \frac{p_0}{2}(p - p_0)^2, \quad (3.2.17)$$

The wave function can now be rewritten as:

$$\Psi_0(z) = e^{i\left(p_0 z - \frac{p_0^3}{6}\right)} \int_C e^{-a(p-p_0)^2} dp, \quad (3.2.18)$$

where  $a = ip_0/2$ . By setting the path  $C$  upon which the integral is carried out to pass through  $p_0$  at an angle such that  $a(p - p_0)$  is real and positive and the magnitude of the integrand rapidly decreases from its maximum at  $p = p_0$ . This gives:

$$\int_C e^{-a(p-p_0)^2} dp \approx \int_{-\infty}^{+\infty} e^{-a\zeta^2} d\zeta = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \quad (3.2.19)$$

For values of  $z$  less than zero, the stationary phase points – where  $p - p_0 = 0$  – exist only on the imaginary plane. Choosing the path of integration to pass through  $p_0$  gives the following asymptotic form:

$$\Psi_0(z) \approx |z|^{\frac{1}{4}} \exp\left(-\frac{(2|z|)^{\frac{3}{2}}}{3}\right) \quad (z \rightarrow -\infty). \quad (3.2.20)$$

For  $z > 0$  there are two stationary points on the real axis due to the two possible values of  $p_0$ . The curve of integration should be chosen to cut the real axis at a  $45^\circ$  angle which ensures that  $a(p - p_0)^2$  is real and positive. The sum of the integrals at both points gives:

$$\Psi_0(z) \approx 2z^{-\frac{1}{4}} \cos\left\{\left[\frac{(2z)^{\frac{3}{2}}}{3} - \frac{\pi}{4}\right]\right\} \quad (z \rightarrow \infty). \quad (3.2.21)$$

We now have an expression for the behaviour of the Airy Ai function for large values of  $z$ . As can be seen, the function is oscillatory and has an overall tendency to zero as  $z^{1/4}$  tends to zero. All eigenfunction has such behaviour and may be considered to be identical, save for the difference in the energy value  $\mathcal{E}$ .

### §3.2.2 Airy Function

The Airy Function is named for George Airy, a nineteenth century British mathematician who studied solutions to the differential equation:

$$\frac{d^2\Phi(z)}{dz^2} - z\Phi(z) = 0.$$

The above form, (3.2.13), for the Airy Function is better denoted as the *Airy-Ai* function, one of two linearly independent solutions to the above differential equation. The other is the *Airy-Bi* function which has the form:

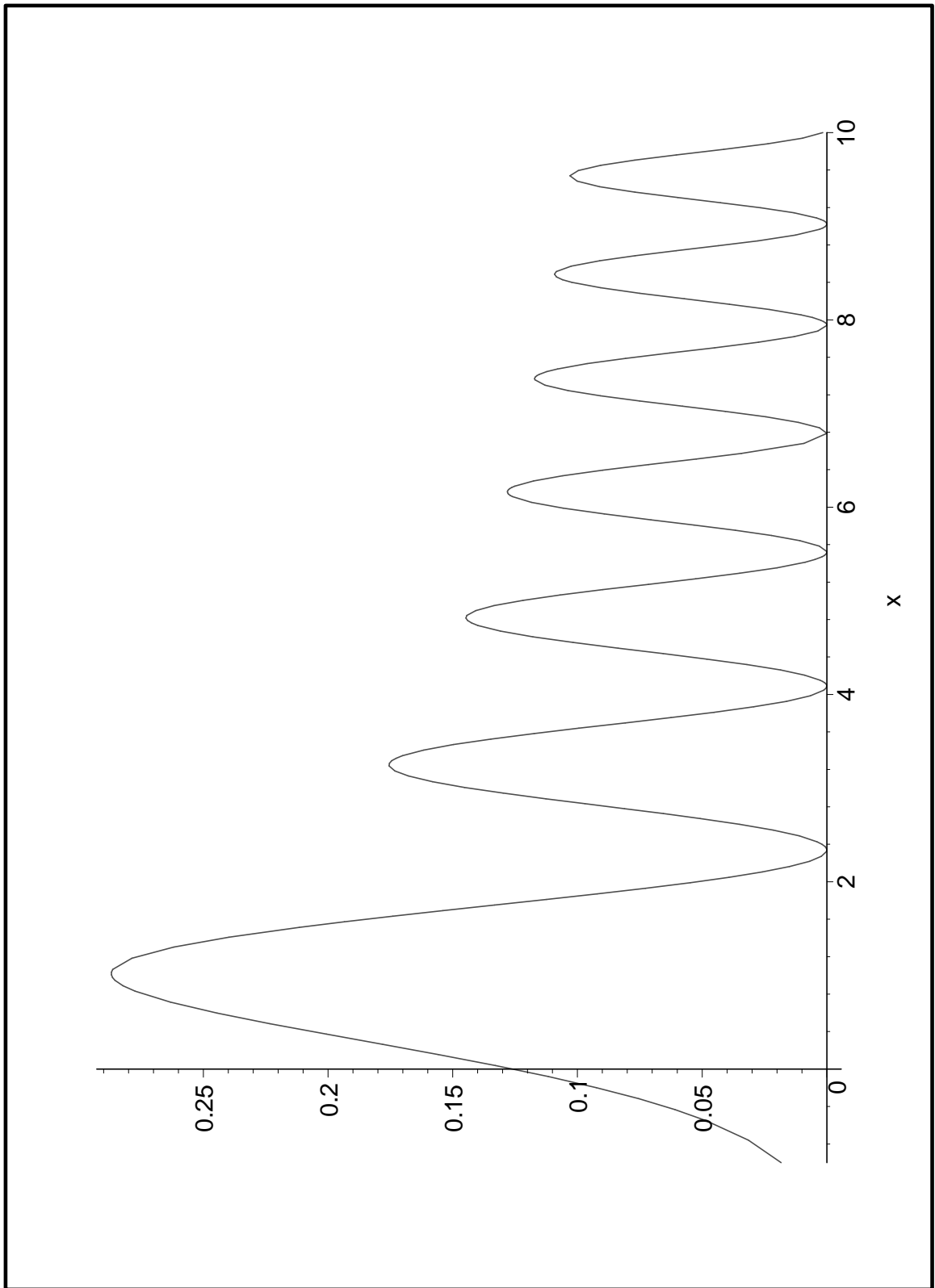
$$Bi(\xi) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(u\xi - \frac{u^3}{3}\right) du.$$

Fig.1 below represents the norm squared of the wave function found in §3.2 ,  $\Phi(-z)$ , and as such represents the probability of the particle existing at each point along the  $z$ -axis. The point at the centre of each “heap” can be viewed as the centre of a volume (or, as this plot is one-dimensional, a length) in which the particle is most likely to appear. The particle can be thought of jumping from one position to the next, in a similar fashion to the quantisation of atomic energy levels of the electron. As can also be seen, there are points around which it is highly unlikely to find the particle. It is these points that the particle is thought to jump past.

As the particle moves through the potential field and is being accelerated, the probability heaps decrease in height – implying that the probability that the particle occupies the volume about these points decreases. This is due to two effects.

The first is dispersion. The wave function or, more correctly, the density function of the particle, will dissipate due to the usual spreading of a wave packet over time. This effect is a well-known artefact of Quantum Mechanics.

The second effect is the acceleration the particle is undergoing along the  $z$ -axis. This means that at each point, in the positive  $z$ -axis direction, the particle is traveling faster than at the preceding point. Hence the particle exists at each increasing point for a shorter period of time. With this in mind, an examination of Fig.1 will show the acceleration of the particle which may not have been obvious at first.



*Fig.1* Plot of norm-squared of Airy Function,  $|\Phi(-x)|^2$ .

### §3.3 Solution of the Time-Dependent Schrödinger equation

The time-dependent Schrödinger equation is:

$$\hat{\mathcal{H}}\Psi(z, t) = i\frac{\partial}{\partial t}\Psi(z, t),$$

where in the system of a scalar particle in one-dimensional potential field, which has as the Hamiltonian operator  $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2} - \hat{z}$ , the time-dependent Schrödinger equation is:

$$\left(\frac{\hat{p}^2}{2} - \hat{z}\right)\Psi(z, t) = i\frac{\partial}{\partial t}\Psi(z, t).$$

The method to find the solution  $\Psi(z, t)$  is as follows. The dynamical solution to the time dependant Schrödinger equation for this system,  $\Psi(z, t)$ , can be expanded as follows:

$$\begin{aligned}\Psi(z, t) &= \langle z|e^{-i\hat{\mathcal{H}}t}|\Psi\rangle = \langle z|e^{-i\hat{\mathcal{H}}t}\hat{I}|\Psi\rangle, \\ &= \int_{-\infty}^{+\infty} \langle z|e^{-i\hat{\mathcal{H}}t}|z'\rangle \langle z'|\Psi\rangle dz', \\ &= \int_{-\infty}^{+\infty} \mathcal{G}(z, z', t)\Psi(z', 0) dz',\end{aligned}$$

where  $\mathcal{G}(z, z', t)$  is the propagator or evolution function for the system. The Green's Function is  $g(z, z', t) = \theta(t)\mathcal{G}(z, z', t)$ . The method is to assume that the propagator has the form:

$$\mathcal{G}(z, z', t) = A(t)e^{i\mathcal{S}(z, z', t)}, \quad (3.3.1)$$

where  $\mathcal{S}(z, t)$  is the action of the system, i.e. the time integral of the system's Lagrangian  $\left(\int_0^t \mathcal{L} dt\right)$ , and  $A(t)$  is some function of time. This assumption on the form of the propagator is shown to be true in the §A.1. The Green's Function method requires that:

$$\lim_{t \rightarrow 0} \mathcal{G}(z, z', t) = \delta(z - z').$$

Also,  $g(z, z', t)$  must satisfy the Schrödinger equation, with the following condition:

$$\left(\hat{\mathcal{H}} - i\frac{\partial}{\partial t}\right)g(z, z', t) = \delta(t)\delta(z - z'). \quad (3.3.2)$$

By assuming  $t \geq 0$  and  $z \neq z'$ , the usual form of the Schrödinger equation holds:

$$\begin{aligned}\hat{\mathcal{H}}\mathcal{G}(z, z', t) &= i\frac{\partial}{\partial t}\mathcal{G}(z, z', t), \\ -\frac{1}{2}\left(\frac{\partial^2}{\partial z^2} + 2z\right)\mathcal{G}(z, z', t) &= i\frac{\partial}{\partial t}\mathcal{G}(z, z', t).\end{aligned}\quad (3.3.3)$$

Using (3.3.1), the left-hand side of (3.3.3) becomes:

$$\begin{aligned}\hat{\mathcal{H}}\mathcal{G}(z, z', t) &= -\frac{1}{2}\left(\frac{\partial^2}{\partial z^2} + 2z\right)A(t)e^{i\mathcal{S}(z, z', t)}, \\ &= -\frac{1}{2}\left[i\frac{\partial^2}{\partial z^2}\mathcal{S}(z, z', t) - \left(\frac{\partial}{\partial z}\mathcal{S}(z, z', t)\right)^2 + 2z\right]A(t)e^{i\mathcal{S}(z, z', t)},\end{aligned}\quad (3.3.4)$$

while the right-hand side becomes:

$$\begin{aligned}i\frac{\partial}{\partial t}\mathcal{G}(z, z', t) &= \left(i\frac{\partial}{\partial t}A(t) - A(t)\frac{\partial}{\partial t}\mathcal{S}(z, z', t)\right)e^{i\mathcal{S}(z, z', t)}, \\ &= \left(i\frac{\partial}{\partial t}\ln A(t) - \frac{\partial}{\partial t}\mathcal{S}(z, z', t)\right)A(t)e^{i\mathcal{S}(z, z', t)}.\end{aligned}\quad (3.3.5)$$

Comparing the real and imaginary parts of (3.3.4) and (3.3.5) gives the following relations:

$$\frac{1}{2}\left(\frac{\partial}{\partial z}\mathcal{S}(z, z', t)\right)^2 - z + \frac{\partial}{\partial t}\mathcal{S}(z, z', t) = 0, \quad (3.3.6a)$$

$$\frac{1}{2}\frac{\partial^2}{\partial z^2}\mathcal{S}(z, z', t) + \frac{\partial}{\partial t}\ln A(t) = 0. \quad (3.3.6b)$$

(3.3.6a) is the Hamilton-Jacobi Equation — for this system — a discussion of which can be found in §3.3.1. The emergence of (3.3.6a) is very important as it shows that this method of using the exponential of the action as the Green's Function works.

There are now two differential equations determining  $\mathcal{G}(z, z', t)$ . The action,  $\mathcal{S}(z, z', t)$ , will be first determined and then, using (3.3.6b), the coefficient function,  $A(t)$ , will be calculated. For the system being studied, the Hamiltonian is:

$$\mathcal{H} = \frac{p^2}{2} - z.$$

From Classical Mechanics, the Hamiltonian is the sum of the Kinetic and Potential Energies of the system being studied. The Lagrangian is defined as Kinetic Energy less the Potential Energy and may be written as:

$$\mathcal{L}(q, \dot{q}) = m\dot{q}^2 - \mathcal{H}(p, q).$$

For the system under consideration, the Lagrangian is:

$$\Rightarrow \mathcal{L}(z, \dot{z}) = \frac{\dot{z}^2}{2} + z.$$

The action for the system is defined as the time integral of the Lagrangian, i.e:

$$\begin{aligned} \mathcal{S}(z, z', t) &= \frac{1}{2} \int_0^t (\dot{z}^2 + 2z) dt \\ &= \frac{1}{2} \left[ \int_0^t \left( \frac{dz}{dt} \right)^2 dt + 2 \int_0^t z dt \right]. \end{aligned} \quad (3.3.7a)$$

The Lagrangian equation of motion, derived from  $\mathcal{L}(z, \dot{z})$  above gives:

$$z(t) = \frac{1}{2}t^2 + vt + z',$$

where  $v$  is the initial velocity and  $z' = z(0)$ . For a more thorough discussion see [36].

By requiring the ratio of the force constant  $k$ , to the mass of the particle  $m$  be one, the acceleration is  $a = 1$  and so the classical path  $z(t)$  becomes:

$$z(t) = \frac{1}{2}t^2 + vt + z'.$$

From this it can be seen that:

$$v = \frac{z - z' - \frac{1}{2}t^2}{t}. \quad (3.3.7b)$$

Completing the integrals in (3.3.7) and eliminating  $v$  from the resulting equation via (3.3.7b), the action for the system is:

$$\mathcal{S}(z, z', t) = \left( \frac{1}{24t} \right) [12(z - z')^2 + 12(z + z')t^2 - t^4]. \quad (3.3.8)$$

This is a solution to the Hamilton-Jacobi equation, (3.3.6a). Using (3.3.6b),  $A(t)$  can be found, up to some normalisation constant, to be:

$$A(t) \approx \frac{1}{\sqrt{t}}.$$

$A(t)$  has been found exactly, see §A.1, for this system to be:

$$A(t) = \sqrt{\frac{1}{2\pi it}}. \quad (3.3.9)$$

Now that the action  $\mathcal{S}(z, z', t)$  and the amplitude function  $A(t)$  have been found, the propagator for this case can be written:

$$\mathcal{G}(z, z', t) = \sqrt{\frac{1}{2\pi it}} \exp \left[ \left( \frac{i}{24t} \right) (12(z - z')^2 + 12(z + z')t^2 - t^4) \right]$$

The Green's Function for the general solution is:

$$g(z, z', t) = \theta(t)\mathcal{G}(z, z', t),$$

where  $\theta(t)$  is the Heaviside Step Function. The derived Green's function also satisfies the necessary condition that:

$$\lim_{t \rightarrow 0^+} g(z, z', t) = \delta(z - z').$$

Note that this limit is taken from the right (i.e. values of  $t > 0$  tending toward zero). Thus, in the case of this limit,  $\theta(t) = 1$  and so  $g(z, z', t) = \theta(t)\mathcal{G}(z, z', t) = \mathcal{G}(z, z', t)$ . For all values of  $t$  less than zero,  $\theta(t) = 0$  and so the limit would also be zero. For this discussion, such times are ignored as it is defined that the propagation of the system begins at  $t = 0$ .

The solution now to the case of an accelerated scalar particle is of the form:

$$\Psi(z, t) = \int_{-\infty}^{+\infty} g(z, z', t)\Psi(z', 0) dz'$$

An initial state describing the particle for the system needs to be provided and inserted into the above integral to find a propagated state for the particle.

### §3.3.1 *Hamilton-Jacobi Theory*

As stated earlier, Quantum Mechanics has developed along similar lines to the classical Hamiltonian formalism of mechanics. This is due to the convenient manner in which the canonical coordinates and momenta can be carried over into Quantum Mechanics as conjugate operators. An alternative formalism for Quantum Mechanics can be based upon the Lagrangian formalism of Classical Mechanics. The Lagrangian formalism cannot be carried over into the quantum theory as efficiently as the Hamiltonian formalism but the main ideas can be, see [37].

The method of using the exponential of the action as the Green's Function to solve the system was first proposed by Dirac. Hamilton-Jacobi theory is concerned with canonical

transformations which have generating functions that solve the Hamilton-Jacobi Equation. Dirac's method, and ultimately that used in this project, works due to the emergence of the Hamilton-Jacobi Equation, (3.3.6a).

There are two major methods in classical mechanics, the Lagrangian Method and the Hamiltonian Method. Although both of these methods are closely related, the Hamiltonian is favoured as it is considered the more elementary, see [38]. The method of canonical transformations is an extension to the Hamiltonian method. When the Hamiltonian for a system has been defined, this definition will be related to the coordinate system used. However, sometimes another coordinate system may be more suitable for the system under consideration and a transformation to this alternative set of coordinates is performed. One reason to perform such a transformation is that one or more of the coordinates or momenta may be cyclic in the new coordinate system. This reduces the number of variables, simplifying the system. These transformations are carried out using Generating Functions, usually denoted  $\mathcal{F}$ . The transformation is carried out as follows. The new transformed Hamiltonian, denoted  $\mathcal{K}$ , is given by the equation:

$$\mathcal{K} = \mathcal{H} + \frac{\partial \mathcal{F}}{\partial t}, \quad (3.3.10)$$

where  $\mathcal{H}$  is the original Hamiltonian.  $\mathcal{H}$  is a function of the old set of coordinates and momenta,  $(q, p)$ , while  $\mathcal{K}$  is a function of the new set of coordinates and momenta,  $(Q, P)$ . There are four different types of generating function. A generating function of Type-1 is a function of the mixed coordinates  $(q, Q)$ , i.e.  $\mathcal{F}_1(q, Q, t)$ . In type-2, the function is  $\mathcal{F}_2(q, P, t)$ ; type-3,  $\mathcal{F}_3(p, Q, t)$ ; and finally type-4,  $\mathcal{F}_4(Q, P, t)$ .

The action for the system has been used as the generating function for a contact transformation. The action is a function of the initial coordinates,  $q = z'$ ; the final coordinates,  $Q = z$ ; and time, and as such is a generating function of Type-1, (i.e.  $\mathcal{F}_1(q, Q, t) \equiv \mathcal{S}(z, z', t)$ ). The initial and final momenta are given by the following formulæ:

$$p = \frac{\partial \mathcal{F}_1}{\partial q} = \frac{\partial}{\partial z'} \mathcal{S}(z, z', t),$$

$$P = -\frac{\partial \mathcal{F}_1}{\partial Q} = -\frac{\partial}{\partial z} \mathcal{S}(z, z', t).$$

The method of the Hamilton-Jacobi theory is to require that the new Hamiltonian,  $\mathcal{K}$ , is identically zero, (i.e.  $K(Q, P, t) = 0$ ). This ensures that there is a simple relationship



between the action and the Hamiltonian, the Hamiltonian is the time derivative of the action. Therefore (3.3.10) becomes:

$$\begin{aligned}
\mathcal{K} &= \mathcal{H} + \frac{\partial}{\partial t} \mathcal{S}(z, z', t) \\
&= \frac{p^2}{2} - z + \frac{\partial}{\partial t} \mathcal{S}(z, z', t), \\
&= \frac{1}{2} \left( \frac{\partial \mathcal{S}}{\partial z} \right)^2 - z + \frac{\partial \mathcal{S}}{\partial t}, \\
&= 0, \\
\Rightarrow \frac{1}{2} \left( \frac{\partial \mathcal{S}}{\partial z} \right)^2 - z + \frac{\partial \mathcal{S}}{\partial t} &= 0,
\end{aligned}$$

where the original momenta  $p$  was replaced by  $\frac{\partial \mathcal{S}}{\partial z}$ . This equation is exactly the Hamilton-Jacobi equation, (3.3.6a). The method then involves solving this equation for  $\mathcal{S}(z, z', t)$  and then finding the required partial derivatives;  $p = \partial \mathcal{S} / \partial z'$  and  $P = -\partial \mathcal{S} / \partial z$ .

### §3.4 Examples of Initial States

In this section, two examples of initial states are to be given, a Gaussian state and a plane-wave state. The calculations are first performed in the coordinate space and the corresponding propagated wave function,  $\Psi(z, t)$  is found for each state.

The general definition of a Gaussian state is:

$$\Psi(z, 0) = \frac{e^{ip_0(z-a)}}{\sqrt{\sqrt{2\pi}\sigma}} e^{-\frac{(z-a)^2}{4\sigma^2}},$$

where  $z = a$  is the position around which the initial state is centered - i.e. most likely to find the initial state in a neighbourhood of this point. In order to simplify the calculations in this project, the initial position is set to zero to simplify the calculations. The general definition of a plane-wave state is:

$$\Psi(q, t) = Ae^{-i(p \cdot q - \mathcal{E}t)}$$

where  $A$  is some constant, typically a normalisation constant, and  $\mathcal{E}$  is the (initial) energy of the state.

From the propagated wave function the density function,  $\rho(z, t)$  will be calculated. The Fourier Transform of the initial state  $\Psi(z', 0)$  will then be used as the initial state for the momentum space, (i.e.  $\tilde{\Psi}(p', 0)$ ).<sup>5</sup> As in the coordinate space, the propagated wave function,  $\tilde{\Psi}(p, t)$  and the density function,  $\tilde{\rho}(p, t)$ , will be found.

#### §3.4.1 Coordinate Space

The initial Gaussian State is:

$$\Psi(z', 0) = \frac{e^{ip_0 z'}}{\sqrt{\sqrt{2\pi}\sigma}} e^{-\frac{z'^2}{4\sigma^2}}. \quad (3.4.1)$$

---

<sup>5</sup> The symbols  $\tilde{\Psi}$  and  $\tilde{\rho}$  are used to highlight the fact that the momentum space wave function is not the same as the coordinate space wave function.

The Gaussian state is used as it is a Minimum Uncertainty State. That is the Gaussian state minimises the uncertainty principle, i.e. rather than an inequality the uncertainty relation is an equality (see [30], pg. 111 – 114):

$$\delta(z)\delta(p) = \frac{1}{2}.$$

For this reason, the chosen Gaussian states reduce the uncertainty effects of quantum mechanics (e.g. dispersion) to make the system similar to its classical analogue.

The required wave function is given by the following integral (as derived in §3.3):

$$\Psi(z, t) = \int_{-\infty}^{+\infty} g(z, z', t) \Psi(z', 0) dz', \quad (3.4.2)$$

where:

$$\begin{aligned} g(z, z', t) &= \theta(t) \langle z | e^{-i\hat{T}t} | z' \rangle, \\ &= \theta(t) A(t) e^{iS(z, z', t)}, \end{aligned}$$

with  $S(z, z', t)$  and  $A(t)$  as given earlier in (3.3.8) and (3.3.9). Assume, for simplicity, that  $t \geq 0$ . Then  $\theta(t) = 1$ . The integral (3.4.2) is evaluated as follows:

$$\begin{aligned} \Psi(z, t) &= \frac{A(t)}{\sqrt{\sqrt{2\pi}\sigma}} \int_{-\infty}^{+\infty} e^{iS(z, z', t)} e^{ip_0 z'} e^{-\frac{z'^2}{4\sigma^2}} dz', \\ &= \frac{A(t)}{\sqrt{\sqrt{2\pi}\sigma}} \int_{-\infty}^{+\infty} e^{az'^2 + bz' + c} dz', \end{aligned}$$

where:

$$\begin{aligned} a &= -\frac{t - 2i\sigma^2}{4t\sigma^2}, \\ b &= i \left( \frac{t}{2} - \frac{z}{t} + p_0 \right), \\ c &= i \frac{12z^2 + 12zt^2 - t^4}{24t}. \end{aligned}$$

The Gaussian integral is:

$$\int_{-\infty}^{+\infty} e^{at^2 + bt + c} dt = \sqrt{-\frac{\pi}{a}} e^{c - \frac{b^2}{4a}}. \quad (3.4.3)$$

Completing the square, using (3.4.3), and recalling that  $A(t) = \frac{1}{\sqrt{2\pi it}}$ , the wave function becomes:

$$\Psi(z, t) = \sqrt{\frac{2\sigma}{\sqrt{2\pi}(it + 2\sigma^2)}} e^{c - \frac{b^2}{4a}}, \quad (3.4.4)$$

with  $a$ ,  $b$  and  $c$  as above. This wave function must satisfy the Schrödinger Equation:

$$\hat{\mathcal{H}}\Psi(z, t) = i \frac{\partial}{\partial t} \Psi(z, t),$$

where:

$$\hat{\mathcal{H}} = -\frac{1}{2} \frac{\partial^2}{\partial z^2} - z.$$

Using the REDUCE mathematical programme, it has been shown that this wave function obeys the Schrödinger equation, see §A.5.8 for the input file nonrelschro.red and §A.5.9 for the output file nonrelschro.out.

Before moving on to the density function for this state, the expression  $b$  above is of particular importance. The expression  $b$  can be rewritten as follows:

$$\begin{aligned} b &= i \left( \frac{t}{2} - \frac{z}{t} + p_0 \right), \\ &= i \left( \frac{t^2}{2} + p_0 t - z \right) / t, \\ &= i (z_{cl}(t) - z) / t, \end{aligned}$$

where  $z_{cl}(t)$  is the classical path of the particle and  $z$  is the usual position in space (limited to the  $z$ -axis). This result shows that the quantum state follows the classical path as it propagates through space, as will be discussed in further detail in §2.4.2.

The density function,  $\rho(z, t)$ , for this state is:

$$\begin{aligned} \rho(z, t) &= |\Psi(z, t)|^2, \\ &= \Psi^*(z, t) \Psi(z, t), \\ &= \frac{2\sigma}{\sqrt{2\pi(t^2 + 4\sigma^4)}} \exp \left[ c^* - \left( \frac{b^2}{4a} \right)^* + c - \frac{b^2}{4a} \right], \\ &= \frac{2\sigma}{\sqrt{2\pi(t^2 + 4\sigma^4)}} \exp \left[ \frac{-2t^2\sigma^2}{(t^2 + 4\sigma^4)} \left( \frac{t}{2} - \frac{z}{t} + p_0 \right)^2 \right]. \end{aligned}$$

or when the second form for  $b$  derived above is used:

$$\rho(z, t) = \frac{2\sigma}{\sqrt{2\pi(t^2 + 4\sigma^4)}} \exp \left[ -\frac{2\sigma^2}{t^2 + 4\sigma^4} (z_{cl}(t) - z)^2 \right]. \quad (3.4.5)$$

The plane-wave case has also been analysed. The initial plane-wave state is:

$$\Phi(z', 0) = \frac{1}{\sqrt{2\pi}} e^{ip_0 z'},$$

where the coefficient is a normalisation constant. The propagated wave function is given by:

$$\Phi(z, t) = \int_{-\infty}^{+\infty} g(z, z', t) \Phi(z', 0) dz',$$

and is evaluated to give:

$$\Phi(z, t) = \frac{1}{\sqrt{2\pi}} \exp \left( i \left[ \left( z - \frac{p_3 t}{2} \right) (t + p_3) - \frac{t^3}{6} \right] \right).$$

Note that it is assumed that  $t \geq 0$ , thus  $\theta(t) = 1$ . Again, using REDUCE, this wave function has been found to satisfy the Schrödinger equation. See §A.5.6 for the input file nonrelplaneschro.red and §A.5.7 for the output file nonrelplaneschro\_out.

The density function for this state is:

$$\rho_{\Phi}(z, t) = \frac{1}{2\pi},$$

which is constant, since the exponent of  $\Phi(z, t)$  is purely imaginary. This density function is virtually meaningless due to the nature of the plane-wave. The momenta of plane-waves are exactly known, implying that the position of this state is infinitely uncertain. The probability of finding the state at any point is a constant value, meaning that the state is spread over all space.

### §3.4.2 Comparison to Classical System

To demonstrate that the propagated Gaussian state follows the classical path, it will suffice to show that the density function is maximised on the classical path. The classical equivalent to the system of a particle being accelerated in a one dimensional field is solved as follows. The Lagrangian for the system was given earlier, in §3.2, as:

$$\mathcal{L}(z, \dot{z}) = \frac{\dot{z}^2}{2} + z,$$

and by solving the Lagrangian for  $z(t)$ , the path of the particle, gives:

$$z(t) = \frac{t^2}{2} + vt + z',$$

where  $\dot{z}(0) = v$  is the initial velocity and  $z(0) = z'$  is the initial position of the particle. As the system is in a dimensionless prescription, the initial velocity and the initial momentum are identical, (i.e.  $v = p_0$ ).

The density function for the coordinate space wave function, the Gaussian state  $\Psi(z, t)$ , is:

$$\begin{aligned}\rho(z, t) &= \frac{2\sigma}{\sqrt{2\pi(t^2 + 4\sigma^2)}} \exp \left[ \frac{-2t^2\sigma^2}{(t^2 + 4\sigma^4)} \left( \frac{t}{2} - \frac{z}{t} + p_0 \right)^2 \right] \\ &= \frac{2\sigma}{\sqrt{2\pi(t^2 + 4\sigma^4)}} \exp \left[ -\frac{\sigma^2 (t^4 - 4t^2z + 4t^3p_0 + 4z^2 - 8tzp_0 + 4t^2p_0^2)}{2(t^2 + 4\sigma^4)} \right].\end{aligned}$$

Substituting the classical path  $z_{cl}$  for  $z$  in the exponent of the density function gives:

$$\rho_{cl} \equiv \rho(z_{cl}, t) = \frac{2\sigma}{\sqrt{2\pi(t^2 + 4\sigma^4)}} \exp \left[ -\frac{2\sigma^2 z'^2}{t^2 + 4\sigma^4} \right]$$

If the path is varied, for instance by varying the powers of  $t$ , it can be seen that this function  $\rho_{cl}$  gives the maximum values for the density function. One can be further convinced of this by setting the initial point of the particle to zero:

$$z' = 0 \quad \Rightarrow \quad \rho_{cl} = \frac{2\sigma}{2\pi\sqrt{t^2 + 4\sigma^4}}.$$

As the exponent is negative, the maximum possible value is zero which results here.

The case of an accelerated plane-wave state,  $\Phi(z, t)$ , is also analysed. The symbol  $\Phi$  has no special significance and is only used here to distinguish the plane-wave state from the Gaussian state. The propagated plane-wave is:

$$\Phi(z, t) = \frac{1}{\sqrt{2\pi}} \exp \left( i \left[ \left( z - \frac{p_3 t}{2} \right) (t + p_3) - \frac{t^3}{6} \right] \right)$$

This wave function has been shown earlier to satisfy the Schrödinger equation. As was seen, the right hand side of the Schrödinger equation is:

$$\left( i \frac{\partial}{\partial t} \Phi(z, t) \right) / \Phi(z, t) = \frac{(p_3 + t)^2}{2} - z.$$

Substituting the classical path for  $z$  as above, the right hand side of the Schrödinger equation becomes:

$$\left( i \frac{\partial}{\partial t} \Phi(z, t) \right) / \Phi(z, t) = \frac{1}{2} p_3^2 - z',$$

which is the initial energy of the system. This means that on the classical path the energy of the system is conserved which is true of the classical system. So while the density function for the plane-wave state is effectively meaningless, the wave function itself does contain important information with regard to the physical system.

### §3.4.3 Momentum Space

An alternative method would be to carry out the equivalent calculations in the momentum space. In the momentum space, the evolution integral is the same after the proper transformations:

$$\tilde{\Psi}(p, t) = \int_{-\infty}^{+\infty} g(p, p', t) \tilde{\Psi}(p', 0) dp',$$

where the initial state is just the Fourier Transform of the Gaussian state as used in the coordinate space in the previous section:

$$\begin{aligned} \tilde{\Psi}(p', 0) &= \langle p' | \tilde{\Psi}(0) \rangle, \\ &= \int_{-\infty}^{+\infty} \langle p' | z' \rangle \langle z' | \Psi(0) \rangle dz', \\ &= \int_{-\infty}^{+\infty} \frac{e^{-ip'z'}}{\sqrt{2\pi}} \frac{e^{ip_0z'}}{\sqrt{\sqrt{2\pi}\sigma}} e^{-\frac{z'^2}{4\sigma^2}} dz', \\ &= \sqrt{\frac{1}{(\sqrt{2\pi})^3 \sigma}} \int_{-\infty}^{+\infty} \exp \left[ \left( \frac{-1}{4\sigma^2} \right) z'^2 + i(p_0 - p') z' \right] dz', \\ &= \sqrt{\frac{1}{(\sqrt{2\pi})^3 \sigma}} 4\pi\sigma^2 e^{-\sigma^2(p_0 - p')^2} = \sqrt{\frac{2\sigma}{\sqrt{2\pi}}} e^{-\sigma^2(p_0 - p')^2}. \end{aligned}$$

The solution is:

$$\begin{aligned} \tilde{\Psi}(p, t) &= \int_{-\infty}^{+\infty} g(p, p', t) \tilde{\Psi}(p', 0) dp', \\ &= \sqrt{\frac{2\sigma}{\sqrt{2\pi}}} \int_{-\infty}^{+\infty} \theta(t) \delta(p - p' - t) e^{-iW(p, t)} e^{-\sigma^2(p_0 - p')^2} dp'. \end{aligned}$$

$W(p, t)$  is found in §A.1 to be:

$$W(p, t) = \frac{t^3}{6} - \frac{pt^2 - p^2t}{2}.$$

As before, let  $\theta(t) = 1$ :

$$\begin{aligned}\tilde{\Psi}(p, t) &= \sqrt{\frac{2\sigma}{\sqrt{2\pi}}} \int_{-\infty}^{+\infty} \delta(p - p' - t) e^{-iW(p, t)} e^{-\sigma^2(p_0 - p')^2} dp', \\ &= \sqrt{\frac{2\sigma}{\sqrt{2\pi}}} e^{-iW(p, t)} e^{-\sigma^2(p_0 - p + t)^2}.\end{aligned}\quad (3.4.6)$$

If this solution is correct, then the wave function must satisfy the Schrödinger Equation:

$$\hat{\mathcal{H}}\tilde{\Psi}(p, t) = i\frac{\partial}{\partial t}\tilde{\Psi}(p, t),$$

where  $\hat{\mathcal{H}} = \frac{p^2}{2} - i\frac{\partial}{\partial p}$ :

$$\begin{aligned}i\frac{\partial}{\partial t}\tilde{\Psi}(p, t) &= \tilde{\Psi}(p, t) \left[ (i)(-i)\frac{\partial}{\partial t}W(p, t) + (i)(-1)\frac{\partial}{\partial t}\sigma^2(p_0 - p + t)^2 \right], \\ &= \left( \frac{t^2}{2} - pt + \frac{p^2}{2} - 2i\sigma^2(p_0 - p + t) \right) \tilde{\Psi}(p, t), \\ -i\frac{\partial}{\partial p}\tilde{\Psi}(p, t) &= \tilde{\Psi}(p, t) \left[ (-i)(-i)\frac{\partial}{\partial p}W(p, t) + (-i)(-1)\frac{\partial}{\partial p}\sigma^2(p_0 - p + t)^2 \right], \\ &= \left( \frac{t^2}{2} - pt - 2i\sigma^2(p - t + p_0)^2 \right) \tilde{\Psi}(p, t).\end{aligned}$$

So:

$$\begin{aligned}\hat{\mathcal{H}}\tilde{\Psi}(p, t) &= \left( \frac{p^2}{2} - i\frac{\partial}{\partial p} \right) \tilde{\Psi}(p, t), \\ &= \left( \frac{p^2}{2} + \frac{t^2}{2} - pt - 2i\sigma^2(p - t + p_0)^2 \right) \tilde{\Psi}(p, t) \\ &= i\frac{\partial}{\partial t}\tilde{\Psi}(p, t).\end{aligned}$$

Therefore the derived expression for  $\tilde{\Psi}(p, t)$  satisfies the Schrödinger Equation.

The momentum-space density function,  $\tilde{\rho}(p, t)$  is given by:

$$\begin{aligned}\tilde{\rho}(p, t) &= |\tilde{\Psi}(p, t)|^2 \\ &= \tilde{\Psi}^*(p, t)\tilde{\Psi}(p, t), \\ &= \sqrt{\frac{2\sigma}{\sqrt{2\pi}}} e^{+iW(p, t)} e^{-\sigma^2(p_0 - p + t)^2} \sqrt{\frac{2\sigma}{\sqrt{2\pi}}} e^{-iW(p, t)} e^{-\sigma^2(p_0 - p + t)^2}, \\ &= \frac{2\sigma}{\sqrt{2\pi}} e^{-2\sigma^2(p_0 - p + t)^2}.\end{aligned}\quad (3.4.7)$$



Note that the exponent can be rewritten as:

$$-2\sigma^2(p_0 - p + t)^2 = -2\sigma^2(p_0 + t - p)^2 = -2\sigma^2(p_{cl}(t) - p(t))^2,$$

where  $p_{cl}(t) = p_0 + t$  is the classical path. Again this demonstrates that the quantum mechanical system - i.e. the state - is centered on the classical path. That is, the most likely volume in which to find the particle at any particular time is centered on the classical path.

As can be seen from the coefficient for  $\tilde{\rho}(p, t)$ , the momentum space wave function does not spread or dissipate. In addition to the fact that the momentum space Schrödinger equation being a first order differential equation, this result shows that the problem is much simpler in momentum space than in the coordinate space.

### §3.5 *Summary*

This chapter has been a discussion of the case of a linearly accelerated particle in Non-Relativistic Quantum Mechanics. The time-independent Schrödinger equation was studied and a solution found, the Airy function. For given initial states (e.g. plane-waves and Gaussian states), propagated wave functions were calculated using the Green's Function method, as described earlier. These states were all shown to be solutions to the Schrödinger equation. A comparison with the classical system was also made, it being shown that the quantum system follows the classical case.

Similar methods to those used here will be applied to the relativistic case, again using plane-waves and Gaussian states as the initial states for the system, in the next chapter. The discussion will follow as close as possible the layout of this chapter to enable direct comparison.

## Chapter 4 Relativistic Theory

### §4.1 Introduction

This chapter will describe the foundation of the Garavaglia Model and outline some arguments for its use over the Klein-Gordon equation. As described earlier, the Garavaglia Model is simply the covariant Schrödinger-type wave equation:

$$\hat{\mathcal{H}}\Psi = i\frac{\partial\Psi}{\partial s}, \quad (4.1.1)$$

where  $\hat{\mathcal{H}}$  is the Hamiltonian operator and  $s$  is the proper time of the particle (i.e. the time as measured in the reference frame of the particle, the proper frame) and the usual time (i.e. the observer's time),  $t$ , is regarded as a coordinate much like  $x$ ,  $y$  and  $z$ .

The system being studied here is that of a charged scalar particle in a potential field. The Hamiltonian for this system is:

$$\hat{\mathcal{H}} = \hat{p}_\mu\hat{p}^\mu + e\hat{\mathcal{A}}_\mu, \quad (4.1.2)$$

with  $e$  the charge on the particle,  $\mathcal{A}_\mu$  the four-potential field such that

$$e\mathcal{A}_\mu = (-V(z), 0, 0, 0).$$

$V(z) = -kz$  is a one-dimensional field along the  $z$ -axis.

In this chapter, the techniques developed in Chapter 3 for the Non-Relativistic Case will be applied to the Relativistic Case. As in Chapter 3, the natural units are used, i.e.  $\hbar = c = 1$ , to simplify the calculations involved. We are working in the Minkowski space, i.e. the metric has nonzero elements  $g_{\mu\nu} = (g_{00}, g_{11}, g_{22}, g_{33}) = (1, -1, -1, -1)$  and  $g_{\mu\nu} = g^{\mu\nu}$ .

The construction of the covariant Schrödinger-type wave equation makes use of the invariance of the square of the energy of the system. The Hamiltonian of the system is simply the total energy of the system, i.e.  $\mathcal{H} = \mathcal{E}$ . Using the usual correspondence principle:

$$\mathcal{E} \rightarrow i\frac{\partial}{\partial t},$$

where  $t \equiv ct$  is the time as measured in the observers reference frame leads to the Klein-Gordon equation, which is known to have a lack of a positive definite density function.

Four-position of the particle as seen by the observer is written as  $q = (q^0, \vec{q})$ , where  $q^0 = ct = t$  as natural units ( $\hbar = c = 1$ ) are being used and  $\vec{q}$  is the cartesian position. The four-position of the particle as seen in its own reference frame, the proper frame, is written as  $(cs, \vec{0}) = (s, \vec{0})$ . Four-velocity,  $u$ , is defined as:

$$u = \left( \frac{dq^0}{ds}, \frac{d\vec{q}}{ds} \right) = \left( \frac{dq^0}{ds}, \vec{\beta} \cdot \frac{dq^0}{ds} \right),$$

where  $\vec{\beta}$  is the conventional velocity vector as measured in the observers reference frame:

$$\vec{\beta} = \frac{d\vec{q}}{dq^0} = \frac{d\vec{q}}{dt}.$$

#### §4.2 Evolution Parameter-Independent Equation

The evolution parameter independent equation should be of similar form to the time-independent equation in the non-relativistic case discussed in §3.1, i.e.:

$$\hat{\mathcal{H}}\Psi = \mathcal{E}'\Psi.$$

where the energy has been denoted as  $\mathcal{E}'$  as its form is not known as yet.

In [1], the classical equation governing the system was found to be:

$$\begin{aligned} (\mathcal{E} - V(z))^2(1 - \dot{z}^2) &= m^2, \\ (\mathcal{E} + kz)^2(1 - \dot{z}^2) &= m^2, \end{aligned} \quad (4.2.1)$$

which becomes:

$$(\mathcal{E} + kz)^2(1 - [\dot{x}^2 + \dot{y}^2 + \dot{z}^2]) = m^2$$

upon inclusion of the  $(x, y)$ -plane. For  $z(0) = \dot{z}(0) = 0$  and with  $V(z) = -kz$ , the solution to (4.2.1) is:

$$z(t) = \frac{m}{k} \left( \sqrt{1 + \left(\frac{kt}{m}\right)^2} - 1 \right). \quad (4.2.2)$$

The quantum analogue of (4.2.1) is found as follows:

$$\begin{aligned} \hat{\mathcal{H}}\Psi(q) &= \frac{m}{2}\Psi(q), \\ \frac{(\hat{p} + e\hat{\mathcal{A}})^2}{2m}\Psi(q) &= \frac{m}{2}\Psi(q), \\ \frac{(\hat{p} + e\hat{\mathcal{A}})^2}{2}\Psi(q) &= \frac{m^2}{2}\Psi(q), \\ \left[ (\hat{p}_0 - V(\hat{z}))^2 - (\vec{\hat{p}})^2 \right] \Psi(q) &= m^2\Psi(q), \\ \left[ (\mathcal{E} + kz)^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \Psi(q) &= m^2\Psi(q), \end{aligned}$$

Thus, considering only the  $(t, z)$ -plane, this equation becomes:

$$\left( (\mathcal{E} + kz)^2 + \frac{\partial^2}{\partial z^2} \right) \Psi(q) = m^2 \Psi(q). \quad (4.2.3)$$

(4.2.3) is the covariant evolution parameter-independent equation. Via the substitutions:

$$\xi = \frac{1}{\sqrt{k}} (\mathcal{E} + kz), \quad \lambda = -\frac{m^2}{k},$$

(4.2.3) becomes:

$$\frac{d^2}{d\xi^2} \Psi(\xi) + (\xi^2 + \lambda) \Psi(\xi) = 0. \quad (4.2.4)$$

This equation is a parabolic cylinder equation with solution given by [20]:

$$\begin{aligned} \Psi(\xi) &= \mathcal{D}_{-\frac{1+i\lambda}{2}} [\pm(1+i)\xi], \\ \mathcal{D}_p [(1+i)\xi] &= \frac{2^{\frac{p+1}{2}}}{\Gamma(-\frac{p}{2})} \int_1^{+\infty} e^{-\frac{i}{2}\xi^2 x} \frac{(x+1)^{\frac{p-1}{2}}}{(x-1)^{1+\frac{p}{2}}} dx. \end{aligned}$$

#### §4.2.1 Asymptotic Expansion of Solution to Evolution Parameter-Independent Equation

The system has been studied in both the non-relativistic and relativistic cases with the equivalent methods applied in both. Here the solutions to the evolution parameter-independent equation (4.2.3) and the time-independent Schrödinger equation (3.2.5) are compared and it will be speculated that the solutions to these equations are linked.

The time-independent Schrödinger equation in the non-relativistic case was studied in §3.2 and found to have Airy Ai function as a solution. In the relativistic case, the evolution parameter-independent equation is:

$$\left[ (\mathcal{E} + kz)^2 - m^2 + \frac{\partial^2}{\partial z^2} \right] \Psi(q) = 0, \quad (4.2.5)$$

which, via the substitutions  $\xi = \frac{1}{\sqrt{k}} (\mathcal{E} + kz)$  and  $\lambda = -m^2/k$ , can be rewritten as:

$$\left[ \frac{d^2}{d\xi^2} + (\xi^2 + \lambda) \right] \Psi(\xi) = 0, \quad (4.2.6)$$

which is a parabolic cylinder equation. A further alteration can be made to this differential equation. Let  $x = \sqrt{2}\xi$  and  $a = -\lambda/2$ , then (4.2.4) becomes [34]:

$$\left[ \frac{d^2}{dx^2} + \left( \frac{1}{4}x^2 - a \right) \right] \Psi(x) = 0. \quad (4.2.7)$$

The important factor in this equation is the constant  $a = m^2/2k$ . It is this factor that is to be used to study the non-relativistic limit. In the non-relativistic limit, the rest mass energy of the particle is much greater than the mechanical energy – the sum of the Kinetic and Potential energies – that the system contains,  $m \gg k$ , and so  $a$  is very large. There are two linearly independent solutions to this differential equation:

$$\begin{aligned} y_1 &= 1 + a \frac{x^2}{2!} + (a^2 - 1) \frac{x^4}{4!} + (a^3 - \frac{7}{2}a) \frac{a^6}{6!} \\ &\quad + (a^4 - 11a^2 + \frac{15}{4}) \frac{x^8}{8!} + (a^5 - 25a^3 + \frac{211}{4}) \frac{x^{10}}{10!} + \dots, \\ y_2 &= x + a \frac{x^3}{3!} + (a^2 - \frac{3}{2}) \frac{x^5}{5!} + (a^3 - \frac{13}{2}a) \frac{x^7}{7!} \\ &\quad + (a^4 - 17a^2 + \frac{63}{4}) \frac{x^9}{9!} + (a^5 - 35a^3 + \frac{531}{4}) \frac{x^{11}}{11!} + \dots, \end{aligned}$$

with the coefficients  $b_n$  of  $x^n/n!$  given by the recursion formula:

$$b_{n+2} = a \cdot b_n - \frac{1}{4}n(n-1)b_{n-2}.$$

A combined solution to (4.7.6) can be given by the Weber function [34]:

$$W(a, \pm x) = \frac{(\cosh(\pi a))^{\frac{1}{4}}}{2\sqrt{\pi}} \left( |\Gamma(1/4 + (1/2)ia)| y_1 \mp |\Gamma(3/4 + (1/2)ia)| y_2 \right).$$

With  $a > 0$ , large, and  $x \in [0, \infty)$ , one specific expansion in terms of the Airy Ai function is possible:<sup>6</sup>

$$x = 2\sqrt{a}\xi, \quad t = (4a)^{\frac{2}{3}} \tau,$$

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<sup>6</sup> See [34], page 693.

$$\tau = - \left( \frac{3}{2} \theta_3 \right)^{\frac{2}{3}},$$

$$\theta_2 = \frac{1}{2} \int_{\xi}^1 \sqrt{s^2 - 1} ds = \frac{1}{4} \arccos(\xi) - \frac{1}{4} \xi \sqrt{1 - \xi^2},$$

$$\tau = + \left( \frac{3}{2} \theta_3 \right)^{\frac{2}{3}},$$

$$\theta_2 = \frac{1}{2} \int_1^{\xi} \sqrt{s^2 - 1} ds = \frac{1}{4} \xi \sqrt{1 - \xi^2} - \frac{1}{4} \arccos(\xi),$$

$$W(a, -x) \approx 2\sqrt{\pi} (4a)^{-\frac{1}{4}} e^{\frac{1}{2}\pi a} \left( \frac{t}{\xi^2 - 1} \right)^{\frac{1}{4}} Ai(-t).$$

It should be noted however that the variable  $t$  depends on  $a$  from (4.2.7) above and not  $x$  from the same equation. This expansion does not support the speculated link between the non-relativistic and relativistic solutions.

A link between the solutions to the evolution parameter independent Schrödinger-equation of the relativistic case and those of the time-independent Schrödinger equation of the non-relativistic case would be a welcome result. It would be expected that the solutions in the relativistic case would become the non-relativistic solutions when moving to the non-relativistic limit. Further examination of the non-relativistic limit is required to confirm whether or not this link exists. This involves further work which is not carried out in this thesis.



### §4.3 Dynamic Equation

The preceding section was concerned with deriving the form of the Hamiltonian and with the energy eigenfunction case. This section will now discuss the time-dependent, or dynamic, system.

The initial state is again a Gaussian state which, in the relativistic case, is:

$$\Psi(q, 0) = \frac{1}{2\pi(\sigma_0\sigma_1\sigma_2\sigma_3)^{\frac{1}{2}}} \exp\left(-ip(0) \cdot q - \frac{(q^0)^2}{4\sigma_0^2} - \frac{(q^1)^2}{4\sigma_1^2} - \frac{(q^2)^2}{4\sigma_2^2} - \frac{(q^3)^2}{4\sigma_3^2}\right)$$

The method to find the propagated wave function,  $\Psi(q, s)$ , is similar to that used in the Non-Relativistic case, via the evolution equation:

$$\Psi(q, s) = \int g(q, q', s) \Psi(q', 0) dq'.$$

The Green's Function for the covariant case is:

$$g(q, q', s) = \frac{-i\theta(s)e^{i(-B^2A+C)}e^{-i\frac{(x-x')^2}{2s}}e^{-i\frac{(y-y')^2}{2s}}}{8\pi^2s \sinh(s/2)}. \quad (4.3.1)$$

where:

$$\begin{aligned} A &\equiv A(s) = -\tanh\left(\frac{s}{2}\right), \\ B &\equiv B(t, t', z, z', s) = \frac{1}{2}\left(z + z' + (t - t') \coth\left(\frac{s}{2}\right)\right), \\ C &\equiv C(z, z', s) = \left(-\cosh(s)\frac{1}{2}(z^2 + z'^2) + zz'\right) \sinh^{-1}(s), \end{aligned}$$

(see [1] & [2]).

This problem can be separated into three independent parts. The initial Gaussian state can be rewritten as being the product of four states, each state being a Gaussian state depending on only one coordinate and time:

$$\Psi(q, 0) = \frac{1}{2\pi(\sigma_0\sigma_1\sigma_2\sigma_3)^{\frac{1}{2}}} \Psi_0(q^0, 0) \Psi_1(q^1, 0) \Psi_2(q^2, 0) \Psi_3(q^3, 0), \quad (4.3.2)$$

where:

$$\Psi_0(q^0, 0) = \exp \left[ +ip_0q^0 - \frac{(q^0)^2}{(4\sigma_0^2)} \right], \quad (4.3.3a)$$

$$\Psi_1(q^1, 0) = \exp \left[ -ip_1q^1 - \frac{(q^1)^2}{(4\sigma_1^2)} \right], \quad (4.3.3b)$$

$$\Psi_2(q^2, 0) = \exp \left[ -ip_2q^2 - \frac{(q^2)^2}{(4\sigma_2^2)} \right], \quad (4.3.3c)$$

$$\Psi_3(q^3, 0) = \exp \left[ -ip_3q^3 - \frac{(q^3)^2}{(4\sigma_3^2)} \right]. \quad (4.3.3d)$$

The Green's Function must also be broken up in a similar fashion. However, in this case, the Green's Function can only be written in the following form:

$$\mathcal{G}(q, q', s) = \frac{-i\theta(s)}{8\pi^2 s \sinh(s/2)} \mathcal{G}_{0,3}(t, t', z, z', s) \mathcal{G}_1(x, x', s) \mathcal{G}_2(y, y', s),$$

where:

$$\mathcal{G}_{0,3}(t, t', z, z', s) = e^{i(C-B^2A)},$$

$$\mathcal{G}_1(x, x', s) = e^{-i\frac{(x-x')^2}{2s}},$$

$$\mathcal{G}_2(y, y', s) = e^{-i\frac{(y-y')^2}{2s}},$$

with  $A$ ,  $B$  and  $C$  as above. The propagated states for the free particle cases, the  $x$  and  $y$  cases, will be found first.

The coefficients of both the initial Gaussian states and the Green's Function are ignored in this section as they have no bearing on the calculations involved in deriving the propagated states. They will be of importance later when it is checked if the wave function satisfies the covariant Schrödinger-type wave equation.

### §4.3.1 Derivation of Covariant Green's Function

The derivation of the covariant Green's function is outlined below, the method is that described in [2]. In the non-relativistic case the propagator for the system is a function of form:

$$\langle q|\hat{U}(t)|q'\rangle = A(t)e^{i\mathcal{S}(q,q',t)}.$$

In the relativistic case a similar function is used but some important changes must be made.

Due to the fact that all possible initial energies must be accounted for, i.e an ambiguity in the value of  $p^0$ , the propagator is assumed to have the form [2]:

$$\mathcal{G}(q, q', s) = \langle q|e^{i\hat{\mathcal{H}}s}|q'\rangle = a_0(s) \int_{-\infty}^{+\infty} e^{i\mathcal{S}(q,q',p^0,s)} dp^0, \quad (4.3.4)$$

The action for the complete system can be written as:

$$\mathcal{S}(q, q', p^0, s) = \mathcal{S}_0(z, z', p^0, s) - \frac{(x - x')^2}{2s} - \frac{(y - y')^2}{2s} - p^0(t - t').$$

The general form of the propagator can be rewritten as:

$$\langle q|e^{i\hat{\mathcal{H}}s}|q'\rangle = a_0(s) e^{-i\frac{(x-x')^2}{2s}} e^{-i\frac{(y-y')^2}{2s}} \int_{-\infty}^{+\infty} e^{i(\mathcal{S}_0(z,z',p^0,s) - p^0(t-t'))} dp^0. \quad (4.3.5)$$

The function  $\mathcal{S}_0(z, z', p^0, s)$  is defined as, see [2]:<sup>7</sup>

$$\mathcal{S}_0(z, z', p^0, s) = \frac{(p^0 - z) \cdot (p^0 + z) - 2p^0 z \cosh(s)}{\sinh(s)}.$$

The exponent in the integrand in (4.3.5) can be rewritten as:

$$\begin{aligned} i(\mathcal{S}_0(z, z', p^0, s) - p^0(t - t')) &= a_p (p^0)^2 + b_p p^0 + c_p, \\ &= a_p (p^0 + B)^2 - B^2 a_p + c_p, \end{aligned}$$

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<sup>7</sup> The following equation is derived from (15) in [2] using the relations (13) found in [2].

where  $B = b_p/(2a_p)$ . The coefficients are:

$$\begin{aligned} a_p &= -i \frac{\cosh(s) - 1}{\sinh(s)} = -i \tanh\left(\frac{s}{2}\right), \\ b_p &= i \frac{-(z + z')(\cosh(s) - 1) - \sinh(s)(t - t')}{\sinh(s)}, \\ c_p &= i \left( -\cosh(s) \frac{1}{2}(z^2 + z'^2) + zz' \right) \sinh^{-1}(s) \end{aligned}$$

Using the usual integral relation, (3.4.3), the exponent of the propagator is:

$$i\Phi(q, q', s) = c_p - \frac{b_p^2}{4a_p},$$

or:

$$i\Phi(q, q', s) = i(-B^2 A + C),$$

where:

$$\begin{aligned} A &= a_p/i, & C &= c_p/i, \\ B &= \frac{1}{2} \left( z + z' + (t - t') \coth\left(\frac{s}{2}\right) \right). \end{aligned}$$

Now incorporating the  $(x, y)$ -plane action terms into the exponent gives the propagator as:

$$\mathcal{G}(q, q', s) = a_0(s)a_1(s)e^{i(-B^2 A + C)} e^{-i\frac{(x-x')^2}{2s}} e^{-i\frac{(y-y')^2}{2s}},$$

with:

$$a_1(s) = \sqrt{\frac{\pi}{\tanh(s/2)}},$$

the coefficient resulting from the integration over  $p^0$  above.

The evolution function must satisfy the covariant Schrödinger-type wave equation:

$$\left[ \frac{1}{2} \left( -\frac{\partial^2}{\partial t^2} + z^2 + 2iz \frac{\partial}{\partial t} + \hat{\nabla}^2 - 2i \frac{\partial}{\partial s} \right) \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = 0. \quad (4.3.6)$$

The Green's function is of the form:

$$\mathcal{G}(q, q', s) \equiv a(s)e^{i\Phi}.$$

The constituent terms in (4.3.6) become:

$$\left[ -\frac{\partial^2}{\partial t^2} \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = -i \frac{\partial^2}{\partial t^2} \Phi + \left( \frac{\partial}{\partial t} \Phi \right)^2,$$

$$\left[ \frac{\partial^2}{\partial x^2} \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = i \frac{\partial^2}{\partial x^2} \Phi - \left( \frac{\partial}{\partial x} \Phi \right)^2,$$

$$\left[ \frac{\partial^2}{\partial y^2} \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = i \frac{\partial^2}{\partial y^2} \Phi - \left( \frac{\partial}{\partial y} \Phi \right)^2,$$

$$\left[ \frac{\partial^2}{\partial z^2} \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = i \frac{\partial^2}{\partial z^2} \Phi - \left( \frac{\partial}{\partial z} \Phi \right)^2,$$

$$\left[ 2iz \frac{\partial}{\partial t} \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = -2z \frac{\partial}{\partial t} \Phi,$$

$$\left[ -2i \frac{\partial}{\partial s} \mathcal{G}(q, q', s) \right] / \mathcal{G}(q, q', s) = -2i \frac{\partial}{\partial s} \ln(a(s)) + 2 \frac{\partial}{\partial s} \Phi.$$

which, after separating real and imaginary parts of the equation, yields two independent differential equations:

$$\frac{1}{2} \left[ \left( \frac{\partial}{\partial t} \Phi \right)^2 - \left( \frac{\partial}{\partial x} \Phi \right)^2 - \left( \frac{\partial}{\partial y} \Phi \right)^2 - \left( \frac{\partial}{\partial z} \Phi \right)^2 + z^2 \right] - z \frac{\partial}{\partial t} \Phi + \frac{\partial}{\partial s} \Phi = 0. \quad (4.3.7a)$$

$$\frac{1}{2} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) \Phi + \frac{\partial}{\partial s} \ln(a(s)) = 0. \quad (4.3.7b)$$

The form and derivation of (4.3.7) is identical to that of the (3.3.6). (4.3.7a) is the covariant equivalent of the classical Hamilton-Jacobi equation, (3.3.6a). (4.3.7b) is the equation that governs the coefficient function, in a similar fashion to (3.3.6b). See §3.3 for the non-relativistic equivalent of this discussion.

On substituting the above expression for  $\Phi(q, s)$  into (4.3.7b), the differential equation

is solved as follows:

$$\begin{aligned}
\frac{\partial}{\partial s} \ln(a(s)) &= - \left( \frac{1}{s} + \coth \left( \frac{s}{2} \right) / 2 \right), \\
\ln(a(s)) &= - \int \left( \frac{1}{s} + \coth \left( \frac{s}{2} \right) / 2 \right) ds, \\
&= - \left[ \ln(s) + \ln \left( \sinh \left( \frac{s}{2} \right) \right) \right] + c, \\
&= - \left[ \ln(s) + \ln \left( \sinh \left( \frac{s}{2} \right) \right) + \ln(n_0) \right], \\
&= - \ln \left( n_0 s \sinh \left( \frac{s}{2} \right) \right) \\
&= \ln \left( \frac{1}{n_0 s \sinh(s/2)} \right).
\end{aligned}$$

In the above, we have taken the constant of integration  $c$  as  $-\ln(n_0)$ , where  $n_0$  is the normalisation constant. Therefore:

$$a(s) = \frac{1}{n_0 s \sinh(s/2)}. \quad (4.3.8)$$

To find  $n_0$ , the following property of the Green's Function is used:

$$\lim_{s \rightarrow 0} \mathcal{G}(q, q', s) = \delta(q - q'), \quad (4.3.9a)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\mathcal{G}(q, q', s)|^2 dq ds = 1. \quad (4.3.9b)$$

From [21],  $n_0$  is  $8\pi^2$  and so the coefficient of the Green's Function is:

$$a(s) = \frac{1}{8\pi^2 s \sinh(s/2)}.$$

### §4.3.2 Free Particle Case

In this section, expressions will be found for the components of the wave-function in the  $(x, y)$ -plane,  $\Psi_1(x, s)$  and  $\Psi_2(y, s)$ , which are free particle systems as noted earlier. Here the coefficients of the initial states and Green's function are ignored and will be returned later. Notice that the corresponding Green's Functions and initial wave-functions are similar, so it suffices to find the evolved state for only one case and then make the correct substitutions to find the second case. The evolved state for the  $x$ -coordinate is found as follows:

$$\begin{aligned}\Psi_1(x, s) &= \int_{-\infty}^{+\infty} \mathcal{G}_1(x, s) \Psi(x', 0) dx', \\ &= \int_{-\infty}^{+\infty} \exp(A_x x'^2 + B_x x' + C_x) dx',\end{aligned}$$

where:

$$\begin{aligned}A_x &= -\frac{i}{2s} - \frac{1}{4\sigma_1^2} = -\frac{s + 2i\sigma_1^2}{4\sigma_1^2 s}, \\ B_x &= i\left(\frac{x}{s} - p_1\right), \\ C_x &= -i\frac{x^2}{2s}.\end{aligned}$$

Using (3.4.3), this becomes:

$$\Psi_1(x, p_1, s) = \sqrt{\frac{4\pi\sigma_1^2 s}{2i\sigma_1^2 + s}} \exp\left[-i\frac{x^2}{2s} + \left(\frac{x}{s} - p_1\right)^2 \left(\sigma_1^2 s \frac{-s + 2i\sigma_1^2}{s^2 + 4\sigma_1^4}\right)\right]. \quad (4.3.10)$$

Since the functions in question are the same with just the changes  $y \rightarrow x$ ,  $p_1 \rightarrow p_2$  and  $\sigma_1 \rightarrow \sigma_2$ , the  $y$ -component wave function is simply:

$$\Psi_2(y, p_2, s) = \sqrt{\frac{4\pi\sigma_2^2 s}{2i\sigma_2^2 + s}} \exp\left[-i\frac{y^2}{2s} + \left(\frac{y}{s} - p_2\right)^2 \left(\sigma_2^2 s \frac{-s + 2i\sigma_2^2}{s^2 + 4\sigma_2^4}\right)\right]. \quad (4.3.11)$$

### §4.3.3 Accelerated Particle Case

The propagated state  $\Psi_{0,3}(t, z, s)$  is found via the evolution equation:

$$\Psi_{0,3}(t, z, s) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{G}_{0,3}(t, t', z, z', s) \Psi_0(t', 0) \Psi_3(z', 0) dt' dz'$$

From here a slightly altered version of the earlier methods will be used. Instead of performing the integrations directly, various relations will be derived that will later simplify the calculations involved.

The current problem is the integration of the following function:

$$\begin{aligned} \mathcal{G}_{0,3}(t, t', z, z', s) \Psi_0(t', 0) \Psi_3(z', 0) &= \exp \left[ +ip_0 q^0 - \frac{(q^0)^2}{(4\sigma_0^2)} \right. \\ &\quad \left. - ip_3 q^3 - \frac{(q^3)^2}{(4\sigma_3^2)} + i(C - B^2 A) \right], \\ &= \exp \left[ +ip_0 t' - \frac{t'^2}{(4\sigma_0^2)} \right. \\ &\quad \left. - ip_3 z' - \frac{z'^2}{(4\sigma_3^2)} + i(C - B^2 A) \right]. \end{aligned}$$

The above exponent can be rewritten as a quadratic expression in  $z$  and  $t$ , this means the following must be true:

$$\mathcal{G}_{0,3}(t, t', z, z', s) \Psi_0(t', 0) \Psi_3(z', 0) = e^\Lambda, \quad (4.3.12)$$

where:

$$\begin{aligned} \Lambda &= \lambda_1 z^2 + \lambda_2 t^2 + \lambda_3 z'^2 + \lambda_4 t'^2 + \lambda_5 z z' + \lambda_6 t t' + \lambda_7 z t' \\ &\quad + \lambda_8 t z' + \lambda_9 z' t' + \lambda_{10} t z + \lambda_{11} t' + \lambda_{12} z'. \end{aligned}$$

The quantities  $\lambda_1$  to  $\lambda_{12}$  will be determined later. In the exponent above, there are no terms to power one in  $t$  or  $z$ , which is why there is no  $\lambda_{13}t + \lambda_{14}z$  term in  $\Lambda$ .

Using  $\Lambda$  as the exponent, the exact same methods as used before will be applied. For the integration over  $z'$ , this expression is rewritten as:

$$\Lambda = A_z z'^2 + B_z z' + C_z,$$



where:

$$\begin{aligned} A_z &= \lambda_3, \\ B_z &= \lambda_5 z + \lambda_8 t + \lambda_9 t' + \lambda_{12}, \\ C_z &= \lambda_1 z^2 + \lambda_2 t^2 + \lambda_4 t'^2 + \lambda_6 t t' + \lambda_7 z t' + \lambda_{10} t z + \lambda_{11} t'. \end{aligned}$$

Using (3.4.3), the integrated exponent is:

$$\begin{aligned} C_z - \frac{B_z^2}{4A_z} &= \lambda_1 z^2 + \lambda_2 t^2 + \lambda_4 t'^2 + \lambda_6 t t' + \lambda_7 z t' + \lambda_{10} t z + \lambda_{11} t' \\ &\quad - \frac{(\lambda_5 z + \lambda_8 t + \lambda_9 t' + \lambda_{12})^2}{4\lambda_3}, \end{aligned}$$

with the coefficient generated by this integral given by:

$$\sqrt{-\frac{\pi}{A_z}} = \sqrt{-\frac{\pi}{\lambda_3}}.$$

Now completing the square of the new exponent in  $t'$  yields:

$$C_z - \frac{B_z^2}{4A_z} = A_t t'^2 + B_t t' + C_t,$$

where:

$$\begin{aligned} A_t &= \lambda_4 - \frac{\lambda_9^2}{4\lambda_3} \\ B_t &= \lambda_6 t + \lambda_7 z + \lambda_{11} - \frac{\lambda_9 (\lambda_5 z + \lambda_8 t + \lambda_{12})}{2\lambda_3} \\ C_t &= \lambda_1 z^2 + \lambda_2 t^2 + \lambda_{10} t z - \frac{(\lambda_5 z + \lambda_8 t + \lambda_{12})^2}{4\lambda_3}. \end{aligned}$$

Performing the integration over  $t'$ , via (3.4.3), yields the exponent:

$$\begin{aligned} C_t - \frac{B_t^2}{4A_t} &= \lambda_1 z^2 + \lambda_2 t^2 + \lambda_{10} t z - \frac{(\lambda_5 z + \lambda_8 t + \lambda_{12})^2}{4\lambda_3} \\ &\quad - \frac{1}{4} \frac{\left( \lambda_6 t + \lambda_7 z + \lambda_{11} - \frac{\lambda_9 (\lambda_5 z + \lambda_8 t + \lambda_{12})}{2\lambda_3} \right)^2}{\lambda_4 - \frac{\lambda_9^2}{4\lambda_3}}. \end{aligned}$$

The coefficient generated by this integration is given by:

$$\sqrt{-\frac{\pi}{A_t}} = \sqrt{\frac{-4\lambda_3\pi}{4\lambda_3\lambda_4 - \lambda_9^2}}.$$

The final form of the coefficient after the two integrations is given by:

$$\begin{aligned}\sqrt{-\frac{\pi}{A_z}}\sqrt{-\frac{\pi}{A_t}} &= \sqrt{\frac{-\pi}{\lambda_3}}\sqrt{\frac{-4\lambda_3\pi}{4\lambda_3\lambda_4 - \lambda_9^2}}, \\ &= \sqrt{\frac{4\pi^2}{4\lambda_3\lambda_4 - \lambda_9^2}}.\end{aligned}$$

The quantities  $\lambda_1$  to  $\lambda_{12}$  are found using the following method.  $\Lambda$  is set equal to the exponent of  $\mathcal{G}_{0,3}(t, t', z, z', s)\Psi_0(t', 0)\Psi_3(z', 0)$ , then the following expressions hold:

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\frac{\partial^2\Lambda}{\partial z^2}, \quad \lambda_2 = \frac{1}{2}\frac{\partial^2\Lambda}{\partial t^2}, \quad \lambda_3 = \frac{1}{2}\frac{\partial^2\Lambda}{\partial z'^2}, \quad \lambda_4 = \frac{1}{2}\frac{\partial^2\Lambda}{\partial t'^2}, \\ \lambda_5 &= \frac{\partial^2\Lambda}{\partial z\partial z'}, \quad \lambda_6 = \frac{\partial^2\Lambda}{\partial t\partial t'}, \quad \lambda_7 = \frac{\partial^2\Lambda}{\partial t'\partial z}, \quad \lambda_8 = \frac{\partial^2\Lambda}{\partial t\partial z'}, \\ \lambda_9 &= \frac{\partial^2\Lambda}{\partial t'\partial z'}, \quad \lambda_{10} = \frac{\partial^2\Lambda}{\partial t\partial z}, \\ \lambda_{11} &= \left.\frac{\partial\Lambda}{\partial t'}\right|_{t=z=z'=t'=0}, \quad \lambda_{12} = \left.\frac{\partial\Lambda}{\partial z'}\right|_{t=z=z'=t'=0}\end{aligned}$$

These coefficients turn out to be the following:

$$\begin{aligned}\lambda_1 &= -\frac{i}{4}\coth\left(\frac{s}{2}\right), & \lambda_2 &= +\frac{i}{4}\coth\left(\frac{s}{2}\right) \\ \lambda_3 &= -\frac{1}{4\sigma_3^2} - \frac{i}{4}\coth\left(\frac{s}{2}\right), & \lambda_4 &= -\frac{1}{4\sigma_0^2} + \frac{i}{4}\coth\left(\frac{s}{2}\right), \\ \lambda_5 &= +\frac{i}{2}\coth\left(\frac{s}{2}\right), & \lambda_6 &= -\frac{i}{2}\coth\left(\frac{s}{2}\right), \\ \lambda_7 &= -\frac{1}{2}i, & \lambda_8 &= \frac{1}{2}i, \\ \lambda_9 &= -\frac{1}{2}i, & \lambda_{10} &= \frac{1}{2}i, \\ \lambda_{11} &= -ip_0, & \lambda_{12} &= ip_3.\end{aligned}$$

The final exponent of the propagated wave function  $\Psi_{0,3}(t, z, s)$  is  $C_t - \frac{B_t^2}{4A_t}$  and using the above expressions it becomes:

$$\begin{aligned}C_t - \frac{B_t^2}{4A_t} &= -\frac{i}{4}\coth\left(\frac{s}{2}\right)(z^2 - t^2) + i\frac{tz}{2} - \frac{\left(\frac{i}{2}\coth\left(\frac{s}{2}\right)z + i\frac{t}{2} + ip_3\right)^2}{4\left(-\frac{1}{4\sigma_3^2} - \frac{i}{4}\coth\left(\frac{s}{2}\right)\right)} \\ &\quad - \frac{1}{4}\frac{\left(-\frac{i}{2}\coth\left(\frac{s}{2}\right)t + -i\frac{z}{2} + -ip_0 - \frac{\left(\frac{1}{2}\coth\left(\frac{s}{2}\right)z + \frac{1}{2}t + p_3\right)}{4\left(-\frac{1}{4\sigma_3^2} - \frac{i}{4}\coth\left(\frac{s}{2}\right)\right)}\right)^2}{-\frac{1}{4\sigma_0^2} + \frac{i}{4}\coth\left(\frac{s}{2}\right) + \frac{1}{16\left(-\frac{1}{4\sigma_3^2} - \frac{i}{4}\coth\left(\frac{s}{2}\right)\right)}}.\end{aligned}\tag{4.3.13}$$

while the coefficient of this integrated wave function is given as:

$$\sqrt{\frac{4\pi^2}{4\lambda_3\lambda_4 - \lambda_7^2}} = \sqrt{\frac{4\pi^2}{4\left(-\frac{1}{4\sigma_3^2} - \frac{i}{4}\coth\left(\frac{s}{2}\right)\right)\left(-\frac{1}{4\sigma_0^2} + \frac{i}{4}\coth\left(\frac{s}{2}\right)\right) - \left(-\frac{1}{2}i\right)^2}}. \quad (4.3.14)$$

So the wave function  $\Psi_{0,3}(t, z, s)$  is given by the following formula:

$$\Psi_{0,3}(t, z, s) = C(s)e^{E(t,z,s)}, \quad (4.3.15)$$

with the coefficient  $C(s) = \sqrt{\frac{\pi^2}{A_z A_t}}$  as given in (4.3.15) and  $E(t, z, s) = C_t - \frac{B_t^2}{4A_t}$  as given in (4.3.13).

#### §4.3.4 Complete Wave Function

The constituent wave functions have been found for each coordinate of the system, so the complete wave function can be constructed now. The complete wave function is the product of each wave function found earlier and the coefficient of the Green's Function and initial Gaussian state:

$$\Psi(q, s) = \frac{1}{2\pi(\sigma_0\sigma_1\sigma_2\sigma_3)^{\frac{1}{2}}} \cdot \frac{-i\theta(s)}{8\pi^2 s \sinh(s/2)} \Psi_{0,3}(t, z, s) \Psi_1(x, s) \Psi_2(y, s). \quad (4.3.16)$$

#### §4.4 Covariant Schrödinger-type Equation

The wave function found in the previous section must be checked to be a solution to the covariant Schrödinger-type equation:

$$\hat{\mathcal{H}}\Psi(q, s) = i \frac{\partial}{\partial s} \Psi(q, s)$$

where:

$$\hat{\mathcal{H}} = \frac{\left(\hat{p}_\mu + e\widehat{\mathcal{A}}_\mu\right)^2}{2m},$$

$$\Psi(q, s) = \Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s).$$

The Hamiltonian operator is derived in §4.3.2.

The Hamiltonian Operator must be expanded:

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{\left(\hat{p} + e\widehat{\mathcal{A}}\right)^2}{2m} \\ &= \frac{\left(\hat{p}_\mu + e\widehat{\mathcal{A}}_\mu\right) \left(\hat{p}^\mu + e\widehat{\mathcal{A}}^\mu\right)}{2m} \\ &= \frac{(\hat{p}_0 + k\hat{z})^2 - \vec{\hat{p}} \cdot \vec{\hat{p}}}{2m} \\ &= \frac{\hat{p}_0^2 + k^2\hat{z}^2 + 2k\hat{p}_0\hat{z} - \hat{p}_1^2 - \hat{p}_2^2 - \hat{p}_3^2}{2m} \end{aligned} \quad (4.4.1)$$

The system is in coordinate space representation, so the operators in (4.4.1) become:

$$\begin{aligned} \hat{p}_0 &= +i \frac{\partial}{\partial t}, & \hat{p}_1 &= -i \frac{\partial}{\partial x}, & \hat{p}_2 &= -i \frac{\partial}{\partial y}, & \hat{p}_3 &= -i \frac{\partial}{\partial z}, \\ & & \hat{z} &= z. \end{aligned}$$

Using these expressions,  $\hat{\mathcal{H}}$  becomes:

$$\hat{\mathcal{H}} = \frac{1}{2m} \left( -\frac{\partial^2}{\partial t^2} + k^2 z^2 + 2ikz \frac{\partial}{\partial t} + \nabla^2 \right), \quad (4.4.2)$$

The mass of the particle may be set to one without the loss of generality of the result. This modification simplifies the calculations involved.

Finally the covariant Schrödinger-type wave equation for this system can be written in whole as:

$$\left[ \hat{\mathcal{H}} - i \frac{\partial}{\partial s} \right] \Psi(q, s) = 0.$$

Thus

$$\left[ \frac{1}{2} \left( -\frac{\partial^2}{\partial t^2} + k^2 z^2 - 2ikz \frac{\partial}{\partial t} + \nabla^2 \right) - i \frac{\partial}{\partial s} \right] \Psi(q, s) = 0. \quad (4.4.3)$$

The next task is the check that the proposed solution derived above satisfies the covariant Schrödinger-type wave equation.

The wave function,  $\Psi(q, s)$ , can be thought of a separable solution to the covariant Schrödinger-type wave equation. The functions that constitute the separable solution have been derived earlier, the complete propagated wave function being a product of each. This complete wave function  $\Psi(q, s)$  cannot be written as a truly separable solution but this does aid in checking the solution and the following can be found.

First act on this wave function with  $\hat{\mathcal{H}}$  and divide by the wave function itself:

$$\begin{aligned} \frac{\hat{\mathcal{H}}\Psi(q, s)}{\Psi(q, s)} &= \frac{1}{2\pi(\sigma_0\sigma_1\sigma_2\sigma_3)^{\frac{1}{2}}} \cdot \frac{-i\theta(s)}{8\pi^2 s \sinh(s/2)} \\ &\quad \times \frac{\hat{\mathcal{H}}(\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s))}{\frac{1}{2\pi(\sigma_0\sigma_1\sigma_2\sigma_3)^{\frac{1}{2}}} \cdot \frac{-i\theta(s)}{8\pi^2 s \sinh(s/2)} \Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s)} \\ &= \frac{\hat{\mathcal{H}}(\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s))}{\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s)} \\ &= \frac{1}{2} \left( \frac{\left( \frac{\partial^2}{\partial z^2} + 2ikz \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} \right) \Psi_{0,3}(t, z, s)}{\Psi_{0,3}(t, z, s)} + k^2 z^2 + \frac{\frac{\partial^2}{\partial x^2} \Psi_1(x, s)}{\Psi_1(x, s)} + \frac{\frac{\partial^2}{\partial y^2} \Psi_2(y, s)}{\Psi_2(y, s)} \right) \end{aligned}$$

Now act on the complete wave function with  $i \frac{\partial}{\partial s}$ :

$$\begin{aligned} \frac{i \frac{\partial}{\partial s} \Psi(q, s)}{\Psi(q, s)} &= i \frac{\frac{\partial}{\partial s} \left( \frac{-i\theta(s)}{8\pi^2 s \sinh(s/2)} \right)}{\frac{-i\theta(s)}{8\pi^2 s \sinh(s/2)}} \\ &\quad + i \frac{\frac{\partial}{\partial s} (\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s))}{\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s)} \\ &= -i \left( \frac{1}{s} + \frac{\coth(s/2)}{2} \right) + i \frac{\frac{\partial}{\partial s} (\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s))}{\Psi_{0,3}(t, z, s)\Psi_1(x, s)\Psi_2(y, s)} \end{aligned}$$

Note that  $\theta(s) = 1$  as  $s > 0$ . This will simplify the differential equations by removing the Heaviside Step Function  $\theta(s)$ . Writing out the covariant Schrödinger-type wave equation now, there is the following expression:

$$\begin{aligned} & \frac{\left(\frac{\partial^2}{\partial z^2} + 2ikz\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - i\frac{\partial}{\partial s}\right)\Psi_{0,3}(t, z, s)}{\Psi_{0,3}(t, z, s)} + k^2 z^2 \\ & + \frac{\left(\frac{\partial^2}{\partial x^2} - i\frac{\partial}{\partial s}\right)\Psi_1(x, s)}{\Psi_1(x, s)} + \frac{\left(\frac{\partial^2}{\partial y^2} - i\frac{\partial}{\partial s}\right)\Psi_2(y, s)}{\Psi_2(y, s)} \\ & - i\left(\frac{1}{s} + \frac{\coth(s/2)}{2}\right) = 0 \end{aligned} \quad (4.4.4)$$

Using REDUCE, it has been found that the complete wave function does indeed satisfy the covariant Schrödinger-type wave equation, (4.4.4). See §A.5.1, §A.5.2 and §A.5.3 for the REDUCE input files `cvwfschro.red`, `relfreeg1.red` and `relfreeg2.red`, respectively, and see §A.5.4 for the resulting REDUCE output file `cvwfschro_out`.

#### §4.5 Comparison of Quantum and Classical Systems

As in §3.4.2, the quantum system and the classical systems for the relativistic case are to be compared to one another and it was seen that the non-relativistic quantum system follows the non-relativistic classical system and the same should be true here. The relativistic discussion is more complicated due to relativistic effects, specifically the coupling of the  $z$  and  $t$  axes, and some changes to the method must be made.

In the non-relativistic case the classical path was simply substituted directly into the exponent of the density function. However for the relativistic case a coordinate transformation is first applied to the wave function,  $\Psi(t, z, s)$ , giving a new quantum state,  $\Phi(u, v, s)$ . The variables  $u$  and  $v$  are defined as:

$$u \equiv u(s) = t + z, \quad v \equiv v(s) = t - z.$$

Using these equations, the substitutions for  $t$  and  $z$  are:

$$t = \frac{u + v}{2}, \quad z = \frac{u - v}{2}$$

These substitutions are made in  $\Psi(t, z, s)$  resulting in a new exponent. The coefficient remains unchanged as it is purely a function of  $s$ . The new state is represented as  $\Phi(u, v, s)$ .

The same transformation is applied to the covariant Schrödinger-type wave equation as a check on the above calculation. As the original state satisfies the covariant Schrödinger-type wave equation, the transformed state should satisfy the transformed covariant Schrödinger-type wave equation.

Using the following results:

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial u}{\partial t} \frac{\partial}{\partial u} + \frac{\partial v}{\partial t} \frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \\ \Rightarrow \frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + 2 \frac{\partial^2}{\partial u \partial v}, \\ \frac{\partial}{\partial z} &= \frac{\partial u}{\partial z} \frac{\partial}{\partial u} + \frac{\partial v}{\partial z} \frac{\partial}{\partial v} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}, \\ \Rightarrow \frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - 2 \frac{\partial^2}{\partial u \partial v}. \end{aligned}$$

and:

$$-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} = -4 \frac{\partial^2}{\partial u \partial v},$$

the Hamiltonian operator becomes:

$$\hat{\mathcal{H}} = \frac{1}{2} \left( -4 \frac{\partial^2}{\partial u \partial v} + \left( \frac{u-v}{2} \right)^2 + 2i \left( \frac{u-v}{2} \right) \cdot \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \right).$$

The transformed wave function,  $\Phi(u, v, s)$ , satisfies the new covariant Schrödinger-type wave equation. A new density function is calculated from  $\Phi(u, v, s)$ :

$$\varrho(u, v, s) = |\Phi(u, v, s)|^2. \quad (4.5.1a)$$

This new density function is of the form:

$$\varrho(u, v, s) = c(s) e^{\lambda(u, v, s)}. \quad (4.5.1b)$$

The variables  $t$  and  $z$  are respectively:

$$\begin{aligned} z &= \xi(s) - p_0, \\ &= p_0 (\cosh(s) - 1) + p_3 \sinh(s) + z', \\ t &= p_0 \sinh(s) + p_3 (\cosh(s) - 1) + t', \end{aligned}$$

see the derivation of covariant Green's function, §4.3.1, and [2] for the origin of these equations. Using the definitions of  $u$  and  $v$ , the transformed variables become:

$$\begin{aligned} u(s) &= t + z, \\ &= (p_0 + p_3) (\cosh(s) + \sinh(s) - 1), \end{aligned} \quad (4.5.2a)$$

$$\begin{aligned} v(s) &= t - z, \\ &= -(p_0 - p_3) (\cosh(s) - \sinh(s) - 1), \end{aligned} \quad (4.5.2b)$$

with, in this discussion, the initial coordinates set to zero.

For the following section the coordinates  $u$  and  $v$  of the transformed quantum state – now  $\Phi(u', v', s)$  – are denoted as  $u'$  and  $v'$  respectively to enable a simple comparison of the classical path  $u$  and  $v$  above. To find the path that the transformed quantum state follows, i.e. where it's density function is maximised for values of  $u'$  and  $v'$ , the following equations are solved for  $u'$  and  $v'$  respectively:

$$\frac{\partial}{\partial u'} \lambda(u', v', s) = 0, \quad (4.5.3a)$$

$$\frac{\partial}{\partial v'} \lambda(u', v', s) = 0, \quad (4.5.3b)$$

which yield:

$$u' = -\frac{(p_0 + p_3) (\cosh(s) + \sinh(s) - 1)}{\cosh(s) + \sinh(s)} = -e^{-s}u, \quad (4.5.4a)$$

$$v' = \frac{(p_0 - p_3) (\cosh(s) - \sinh(s) - 1)}{\cosh(s) - \sinh(s)} = -e^s v. \quad (4.5.4b)$$

The preceding derivation was carried out in REDUCE. See the REDUCE input file `uvclass-path.red` in §A.5.4 and the output file

These equations are scaled versions of the classical path derived earlier. This scaling effect results from the acceleration of the state to relativistic velocities. Due to time-dilation and length-contraction, the wave packet is deformed. The new coordinates are analogous to the original system and have the same space-time metric and as such both systems are similar.

In the original system the space-time measure is:

$$\begin{aligned} s^2 &= t^2 - z^2 = (t + z)(t - z), \\ &= uv. \end{aligned}$$



In the new system (deformed or primed system), the space time measure is:

$$\begin{aligned} s'^2 &= t'^2 - z'^2 = (t' + z')(t' - z'), \\ &= u'v' = (-e^{-s}u)(-e^sv) = uv, \\ \Rightarrow s'^2 &= s^2. \end{aligned}$$

The wave function for the free particle propagating along the  $x$ -axis has the exponent:

$$\exp \left[ -i \frac{x^2}{2s} + \left( \frac{x}{s} - p_1 \right)^2 \left( \sigma_1^2 s \frac{-s + 2i\sigma_1^2}{s^2 + 4\sigma_1^4} \right) \right],$$

and so its associated density function  $\rho(x, s)$  has the exponent:

$$\exp \left[ -\frac{2\sigma_1}{s^2 + 4\sigma_1^4} (x - p_1 s)^2 \right].$$

The classical path is  $x_{cl} = x(s) = p_1 s$  and so this density function can be written as:

$$\rho(x, p_1, s) \approx \exp \left[ -\frac{2\sigma_1}{s^2 + 4\sigma_1^4} (x - x_{cl})^2 \right].$$

## Chapter 5 Conclusion

The aim of this work has been to demonstrate the properties of the Garavaglia Model a covariant formalism of Relativistic Quantum Mechanics proposed in [1] and [2]. and to find solutions to the covariant Schrödinger-type wave equation. The method followed was to demonstrate standard results in the Non-Relativistic Case and show that these results hold also in the Relativistic Case.

To show consistency between the Garavaglia Model and Classical Mechanics, the wave functions which describe the particle in the Non-Relativistic and Relativistic cases must propagate along the classical path. The method and results which showed the particle's propagation along the classical path for the Non-Relativistic Case are discussed in §3.4.2, which is a standard result. The same method was applied to a free particle in the Relativistic Case. However, there exists a coupling of the  $t$  and  $z$  axes due to Lorentz transformation. This means that a modification of the method is required. Rather than directly substituting the classical path into the exponent of the density function (§3.4.2), a transformation is performed on the coordinates. This yields a transformed covariant Schrödinger-type wave equation and a transformed wave function. Once the classical path was determined, it was left to show that the particle propagated along this path.

In order to determine the functions that describe the maximum of the probability density function (4.5.4), differentiation was applied to its exponent. In order to show that (4.5.4) is equivalent to the classical path (4.5.2), the ratio of these functions was calculated. The result ( $-e^{\pm s}$ ) shows that the path of the quantum particle is a scaled version of the classical path. To show that these paths are equivalent, the space-time measurement was determined for both paths. It was shown in §4.5 that these metrics are identical. Therefore it can be concluded that as the accelerated relativistic particle propagates, it is most likely to be found in the neighbourhood of the classical path, the most probable path the particle follows is the classical path.

In these analyses the consistency between the Garavaglia Model and Classical Mechanics has been established for the Non-Relativistic and Relativistic cases.

The Garavaglia Model provides a neater method for the analysis of Relativistic Quantum Mechanics which is not sufficiently provided by the Klein-Gordon equation. Though the Klein-Gordon equation is useful in certain specific cases, those of energy eigenfunction

cases for instance, the lack of a positive definite density function shows the problem inherent with the Klein-Gordon equation. There is, however, an important limitation in the Garavaglia Model. There are similar methods to the Garavaglia Model which seek to overcome the problems with the Klein-Gordon equation, such as the Parameterized Relativistic Dynamics discussed earlier. The PRD is constructed in such a way so as to ensure that the proper-time is independent of the system which is undergoing the interaction. This is not a case with the Garavaglia Model. The proper time,  $s$ , as used in the Garavaglia Model is the proper time of the particle undergoing the interaction. This means that the dynamic coordinates in the observers frame of reference are connected to the proper time of the particle. The resulting differential equation, the covariant Schrödinger-type wave equation, does not therefore depend on five parameters but rather four – after expressing  $s$  as a function of  $(t, x, y, z)$ .

This model provides results that correspond to well established results in the Non-Relativistic Case. For this reason, the Garavaglia Model is useful as an exploration of some of the basic concepts of Relativistic Quantum Mechanics.

The work described above was performed as an exploration of the Garavaglia Model as established in [1] and [2]. New results presented here are:

1. The derivation of the propagated states using the Green's function method, with the Green's function as derived in [2]. This work forms the basis of the demonstration of the Garavaglia Model. The propagated state for the free case was also derived.
2. It was shown in §4.4 that these states obey the covariant Schrödinger-type wave equation. See also the REDUCE files `cvwf_schro.red`, `relfreeg1.red` and `relfreeg2.red` in §A.5 and the associated output file.
3. It was demonstrated that these propagated states are maximised on the classical path.

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## Appendix

The appendix contains calculations which are necessary for much of the main body of work.

### §A.1: Green's Function Proofs.

In this section some proofs are included which are important to the method of Green's Functions used in this project. Note that this discussion is concerned with the non-relativistic case.

To show  $\langle z|e^{-i\hat{\mathcal{H}}t}|z'\rangle$  can be expressed in the form  $A(t)e^{iS(z,z',t)}$ , where  $S(z,z',t)$  is the action of the system and  $A(t)$  is some function of time.

The proposed propagator,  $\langle z|e^{-i\hat{\mathcal{H}}t}|z'\rangle$ , can be expanded as:

$$\langle z|e^{-i\hat{\mathcal{H}}t}|z'\rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle z|p\rangle \langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle \langle p'|z'\rangle dp' dp, \quad (A.1)$$

by making use of the following identity function twice:

$$\hat{\mathcal{I}} = \int_{-\infty}^{+\infty} |p\rangle \langle p| dp.$$

First the equivalent result from momentum space needs to be found, i.e. an expression for  $\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle$  must be found.  $\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle$  can be expanded as follows by using results from §3.3:

$$\begin{aligned} \langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle &= \int_{-\infty}^{+\infty} \langle p|e^{-i\hat{\mathcal{H}}t}|\alpha\rangle \langle \alpha|p'\rangle d\alpha \\ &= \int_{-\infty}^{+\infty} e^{-i\alpha t} \langle p|\alpha\rangle \langle \alpha|p'\rangle d\alpha \\ &= \int_{-\infty}^{+\infty} e^{-i\alpha t} \frac{e^{i\left(p\alpha - \frac{p^3}{6}\right)}}{\sqrt{2\pi}} \frac{e^{-i\left(p'\alpha - \frac{p'^3}{6}\right)}}{\sqrt{2\pi}} d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\alpha(p-p'-t)} e^{i\frac{p'^3-p^3}{6}} d\alpha \\ &= \delta(p-p'-t) e^{i\frac{p'^3-p^3}{6}}. \end{aligned} \quad (A.2)$$

This is zero for all values of  $p'$  not equal to  $p-t$ . Therefore the momentum space propagator may be written as:

$$\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle = \delta(p - p' - t)e^{-iW(p,t)}, \quad (\text{A.3})$$

where:

$$W(p,t) = \frac{t^3}{6} - \frac{pt^2 - p^2t}{2}.$$

The propagator  $\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle$  must obey the Schrödinger equation:

$$\left( \hat{\mathcal{H}} - i\frac{\partial}{\partial t} \right) \langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle = 0. \quad (\text{A.4})$$

For this system in momentum space, the Hamiltonian operator,  $\hat{\mathcal{H}}$ , is  $\hat{\mathcal{H}} = \frac{p^2}{2} - i\frac{\partial}{\partial p}$ . Expanding these terms gives:

$$\begin{aligned} -i\frac{\partial}{\partial p}\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle &= -i\frac{\partial}{\partial p}\left[\delta(p - p' - t)e^{-iW(p,t)}\right] \\ &= \left[-i\delta'(p - p' - t) - \delta(p - p' - t)\left(pt - \frac{t^2}{2}\right)\right]e^{-iW(p,t)} \end{aligned}$$

and:

$$\begin{aligned} i\frac{\partial}{\partial t}\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle &= i\frac{\partial}{\partial t}\left[\delta(p - p' - t)e^{-iW(p,t)}\right], \\ &= \left[-i\delta'(p - p' - t) + \delta(p - p' - t)\left(\frac{t^2}{2} - pt + \frac{p^2}{2}\right)\right]e^{-iW(p,t)}. \end{aligned}$$

Also:

$$\frac{p^2}{2}\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle = \frac{p^2}{2}\delta(p - p' - t)e^{-iW(p,t)}.$$

Using these expansions:

$$\begin{aligned} \hat{\mathcal{H}}\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle &= \left(\frac{p^2}{2} - i\frac{\partial}{\partial p}\right)\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle, \\ &= \left[\left(\frac{p^2}{2} - pt + \frac{t^2}{2}\right)\delta(p - p' - t) - i\delta'(p - p' - t)\right]e^{-iW(p,t)} \\ &= i\frac{\partial}{\partial t}\langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle. \end{aligned}$$

Therefore:

$$\left( \hat{\mathcal{H}} - i\frac{\partial}{\partial t} \right) \langle p|e^{-i\hat{\mathcal{H}}t}|p'\rangle = 0.$$



The proposed momentum space evolution function is a solution to the Schrödinger Equation and so can be taken to be correct.

Return now to the original problem. Inserting (A.3) into (A.1) and recalling that  $\langle z|p\rangle = \frac{e^{izp}}{\sqrt{2\pi}}$  and  $\langle p'|z'\rangle = \frac{e^{-iz'p'}}{\sqrt{2\pi}}$ , gives:

$$\begin{aligned}\langle z|e^{-i\hat{H}t}|z'\rangle &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{izp} \delta(p-p'-t) e^{-iW(p,t)} e^{-ip'z'} dp' dp \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{izp} e^{-iW(p,t)} e^{-i(p-t)z'} dp.\end{aligned}\tag{A.5}$$

The exponent in (A.5) can be manipulated to become:

$$\left(-i\frac{t}{2}\right)p^2 + \left(i\left[\frac{t^2}{2} + (z-z')\right]\right)p + i\left(-\frac{t^3}{6} + tz'\right).$$

Applying (3.4.3) to (A.5) gives:

$$\langle z|e^{-i\hat{H}t}|z'\rangle = \sqrt{\frac{1}{2\pi it}} e^{i\frac{1}{24i}(12(z-z')^2 + 12(z+z')t^2 - t^4)}.\tag{A.6}$$

This equation can now be written as:

$$\langle z|e^{-i\hat{H}t}|z'\rangle = A(t)e^{iS(z,z',t)},$$

which is the desired form for the propagator for the accelerated particle.

### §A.1.1: Path Integral Method

The above results will now be confirmed by use of the Path Integral method, see [37] and [39] for the initial development of this method. In Feynman's notation, the propagated state  $\Psi(q, t)$  was found from an initial state  $\Psi(q', t')$  by means of the following integration:

$$\Psi(q, t) = \int_{q_0}^{q_N} K(q, q'; t, t') \Psi(q', t') dq',$$

with  $K(q, q'; t, t')$  is called the *Feynman Kernel* and is the propagator for the system in question. While this is identical to the method as used earlier, the difference comes about in the method used to derive the kernel for this integration. As before, the system in question is one with a potential of the form  $V(z) = -kz$ , where  $k$  is some constant. Again assume  $k$  to be unity by using the same dimensionless system as defined in §2.1. The calculation which needs to be made is:

$$\langle z | e^{-i\hat{H}t} | z' \rangle = \int_{z'}^z \exp\left(i\mathcal{S}[z(t')]\right) \mathcal{D}[z(t')],$$

where the term  $\mathcal{D}[z(t')]$  signifies the integral is over all possible paths from  $z'$  to  $z$ . The path followed by the particle can be written as:

$$z(t') = z_{cl}(t') + y(t'),$$

where  $z_{cl}(t')$  is the classical path and  $y(t')$  is the variation from  $z_{cl}(t')$ . All paths must have the same endpoints, i.e.  $y(0) = y(t) = 0$  which implies that  $z(0) = z'$  and  $z(t) = z$ . It can be seen that:

$$\dot{z}(t') = \dot{z}_{cl}(t') + \dot{y}(t').$$

This path must first be discretised by setting up an array of points in time,  $t_j$ , where  $0 \leq j \leq N$  for some integer  $N$ :<sup>8</sup>

$$\begin{aligned} z_j = z(t'_j) &= z_{cl}(t'_j) + y(t'_j) = z_{cl}(t'_j) + y_j, \\ t'_{j+1} - t'_j &= \epsilon, \\ N\epsilon &= t. \end{aligned}$$

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<sup>8</sup> Obviously,  $z(t_0) = z(0) = z'$  and  $z(t_N) = z(t) = z$ .

Also note that  $dz_j = dy_j$  and so the following can be written:

$$\int_{z'}^z \mathcal{D}[z(t')] = \int_0^1 \mathcal{D}[y(t')].$$

The path integral is now of the form:

$$\int_{z'}^z \exp(i\mathcal{S}[z(t')]) \mathcal{D}[z(t')] = \int_0^1 \exp(i\mathcal{S}[z_{cl}(t') + y(t')]) \mathcal{D}[y(t')].$$

Now the action for the new path,  $\mathcal{S}[z_{cl}(t') + y(t')]$ , must be found and be expanded in a Taylor Series about the classical path  $z_{cl}(t')$ :

$$\begin{aligned} \mathcal{S}[z_{cl}(t') + y(t')] &= \int_0^t \mathcal{L}(z_{cl}(t') + y(t'), \dot{z}_{cl}(t') + \dot{y}(t')) dt, \\ &= \int_0^t \left( \mathcal{L}(z_{cl}(t'), \dot{z}_{cl}(t')) + \left[ \frac{\partial \mathcal{L}}{\partial z} \Big|_{z_{cl}(t')} y(t') + \frac{\partial \mathcal{L}}{\partial \dot{z}} \Big|_{z_{cl}(t')} \dot{y}(t') \right] \right. \\ &\quad \left. + \frac{1}{2} \left[ \frac{\partial^2 \mathcal{L}}{\partial z^2} \Big|_{z_{cl}(t')} y(t')^2 + 2 \frac{\partial^2 \mathcal{L}}{\partial z \partial \dot{z}} \Big|_{z_{cl}(t')} y(t') \dot{y}(t') + \frac{\partial^2 \mathcal{L}}{\partial \dot{z}^2} \Big|_{z_{cl}(t')} \dot{y}(t')^2 \right] \right) dt'. \end{aligned}$$

The first term,  $\mathcal{L}(z_{cl}(t'), \dot{z}_{cl}(t'))$ , will integrate to become the action for the classical path.

Using integration by parts, the second term integrates to zero in keeping with Euler-Lagrange Equations<sup>9</sup>:

$$\begin{aligned} \int_0^t \left[ \frac{\partial \mathcal{L}}{\partial z} y(t') + \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{y}(t') \right] dt' &= \left[ \int_0^t \frac{\partial \mathcal{L}}{\partial z} y(t') dt' \right] + \int_0^t \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{y}(t') dt', \\ &= \left[ \int_0^t y(t') \left( \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{z}} \right) dt' \right] + \int_0^t \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{y}(t') dt', \\ &= \left[ y(t) \frac{\partial \mathcal{L}}{\partial \dot{z}} \Big|_0^t - \int_0^t \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{y}(t') dt' \right] + \int_0^t \frac{\partial \mathcal{L}}{\partial \dot{z}} \dot{y}(t') dt', \\ &= 0. \end{aligned}$$

The integrals cancel with each other and, by definition, the function  $y(t)$  is zero at the end points. Similarly, the third term:

$$\frac{\partial^2 \mathcal{L}}{\partial z^2} \Big|_{z_{cl}(t')} y(t')^2 + 2 \frac{\partial^2 \mathcal{L}}{\partial z \partial \dot{z}} \Big|_{z_{cl}(t')} y(t') \dot{y}(t') + \frac{\partial^2 \mathcal{L}}{\partial \dot{z}^2} \Big|_{z_{cl}(t')} \dot{y}(t')^2,$$

---

<sup>9</sup> Note that the notation denoting the classical path has been dropped here for convenience only.

in the integral will sum to zero. So the action for the new path is simply the action on the classical path, i.e.:

$$\mathcal{S}[z_{cl}(t') + y(t')] = \mathcal{S}[z_{cl}(t')].$$

For simplicity, write  $z_{cl}(t') \equiv z_{cl}$  and  $\mathcal{S}[z_{cl}(t')] \equiv \mathcal{S}_{cl}$ .

So the path integral is now:

$$\begin{aligned} \int_{z'}^z \exp(i\mathcal{S}[z(t')]) \mathcal{D}[z(t')] &= \exp(i\mathcal{S}_{cl}) \int_0^0 \mathcal{D}[y(t')], \\ &= \exp(i\mathcal{S}_{cl}) F(t). \end{aligned}$$

As the remaining path integral depends only on time, it is easily seen that the potential field has played no direct part in this calculation, the influence of the potential field is accounted for in the action,  $\mathcal{S}(z, z', t)$ . From this it is reasonable to assume that the remaining unknown function,  $F(t)$ , can be found by evaluating the propagator for the free particle, which will be performed by using the path integral method.

#### *Path Integral for Free Particle Propagator.*

Using the same method as in §3.3, the action for the free particle can easily be found to be:

$$\mathcal{S}(z, z', t) = \frac{(z - z')^2}{2t}.$$

The path integral is now, after changing the above expression to a discrete form – following the same method as outlined above:

$$\begin{aligned} \langle z | e^{-i\hat{H}t} | z' \rangle &= \int_{z_0}^{z_N} \exp(i\mathcal{S}(z, z', t)) \mathcal{D}[z(t)], \\ &= \lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} A \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \exp \left[ \frac{i}{2\epsilon} \sum_{j=0}^{N-1} (z_{j+1} - z_j)^2 \right] dz_1 \dots dz_{N-1}, \end{aligned}$$

where  $z_N \equiv z$ ,  $z_0 \equiv z'$ ,  $N\epsilon = t$  and  $A$  is some normalization constant to be determined later. First perform the integration over  $z_1$ :

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \exp \left[ \frac{i}{2\epsilon} ((z_2 - z_1)^2 + (z_1 - z_0)^2) \right] dz_1, \\ &= \int_{-\infty}^{+\infty} \exp [A_1 z_1^2 + B_1 z_1 + C_1] dx_1, \end{aligned}$$

where:

$$\begin{aligned} A_1 &= \frac{i}{\epsilon}, \\ B_1 &= \frac{-i(z_0 + z_2)}{\epsilon}, \\ C_1 &= \frac{i(z_2^2 + z_0^2)}{2\epsilon}. \end{aligned}$$

It should be noted here that while the label  $j$  ranges from 0 to  $N$ , the integrals  $I_k$  range from  $I_1$  to  $I_{N-1}$ . Using (3.4.3), the integral  $I_1$  is:

$$I_1 = \sqrt{i\pi\epsilon} \exp \left[ \frac{i(z_2 - z_0)^2}{4\epsilon} \right].$$

The next step is to integrate over  $z_2$ . The second integral is:

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} \exp \left[ \frac{i}{2\epsilon}(z_2 - z_1)^2 \right] I_1 dz_2, \\ &= \sqrt{i\pi\epsilon} \int_{-\infty}^{+\infty} \exp [A_2 z_2^2 + B_2 z_2 + C_2] dz_2, \end{aligned}$$

where:

$$\begin{aligned} A_2 &= \frac{3i}{4\epsilon}, \\ B_2 &= \frac{-i(z_0 + 2z_3)}{2\epsilon}, \\ C_2 &= \frac{i(z_0^2 + 2z_3^2)}{2\epsilon}. \end{aligned}$$

So the integral now becomes:

$$I_2 = \sqrt{\frac{4}{3}(i\pi\epsilon)^2} \exp \left[ \frac{i(z_3 - z_0)^2}{6\epsilon} \right].$$

Integrating over  $z_3$ :

$$\begin{aligned} I_3 &= \int_{-\infty}^{+\infty} \exp \left[ \frac{i(z_4 - z_3)^2}{2\epsilon} \right] I_2 dz_3, \\ &= \sqrt{\frac{4}{3}(i\pi\epsilon)^2} \int_{-\infty}^{+\infty} \exp [A_3 z_3^2 + B_3 z_3 + C_3] dz_3, \end{aligned}$$

where:

$$\begin{aligned} A_3 &= \frac{2i}{3\epsilon}, \\ B_3 &= \frac{-i(z_0 + 3z_4)}{3\epsilon}, \\ C_3 &= \frac{i(z_0^2 + 3z_4^2)}{6\epsilon}. \end{aligned}$$

Again using (3.4.3), the integral becomes

$$I_3 = \sqrt{2(i\pi\epsilon)^3} \exp \left[ \frac{i(z_4 - z_0)^2}{8\epsilon} \right].$$

Going one step further, integrate over  $z_4$ :

$$\begin{aligned} I_4 &= \int_{-\infty}^{+\infty} \exp \left[ \frac{i(z_5 - z_4)^2}{2\epsilon} \right] I_3 dz_4, \\ &= \sqrt{2(i\pi\epsilon)^3} \int_{-\infty}^{+\infty} \exp [A_4 z_4^2 + B_4 z_4 + C_4] dz_4, \end{aligned}$$

where:

$$\begin{aligned} A_4 &= \frac{5i}{8\epsilon}, \\ B_4 &= -i \frac{(z_0 + 4z_5)}{4\epsilon}, \\ C_4 &= i \frac{(z_0^2 + 4z_5^2)}{8\epsilon}. \end{aligned}$$

Using (3.4.3), the integrated function is:

$$I_4 = \sqrt{\frac{16}{5}} (i\pi\epsilon)^4 \exp \left[ i \frac{(z_5 - z_0)^2}{10\epsilon} \right].$$

It can be seen that the coefficients are of the form  $\sqrt{\frac{(2i\pi\epsilon)^j}{j+1}}$  and the exponents are of the form  $\frac{i(z_{j+1} - z_0)^2}{2(j+1)\epsilon}$ . So for  $j = N - 1$ :

$$I_N = \sqrt{\frac{(2i\pi\epsilon)^{N-1}}{N}} \exp \left[ \frac{i(z_N - z_0)^2}{2t} \right].$$

The propagator is now of the form:

$$\langle z | e^{-i\hat{\mathcal{H}}t} | z' \rangle = \lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \mathcal{A} \sqrt{\frac{(2i\pi\epsilon)^{N-1}}{N}} \exp \left[ \frac{i(z_N - z_0)^2}{N} \right], \quad (A.7)$$

where  $\mathcal{A}$  is some normalisation factor. Before moving on, it should be stated that the aim here is to find a coefficient function  $F(t)$ , such that the propagator is of the form:

$$\langle z | e^{-i\hat{\mathcal{H}}t} | z' \rangle = F(t) \exp \left[ \frac{i(z - z')^2}{2t} \right]. \quad (A.7b)$$

The method to find the factor  $\mathcal{A}$ , and hence  $F(t)$ , is due to Feynman and is outlined as follows. The wave function for the particle at a time  $t_2$  and position  $z_2$  can be written as:

$$\Psi(z_2, t_2) = \int_{-\infty}^{+\infty} K(z_2, t_2; z_1, t_1) \Psi(z_1, t_1) dz_1.$$

It is also known that:

$$K(z_2, t_2; z_1, t_1) \sim \exp [i\mathcal{S}(z_2, t_2; z_1, t_1)].$$

An infinitesimal time  $\epsilon$  for  $t_2 - t_1$  is chosen and the motion in this time is considered. The particle moves in the time  $\epsilon$  from original point  $z$  to point  $y$ . When the Lagrangian,  $\mathcal{L}$ , is integrated over this short interval the action is approximately  $\epsilon\mathcal{L}$ . Take for this case the motion of a particle moving in a slowly varying potential field, i.e.:

$$\mathcal{L} = \frac{\dot{z}^2}{2} - V(z, t).$$

So now write:

$$\begin{aligned} \Psi(z, t + \epsilon) &= \int_{-\infty}^{+\infty} \frac{1}{\mathcal{A}} \exp [i\epsilon\mathcal{L}] \Psi(y, t) dy, \\ &= \int_{-\infty}^{+\infty} \frac{1}{\mathcal{A}} \exp \left[ \frac{i(z-y)^2}{\epsilon} - i\epsilon V \left( \frac{z+y}{2}, t \right) \right] \Psi(y, t) dy. \end{aligned} \quad (A.8)$$

The position  $y$  must be close to  $z$ , i.e. say  $y = z + \eta$ , with  $\eta \ll 1$ . Otherwise the integration will be very small due to the large oscillation in the  $(z-y)^2$  term, while the other terms in the exponent remain smoothly varying properties. Only values of  $y$  close to  $z$  will contribute to the integral. So, using the above substitution for  $y$ , (A.8) becomes:

$$\Psi(z, t + \epsilon) = \int_{-\infty}^{+\infty} \frac{1}{\mathcal{A}} \exp \left[ i\frac{\eta^2}{2\epsilon} - i\epsilon V \left( z + \frac{\eta}{2}, t \right) \right] \Psi(z + \eta, t) d\eta. \quad (A.9)$$

By requiring that  $\eta$  of the order  $\sqrt{\epsilon}$ ,  $\eta^2$  is of order one in  $\epsilon$ . The next step is to perform a Taylor Expansion on the equation above to first order in  $\epsilon$  and so second order in  $\eta$ . Also note that it can be assumed that  $V(z + \frac{\eta}{2}, t) \equiv V(z, t)$  since  $\frac{\eta}{2}$  can be considered sufficiently small:

$$\exp [-i\epsilon V(z, t)] \approx 1 - i\epsilon V(z, t) + O(\epsilon^2)$$

Taylor expanding  $\Psi(z + \eta, t)$  yields:

$$\Psi(z + \eta, t) \approx \Psi(z, t) + \eta \frac{\partial}{\partial x} \Psi(z, t) + \frac{\eta^2}{2} \frac{\partial^2}{\partial x^2} \Psi(z, t).$$

The term  $\exp\left[i\frac{\eta^2}{2\epsilon}\right]$  was not expanded as it is of order zero in  $\epsilon$ . (A.9) can now be rewritten as follows:

$$\begin{aligned} \Psi(z, t) + \epsilon \frac{\partial}{\partial t} \Psi(z, t) &= \int_{-\infty}^{+\infty} \frac{1}{\mathcal{A}} \exp\left[\frac{i\eta^2}{2\epsilon}\right] [1 - i\epsilon V(z, t)] \\ &\quad \times \left[ \Psi(z, t) + \eta \frac{\partial}{\partial z} \Psi(z, t) + \frac{1}{2} \eta^2 \frac{\partial^2}{\partial z^2} \Psi(z, t) \right] d\eta. \end{aligned}$$

The first term on the right hand side of this equation turns out to be:

$$\frac{1}{\mathcal{A}} \left( \int_{-\infty}^{+\infty} \exp\left[\frac{i\eta^2}{2\epsilon}\right] d\eta \right) \Psi(z, t) = \frac{1}{\mathcal{A}} (2\pi i\epsilon)^{\frac{1}{2}} \Psi(z, t).$$

$\mathcal{A}$  must be chosen such that the above equation equals  $\Psi(z, t)$ , i.e.  $\frac{\sqrt{2\pi i\epsilon}}{\mathcal{A}} = 1$ . It's obvious to see that the correct choice is  $\mathcal{A} = \sqrt{2\pi i\epsilon}$ . There are  $N$  points, or  $N$  kernels, so

$$A = \mathcal{A}^{-N} = (2\pi i\epsilon)^{-\frac{N}{2}}. \quad (\text{A.10})$$

The factor,  $F(t)$  in (A.7b) becomes now:

$$\begin{aligned} F(t) &= A \sqrt{\frac{(2i\pi\epsilon)^{N-1}}{N}} = \sqrt{\frac{1}{2i\pi(\epsilon N)}}, \\ &= \sqrt{\frac{1}{2\pi i t}}, \end{aligned} \quad (\text{A.11})$$

which is what was expected.



## §A.2: Further Calculations: Free Particle Non-Relativistic Case

Above in §3.4 the wave function and density function were derived for an initial Gaussian state in a potential field,  $V(z) = -kz$ , for the motion along the  $z$ -axis. The motion along the  $x$  and  $y$  axes was not included. The motion along the  $x$  and  $y$  axes is that of a free particle and is now be studied so that the complete wave function and the complete density function can be calculated.

### §A.2.1: Wave Functions and Density Functions for the Free Particle

The following will comprise calculations for the wave function for the free particle on the  $x$  and  $y$  axes and the respective density functions, in both coordinate and momentum spaces.

Above the evolution function for the free particle case was found using the method of path integrals. This will now be used to find a propagated wave function for an initial Gaussian state for both the  $x$  and  $y$  axes. This will then be used to find a complete three dimensional, time dependent, density function for the coordinate and momentum spaces in the non-relativistic case.

### §A.2.2: Coordinate Space

The derived evolution function, for the  $x$ -axis, is:

$$\langle x|e^{-i\hat{\mathcal{H}}t}|x'\rangle = \sqrt{\frac{1}{2\pi it}} \exp\left[\frac{i(x-x')^2}{2t}\right], \quad (\text{A.12})$$

and for the  $y$ -axis:

$$\langle y|e^{-i\hat{\mathcal{H}}t}|y'\rangle = \sqrt{\frac{1}{2\pi it}} \exp\left[\frac{i(y-y')^2}{2t}\right]. \quad (\text{A.13})$$

The initial Gaussian states are:<sup>10</sup>

$$\Psi(x', 0) = \frac{e^{ip_{x0}x'}}{\sqrt{\sqrt{2\pi}\sigma_x}} e^{-\frac{x'^2}{4\sigma_x^2}}$$

$$\Psi(y', 0) = \frac{e^{ip_{y0}y'}}{\sqrt{\sqrt{2\pi}\sigma_y}} e^{-\frac{y'^2}{4\sigma_y^2}}$$

As above, the propagated wave function, for the  $x$ -axis, is given by the integral:

$$\begin{aligned}\Psi(x, t) &= \theta(t) \langle x | e^{-i\hat{H}t} | \Psi \rangle \\ &= \theta(t) \int_{-\infty}^{+\infty} \langle x | e^{-i\hat{H}t} | x' \rangle \langle x' | \Psi(0) \rangle dx', \\ &= \theta(t) \int_{-\infty}^{+\infty} \sqrt{\frac{1}{2\pi it}} e^{i\frac{(x-x')^2}{2t}} \frac{e^{ip_{x0}x'}}{\sqrt{\sqrt{2\pi}\sigma_x}} e^{-\frac{x'^2}{4\sigma_x^2}} dx', \\ &= \theta(t) \sqrt{\frac{1}{2\pi it} \cdot \frac{1}{\sqrt{2\pi}\sigma_x}} \int_{-\infty}^{+\infty} e^{A_{x'}x'^2 + B_{x'}x' + C_{x'}} dx',\end{aligned}$$

where:

$$A_{x'} = -\frac{t - 2i\sigma_x^2}{4t\sigma_x^2},$$

$$B_{x'} = i\frac{p_{x0}t - x}{t},$$

$$C_{x'} = i\frac{x^2}{2t}.$$

As before, assume  $\theta(t) = 1$ . Using (3.4.3) the propagated wave function is:

$$\Psi(x, t) = \sqrt{-i\frac{2\sigma_x(t + 2i\sigma_x^2)}{\sqrt{2\pi}(t^2 + 4\sigma_x^4)}} \exp\left[\frac{i(x^2 - 4i\sigma_x^2xp_{x0} + 2i\sigma_x^2p_{x0}^2t)(t + 2i\sigma_x^2)}{2(t^2 + 4\sigma_x^4)}\right] \quad (A.14)$$

The  $y$ -axis case is identical to the  $x$ -axis case, so the propagated wave function  $\Psi(y, t)$  can just be written down once the correct substitutions are made to the above equation:

$$\Psi(y, t) = \sqrt{-i\frac{2\sigma_y(t + 2i\sigma_y^2)}{\sqrt{2\pi}(t^2 + 4\sigma_y^4)}} \exp\left[\frac{i(y^2 - 4i\sigma_y^2yp_{y0} + 2i\sigma_y^2p_{y0}^2t)(t + 2i\sigma_y^2)}{2(t^2 + 4\sigma_y^4)}\right] \quad (A.15)$$

The complete wave function for the coordinate space will be simply the product of the individual wave functions, i.e.:

$$\Psi(q, t) = \Psi(x, t)\Psi(y, t)\Psi(z, t).$$

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<sup>10</sup> The quantities  $p_{x0}$  and  $p_{y0}$  are the initial momenta along the respective axes.

Density functions for both these wave functions can now be found. Once these have been found, the complete density function will be a product of the individual density functions, i.e.:

$$\rho(q, t) = \rho(x, t)\rho(y, t)\rho(z, t).$$

The density function for  $\Psi(x, t)$  is found as follows:

$$\begin{aligned}\rho(x, t) &= |\Psi(x, t)|^2 \\ &= \Psi^*(x, t)\Psi(x, t) \\ &= \frac{2\sigma_x}{\sqrt{2\pi(t^2 + 4\sigma_x^4)}} \exp\left[-\frac{2\sigma_x^2(p_{x0}t - x)^2}{(t^2 + 4\sigma_x^4)}\right]\end{aligned}\quad (\text{A.16})$$

As before with the wave functions, the  $y$ -axis density function can just be written down after the correct substitutions are made:

$$\rho(y, t) = \frac{2\sigma_y}{\sqrt{2\pi(t^2 + 4\sigma_y^4)}} \exp\left[-\frac{2\sigma_y^2(p_{y0}t - y)^2}{(t^2 + 4\sigma_y^4)}\right]\quad (\text{A.17})$$

The classical path for the free particle along the  $x$ -axis,  $x_{cl}(t)$ , is found in  $B_{x'}$  as:

$$x_{cl}(t) = p_{x0}t.$$

Using this the corresponding density function,  $\rho(x, t)$  can be written as:

$$\rho(x, t) = \frac{2\sigma_x}{\sqrt{2\pi(t^2 + 4\sigma_x^4)}} \exp\left[-\frac{2\sigma_x^2(x_{cl}(t) - x)^2}{(t^2 + 4\sigma_x^4)}\right].$$

This shows that the probability density function  $\rho(z, t)$  is maximised on the classical path. This holds also for the  $y$ -axis.

### §A.2.3: Momentum Space

The momentum space wave function is simply the Fourier Transform of the coordinate space wave function:

$$\begin{aligned}\langle p|e^{-i\hat{r}t}|\tilde{\Psi}\rangle &= \int_{-\infty}^{+\infty} \langle p|q\rangle\langle q|e^{-i\hat{r}t}|\Psi\rangle dq, \\ &= \int_{-\infty}^{+\infty} \frac{e^{ipq}}{\sqrt{2\pi}}\Psi(q, t) dq.\end{aligned}$$

The momentum space wave function will now be found by generating the Fourier Transform of the coordinate space wave function:

$$\begin{aligned}\tilde{\Psi}(p_x, t) &= \int_{-\infty}^{+\infty} \frac{e^{-ip_x x}}{\sqrt{2\pi}} \sqrt{-i \frac{2\sigma_x(t + 2i\sigma_x^2)}{\sqrt{2\pi}(t^2 + 4\sigma_x^4)}} \\ &\quad \times \exp \left[ \frac{i(x^2 - 4i\sigma_x^2 x p_{x0} + 2i\sigma_x^2 p_{x0}^2 t)(t + 2i\sigma_x^2)}{2(t^2 + 4\sigma_x^4)} \right] dx, \\ &= \sqrt{-i \frac{2\sigma_x(t + 2i\sigma_x^2)}{\sqrt{8\pi^3}(t^2 + 4\sigma_x^4)}} \int_{-\infty}^{+\infty} e^{A_x x^2 + B_x x + C_x} dx,\end{aligned}$$

where:

$$A_x = \frac{i}{2} \left( \frac{t + 2i\sigma_x^2}{t^2 + 4\sigma_x^4} \right),$$

$$B_x = -ip_x + 2\sigma_x^2 p_{x0} \left( \frac{t + 2i\sigma_x^2}{t^2 + 4\sigma_x^4} \right),$$

$$C_x = -\sigma_x^2 p_{x0}^2 t \left( \frac{t + 2i\sigma_x^2}{t^2 + 4\sigma_x^4} \right).$$

Using (3.4.3), the momentum space wave function for the free particle along the  $p_x$ -axis is:

$$\tilde{\Psi}(p_x, t) = \sqrt{\frac{2\sigma_x}{\sqrt{2\pi}}} \exp \left[ -\sigma_x^2 (p_x - p_{x0})^2 - i \frac{p_x t}{2} \right]. \quad (\text{A.18})$$

As above, the corresponding wave function along the  $p_y$ -axis can be found by making the relevant substitutions in (A.18):

$$\tilde{\Psi}(p_y, t) = \sqrt{\frac{2\sigma_y}{\sqrt{2\pi}}} \exp \left[ -\sigma_y^2 (p_y - p_{y0})^2 - i \frac{p_y t}{2} \right]. \quad (\text{A.19})$$

The relevant density functions are found via the same method as used for the case of the coordinate space wave functions. Along the  $p_x$ -axis:

$$\rho(p_x, t) = \frac{2\sigma_x}{\sqrt{2\pi}} e^{-2\sigma_x^2 (p_x - p_{x0})^2}. \quad (\text{A.20})$$

and along the  $p_y$ -axis:

$$\rho(p_y, t) = \frac{2\sigma_y}{\sqrt{2\pi}} e^{-2\sigma_y^2(p_y - p_{y0})^2}. \quad (\text{A.21})$$

As was seen in §3.4.3 in the case of the accelerated particle, the momentum space density function does not spread for the free particle and this result acts as a confirmation of this behaviour of the momentum space wave function.

#### §A.2.4: Schrödinger equation

The wave functions derived above for the free particle along the  $x$  and  $y$  axes must be solutions to the Schrödinger equation if they are correct. As both the  $x$  and  $y$  cases are identical, given the correct substitutions, only one wave function need be checked as a solution for the Schrödinger equation. Here the  $x$ -directional wave function,  $\Psi(x, t)$  and its momentum space counterpart,  $\tilde{\Psi}(p_x, t)$ , will be checked to be a solution.

For the free particle case, the Hamiltonian, in the coordinate space, is:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2} = \frac{1}{2} \frac{\partial^2}{\partial x^2},$$

and in the momentum space is:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2} = \frac{p^2}{2}$$

and so the Schrödinger equation is, for the coordinate space and momentum space respectively:

$$\begin{aligned} \left( \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right) \Psi(x, t) &= 0, \\ \left( \frac{p^2}{2} - i \frac{\partial}{\partial t} \right) \tilde{\Psi}(p_x, t) &= 0. \end{aligned}$$

To simplify the calculation, the Schrödinger equation is also divided by the wave function itself. So the Schrödinger equation for the coordinate space is evaluated as follows:

$$\frac{\left( \frac{1}{2} \frac{\partial^2}{\partial x^2} \Psi(x, t) \right)}{\Psi(x, t)} - \frac{i \frac{\partial}{\partial t} \Psi(x, t)}{\Psi(x, t)} = 0.$$

The momentum space Schrödinger equation is:

$$\frac{\left( \frac{p^2}{2} \tilde{\Psi}(p_x, t) \right)}{\tilde{\Psi}(p_x, t)} - \frac{i \frac{\partial}{\partial t} \tilde{\Psi}(p_x, t)}{\tilde{\Psi}(p_x, t)} = 0.$$

The momentum space Schrödinger equation becomes:

$$\begin{aligned}\frac{\left(\frac{p_x^2}{2}\tilde{\Psi}(p_x, t)\right)}{\tilde{\Psi}(p_x, t)} &= \frac{p_x^2}{2}, \\ \frac{i\frac{\partial}{\partial t}\tilde{\Psi}(p_x, t)}{\tilde{\Psi}(p_x, t)} &= \frac{p_x^2}{2}, \\ \Rightarrow \left(\hat{\mathcal{H}} - i\frac{\partial}{\partial t}\right)\tilde{\Psi}(p_x, t) &= 0.\end{aligned}$$

§A.3: *Notes*

This project was typeset using  $\text{\TeX}$ . Calculations in this project were carried using REDUCE.

#### §A.4: REDUCE Files

The following files demonstrate calculations performed in this work. The files ending *.red* are the input files for REDUCE. The files ending *\_out* are the files output by REDUCE. REDUCE is an open source program and may be obtained for free under the GPL licence from Sourceforge.net.<sup>11</sup>

##### §A.4.1: Input File: *cvwf\_scrho.red*

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<sup>11</sup> <http://reduce-algebra.sourceforge.net/>

§A.4.2: *Input File: relfreeg1.red*



§A.4.3: *Input File: relfreeg2.red*

§A.4.4: *Input File: uvclasspath.red*

§A.4.5: *Output File: cvwf\_scrho\_out*

§A.4.6: *Input File: nonrelplaneschro.red*

§A.4.7: *Output File: nonrelplaneschro\_out*

§A.4.8: *Input File: nonrelscrho.red*

**§A.4.9:** *Output File: nonrelscrho\_out*

§A.4.10: *Input File: relations.red*