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Poisson structure and Action-Angle variables for the Camassa-Holm equation

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Abstract

The Poisson brackets for the scattering data of the Camassa-Holm equation are computed. Consequently, the action-angle variables are expressed in terms of the scattering data.

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1 Introduction

The Camassa-Holm equation (CH)

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

where ω is a real constant, firstly appeared in [22] as an equation with a bi-Hamiltonian structure. In [7] it was pushed forward as a model, describing the unidirectional propagation of shallow water waves over a flat bottom, see also [25]. CH is a completely integrable equation [2, 16, 9, 11, 28, 33], describing permanent and breaking waves [12, 30, 10]. Its solitary waves are stable solitons if $\omega > 0$ [3, 18, 26] or peakons if $\omega = 0$ [17]. CH arises also as an equation of the geodesic flow for the H^1 right-invariant metric on the Bott-Virasoro group (if $\omega > 0$) [31, 13] and on the diffeomorphism group (if $\omega = 0$) [14, 15]. The bi-Hamiltonian form of (1) is [7, 22]:

$$m_t = -(\partial - \partial^3)\frac{\delta H_2[m]}{\delta m} = -(2\omega\partial + m\partial + \partial m)\frac{\delta H_1[m]}{\delta m}. \tag{2}$$

where

$$m = u - u_{xx} \tag{3}$$

and the Hamiltonians are

$$H_1[m] = \frac{1}{2}\int mudx \tag{4}$$

$$H_2[m] = \frac{1}{2}\int (u^3 + uu_x^2 + 2\omega u^2)dx. \tag{5}$$

The integration is from $-\infty$ to ∞ in the case of Schwartz class functions, and over one period in the periodic case.

In general, there exists an infinite sequence of conservation laws (multi-Hamiltonian structure) $H_n[m]$, $n = 0, \pm 1, \pm 2, \ldots$, including (4) and (5), such that [29]

$$(\partial - \partial^3)\frac{\delta H_n[m]}{\delta m} = (2\omega\partial + m\partial + \partial m)\frac{\delta H_{n-1}[m]}{\delta m}. \tag{6}$$

The CH equation can be written as

$$m_t = \{m, H_1\}, \tag{7}$$

where the Poisson bracket is defined as

$$\{A, B\} \equiv \int \frac{\delta A}{\delta m}(-2\omega\partial - m\partial - \partial m)\frac{\delta B}{\delta m}dx, \tag{8}$$

or in more obvious antisymmetric form

$$\{A,B\} = -\int (\omega+m)\Big(\frac{\delta A}{\delta m}\partial\frac{\delta B}{\delta m} - \frac{\delta B}{\delta m}\partial\frac{\delta A}{\delta m}\Big)dx. \tag{9}$$

CH has an infinite number of conserved quantities. Schemes for the computation of the conservation laws can be found in [21, 33, 29, 8, 23].

The equation (1) admits a Lax pair [7, 11]

$$\Psi_{xx} = \Big(\frac{1}{4} + \lambda(m+\omega)\Big)\Psi \tag{10}$$

$$\Psi_t = \Big(\frac{1}{2\lambda} - u\Big)\Psi_x + \frac{u_x}{2}\Psi + \gamma\Psi \tag{11}$$

where γ is an arbitrary constant. We will use this freedom for a proper normalization of the eigenfunctions.

We consider the case where m is a Schwartz class function, $\omega > 0$ and $m(x,0)+\omega > 0$ (see [9, 16] for a discussion of the periodic case). Then $m(x,t)+\omega > 0$ for all t [11]. Let $k^2 = -\frac{1}{4} - \lambda\omega$, i.e.

$$\lambda(k) = -\frac{1}{\omega}\Big(k^2 + \frac{1}{4}\Big). \tag{12}$$

The spectrum of the problem (10) under these conditions is described in [11]. The continuous spectrum in terms of k corresponds to k – real. The discrete spectrum (in the upper half plane) consists of finitely many points $k_n = i\kappa_n$, $n = 1,\dots,N$ where κ_n is real and $0 < \kappa_n < 1/2$.

For all real $k \neq 0$ a basis in the space of solutions of (10) can be introduced, fixed by its asymptotic when $x \to \infty$ [11] (see also [37]):

$$\psi_1(x,k) = e^{-ikx} + o(1), \qquad x \to \infty; \tag{13}$$

$$\psi_2(x,k) = e^{ikx} + o(1), \qquad x \to \infty. \tag{14}$$

Another basis can be introduced, fixed by its asymptotic when $x \to -\infty$:

$$\varphi_1(x,k) = e^{-ikx} + o(1), \qquad x \to -\infty; \tag{15}$$

$$\varphi_2(x,k) = e^{ikx} + o(1), \qquad x \to -\infty. \tag{16}$$

For all real $k \neq 0$ if $\Psi(x,k)$ is a solution of (10), then $\Psi(x,-k)$ is also a solution, thus

$$\varphi_1(x,k) = \varphi_2(x,-k), \qquad \psi_1(x,k) = \psi_2(x,-k). \tag{17}$$

Due to the reality of m in (10) for any k we have

$$\varphi_1(x,k) = \bar{\varphi}_2(x,\bar{k}), \qquad \psi_1(x,k) = \bar{\psi}_2(x,\bar{k}) \tag{18}$$

The vectors of each of the bases are a linear combination of the vectors of the other basis:

$$\varphi_i(x,k) = \sum_{l=1,2} T_{il}(k)\psi_l(x,k) \tag{19}$$

where the matrix $T(k)$ defined above is called the scattering matrix. For real $k \neq 0$, instead of $\varphi_1(x,k)$, $\varphi_2(x,k)$, $\psi_1(x,k)$, $\psi_2(x,k)$ due to (18), for simplicity we can write correspondingly $\varphi(x,k)$, $\bar{\varphi}(x,k)$, $\psi(x,k)$, $\bar{\psi}(x,k)$. Thus $T(k)$ has the form

$$T(k) = \begin{pmatrix} a(k) & b(k) \\ \bar{b}(k) & \bar{a}(k) \end{pmatrix} \tag{20}$$

and clearly

$$\varphi(x,k) = a(k)\psi(x,k) + b(k)\bar{\psi}(x,k). \tag{21}$$

The Wronskian $W(f_1,f_2) \equiv f_1\partial_x f_2 - f_2\partial_x f_1$ of any pair of solutions of (10) does not depend on x. Therefore

$$W(\varphi(x,k),\bar{\varphi}(x,k)) = W(\psi(x,k),\bar{\psi}(x,k)) = 2ik \tag{22}$$

From (21) and (22) it follows that

$$|a(k)|^2 - |b(k)|^2 = 1, \tag{23}$$

i.e. $\det(T(k)) = 1$. Computing the Wronskians $W(\varphi,\bar{\psi})$ and $W(\psi,\varphi)$ and using (21), (22) we obtain:

$$a(k) = (2ik)^{-1}\Big(\bar{\psi}_x(x,k)\varphi(x,k) - \bar{\psi}(x,k)\varphi_x(x,k)\Big), \tag{24}$$

$$b(k) = (2ik)^{-1}\Big(\varphi_x(x,k)\psi(x,k) - \varphi(x,k)\psi_x(x,k)\Big). \tag{25}$$

In analogy with the spectral problem for the KdV equation, the quantities $\mathcal{T}(k) = a^{-1}(k)$ and $\mathcal{R}(k) = b(k)/a(k)$ represent the transmission and reflection coefficients respectively. Indeed, the asymptotic of the eigenfunction $\varphi(x,k)/a(k)$ when $x \to \infty$ is

$$\frac{\varphi(x,k)}{a(k)} = e^{-ikx} + \mathcal{R}(k)e^{ikx} + o(1), \tag{26}$$

i.e. a superposition of incident (e^{-ikx}) and reflected ($\mathcal{R}(k)e^{ikx}$) waves. For $x \to -\infty$ we have a transmitted wave:

$$\frac{\varphi(x,k)}{a(k)} = \mathcal{T}(k)e^{-ikx} + o(1) \tag{27}$$

From (23) it follows that the scattering matrix is unitary, i.e.

$$|\mathcal{T}(k)|^2 + |\mathcal{R}(k)|^2 = 1. \tag{28}$$

In what follows we will show that the entire information about $T(k)$ in (20) is provided by $\mathcal{R}(k)$ for $k > 0$ only. It is sufficient to know $\mathcal{R}(k)$ only on the half line $k > 0$, since from (17) and (21), $\bar{a}(k) = a(-k)$, $\bar{b}(k) = b(-k)$ and thus $\mathcal{R}(-k) = \bar{\mathcal{R}}(k)$. Also, from (28)

$$|a(k)|^2 = (1 - |\mathcal{R}(k)|^2)^{-1}, \tag{29}$$

i.e. $|\mathcal{R}(k)|$ determines $|a(k)|$. In the next section we will show that $|a(k)|$ uniquely determines $\arg(a(k))$ as well.

At the points of the discrete spectrum, $a(k)$ has simple zeroes [11], therefore the Wronskian $W(\varphi, \bar{\psi})$ (24) is zero. Thus φ and $\bar{\psi}$ are linearly dependent:

$$\varphi(x, i\kappa_n) = b_n \bar{\psi}(x, -i\kappa_n). \tag{30}$$

In other words, the discrete spectrum is simple, there is only one (real) eigenfunction $\varphi^{(n)}(x)$, corresponding to each eigenvalue $i\kappa_n$, and we can take this eigenfunction to be

$$\varphi^{(n)}(x) \equiv \varphi(x, i\kappa_n) \tag{31}$$

Moreover, one can argue (see [37]), that (cf. (30), (18) and (21))

$$b_n = b(i\kappa_n) \tag{32}$$

The asymptotic of $\varphi^{(n)}$, according to (15), (14), (30) is

$$\varphi^{(n)}(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \qquad x \to -\infty; \tag{33}$$
$$\varphi^{(n)}(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \qquad x \to \infty. \tag{34}$$

The sign of b_n obviously depends on the number of the zeroes of $\varphi^{(n)}$. Suppose that $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N < 1/2$. Then from the oscillation theorem for the Sturm-Liouville problem [5], $\varphi^{(n)}$ has exactly $n-1$ zeroes. Therefore

$$b_n = (-1)^{n-1}|b_n|. \tag{35}$$

The set

$$\mathcal{S} \equiv \{\mathcal{R}(k) \quad (k > 0), \quad \kappa_n, \quad |b_n|, \quad n = 1, \ldots N\} \tag{36}$$

is called scattering data. In what follows we will compute the Poisson brackets for the scattering data and we will also express the Hamiltonians for the CH equation in terms of the scattering data. The derivation is similar to that for other integrable systems, e.g. [37, 35, 36, 19, 6].

The time evolution of the scattering data can be easily obtained as follows. From (21) with $x \to \infty$ one has

$$\varphi(x,k) = a(k)e^{-ikx} + b(k)e^{ikx} + o(1). \tag{37}$$

The substitution of $\varphi(x,k)$ into (11) with $x \to \infty$ gives

$$\varphi_t = \frac{1}{2\lambda}\varphi_x + \gamma\varphi \tag{38}$$

From (37), (38) with the choice $\gamma = ik/2\lambda$ we obtain

$$\dot{a}(k,t) = 0, \tag{39}$$
$$\dot{b}(k,t) = \frac{ik}{\lambda} b(k,t), \tag{40}$$

where the dot stands for derivative with respect to t. Thus

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)e^{\frac{ik}{\lambda}t}; \tag{41}$$

$$\mathcal{T}(k,t) = \mathcal{T}(k,0), \qquad \mathcal{R}(k,t) = \mathcal{R}(k,0)e^{\frac{ik}{\lambda}t}. \tag{42}$$

In other words, $a(k)$ is independent on t and will serve as a generating function of the conservation laws.

The time evolution of the data on the discrete spectrum is found as follows. $i\kappa_n$ are zeroes of $a(k)$, which does not depend on t, and therefore $\dot{\kappa}_n = 0$. From (32) and (40) one can obtain

$$\dot{b}_n = \frac{4\omega\kappa_n}{1-4\kappa_n^2} b_n. \tag{43}$$

The conservation laws are expressed through the scattering data in Section 2 and the Poisson brackets for the scattering data are computed in Section 3.

2 Conservation laws and scattering data

The solution of (10) can be represented in the form

$$\varphi(x,k) = \exp\Big(- ikx + \int_{-\infty}^{x} \chi(y,k)dy\Big). \tag{44}$$

For Im $k > 0$ and $x \to \infty$, $\varphi(x,k)e^{ikx} = a(k)$, i.e.

$$\ln a(k) = \int_{-\infty}^{\infty} \chi(x,k)dx, \qquad \text{Im } k > 0. \tag{45}$$

Since $a(k)$ does not depend on t, the expressions $\int_{-\infty}^{\infty} \chi(x,k)dx$ represent integrals of motion for all k. The equation for $\chi(x,k)$ follows from (10) and (44)

$$\chi_x(x,k) + \chi^2 - 2ik\chi = -\frac{1}{\omega}\Big(k^2 + \frac{1}{4}\Big)m(x) \tag{46}$$

and admits a solution with the asymptotic expansion

$$\chi(x,k) = p_1 k + p_0 + \sum_{n=1}^{\infty} \frac{p_{-n}}{k^n}. \tag{47}$$

The substitution of (47) into (46) gives the following quadratic equation for p_1:

$$p_1^2 - 2ip_1 + \frac{m}{\omega} = 0, \tag{48}$$

with solutions

$$p_1 = i\Big(1 \pm \sqrt{1 + \frac{m}{\omega}}\Big) \tag{49}$$

Since $\int_{-\infty}^{\infty} p_1(x)dx$ is an integral of the CH equation, presumably finite, we take the minus sign in (49). One can easily see that p_0 and all p_{-2n} are total derivatives [23] and thus we have the expansion

$$\ln a(k) = -i\alpha k + \sum_{n=1}^{\infty} \frac{I_{-n}}{k^n}, \tag{50}$$

where α is a positive constant (integral of motion):

$$\alpha = \int_{-\infty}^{\infty} \Big(\sqrt{1 + \frac{m(x)}{\omega}} - 1\Big) dx, \tag{51}$$

and $I_{-n} = \int_{-\infty}^{\infty} p_{-n} dx$ are the other integrals, whose densities, p_{-n} can be obtained reccurently from (46), (47) [23]. For example

$$p_0 = \frac{q_x}{4q}, \qquad q \equiv m + \omega, \tag{52}$$

$$p_{-1} = \frac{1}{8} p_1 + i\frac{\sqrt{\omega}}{8}\Big[\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{\omega}} + \frac{q_x^2}{4q^{5/2}} + \Big(\frac{q_x}{q^{3/2}}\Big)_x\Big], \tag{53}$$

etc., i.e.

$$I_{-1} = -\frac{1}{8} i\alpha + i\frac{\sqrt{\omega}}{8} \int_{-\infty}^{\infty} \Big(\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{\omega}} + \frac{q_x^2}{4q^{5/2}}\Big) dx, \qquad \ldots \tag{54}$$

The asymptotic of $a(k)$ for Im $k > 0$ and $|k| \to \infty$ from (50) is $a(k) \to e^{-i\alpha k}$, or

$$e^{i\alpha k} a(k) \to 1, \qquad \text{Im } k > 0, \qquad |k| \to \infty. \tag{55}$$

Now let us consider the function

$$a_1(k) \equiv e^{i\alpha k} \prod_{n=1}^{N} \frac{k + i\kappa_n}{k - i\kappa_n} a(k). \tag{56}$$

This function is analytic for Im $k > 0$, but does not have any zeroes there. This is due to the fact [11] that $a(k)$ has at most simple zeroes at the points of the discrete spectrum $i\kappa_n$. Therefore $\ln a_1(k)$ is analytic in the upper half plane and due to (55) $\ln a_1(k) \to 0$ for $|k| \to \infty$. Moreover, on the real line $|a_1(k)| = |a(k)|$, and the Kramers-Kronig dispersion relation [24] for the function

$$\ln a_1(k) = \ln |a(k)| + i \arg a_1(k) \tag{57}$$

gives $\arg a_1(k)$ (the symbol P means the principal value):

$$\arg a_1(k) = -\frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\ln |a(k')|}{k' - k} dk' \tag{58}$$

Therefore, from (57), (58), for real k:

$$\ln a_1(k) = \ln|a(k)| - \frac{i}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\ln|a(k')|}{k'-k}dk' \tag{59}$$

With the help of the Sohotski-Plemelj formula we have (cf. [37, 24])

$$\ln a_1(k) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\ln|a(k')|}{k'-k-i0}dk', \tag{60}$$

or with (56)

$$\ln a(k) = -i\alpha k + \sum_{n=1}^{N}\ln\frac{k-i\kappa_n}{k+i\kappa_n} + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\ln|a(k')|}{k'-k-i0}dk'. \tag{61}$$

We will argue that (60), (61) are valid not only for real k, but also when k is in the upper half plane. Indeed, from the Cauchy theorem (the closed contour Γ consists from the real axis and the infinite semicircle in the upper half plane, where $\ln a_1(k) = 0$) for the function $\ln a_1(k)$ and $\mathrm{Im}\ k > 0$ we have:

$$\ln a_1(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln a_1(k')}{k'-k}dk'. \tag{62}$$

The substitution of (60) into (62) gives

$$\ln a_1(k) = \frac{1}{2(\pi i)^2}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\frac{dk'}{(k'-k)(k''-k'-i0)}\Big)\ln|a(k'')|dk'' \tag{63}$$

Computing the integral in the brackets with the residue theorem, the contour Γ being as before (note that the pole at $k' = k'' - i0$ is outside the contour, since k'' is real) we find

$$\ln a_1(k) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\ln|a(k'')|dk''}{k''-k}, \qquad \mathrm{Im}\ k > 0. \tag{64}$$

Therefore, from (56) and (64) when k is in the upper half plane:

$$\ln a(k) = -i\alpha k + \sum_{n=1}^{N}\ln\frac{k-i\kappa_n}{k+i\kappa_n} + \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\ln|a(k')|}{k'-k}dk' \tag{65}$$

Equation (46) can also be written in the form

$$\chi_x(x,k) + (\chi - ik)^2 = \frac{1}{4} + \lambda(k)(m(x)+\omega) \tag{66}$$

and admits a solution with the asymptotic expansion

$$\chi(x,k) = \frac{1}{2} + ik + \sum_{n=1}^{\infty}p_n\lambda(k)^n. \tag{67}$$

Since $\lambda(i/2)=0$, then $\chi(x,i/2)=0$ and therefore, due to (45) $\ln a(i/2)=0$. Now (65) for $k=i/2$ gives the integral α (51) in terms of the scattering data:

$$\alpha=\sum_{n=1}^{N}\ln\Big(\frac{1+2\kappa_n}{1-2\kappa_n}\Big)^2-\frac{8}{\pi}\int_0^\infty\frac{\ln|a(\widetilde{k})|}{4\widetilde{k}^2+1}d\widetilde{k}. \tag{68}$$

The integral $\sqrt{\omega}\alpha$ is equal to H_{-1}, cf. (6), [29, 8, 23]:

$$\begin{aligned} H_{-1} &\equiv \int_{-\infty}^{\infty}\Big(\sqrt{\omega+m(x)}-\sqrt{\omega}\Big)dx \\ &= \sqrt{\omega}\sum_{n=1}^{N}\ln\Big(\frac{1+2\kappa_n}{1-2\kappa_n}\Big)^2-\frac{8\sqrt{\omega}}{\pi}\int_0^\infty\frac{\ln|a(\widetilde{k})|}{4\widetilde{k}^2+1}d\widetilde{k}. \end{aligned}$$

From (29) we know that $|\mathcal{R}(k)|$ determines $|a(k)|$. But $|a(k)|$ determines uniquely $a(k)$ for real k due to (61) and (68). Therefore $|\mathcal{R}(k)|$ determines uniquely $a(k)$ for real k. Since $b(k)$ is simply $a(k)\mathcal{R}(k)$, then as expected, the entire information about $T(k)$ (20) is provided by $\mathcal{R}(k)$ for $k>0$.

The integrals I_{-n} can be expressed by the scattering data from (50) and (65) taking the expansion at $|k|\to\infty$: $I_{-2n}=0$;

$$I_{-(2n+1)}=\frac{2i(-1)^{n+1}}{2n+1}\sum_{n=1}^{N}\kappa_n^{2n+1}+\frac{2i}{\pi}\int_0^\infty\widetilde{k}^{2n}\ln|a(\widetilde{k})|d\widetilde{k}. \tag{69}$$

For example, from I_{-1}, expressed from (54), (69) and using (68), the conservation law H_{-2} (see (6)) can be expressed by the scattering data:

$$\begin{aligned} H_{-2}\equiv&-\frac{1}{4}\int_{-\infty}^{\infty}\Big(\frac{1}{\sqrt{q}}-\frac{1}{\sqrt{\omega}}+\frac{q_x^2}{4q^{5/2}}\Big)dx= \\ &-\frac{1}{4\sqrt{\omega}}\sum_{n=1}^{N}\Big(\ln\Big(\frac{1+2\kappa_n}{1-2\kappa_n}\Big)^2-16\kappa_n\Big)-\frac{2}{\pi\sqrt{\omega}}\int_0^\infty\frac{8\widetilde{k}^2+1}{4\widetilde{k}^2+1}\ln|a(\widetilde{k})|d\widetilde{k}. \end{aligned}$$

From (66) and (67) we obtain (recall that $q=m+\omega=u-u_{xx}+\omega$),

$$p_1+p_1'=q,\qquad p_1=u-u_x+\omega, \tag{70}$$

which leads to the integral:

$$H_0=\int_{-\infty}^{\infty}m dx, \tag{71}$$

i.e.

$$\int_{-\infty}^{\infty} p_1 dx = H_0 + \int_{-\infty}^{\infty} \omega dx \tag{72}$$

The infinite contribution $\int_{-\infty}^{\infty} dx$ does not make sense, but all such contributions cancel the also infinite term $\int_{-\infty}^{\infty}(1/2 + ik)dx$ from (67) when it is substituted in (45).

The next equation from (66) and (67) is

$$p_2 + p_2' + p_1^2 = 0, \tag{73}$$

and hence, formally (recall (4), (71))

$$\int_{-\infty}^{\infty} p_2 dx = -\int_{-\infty}^{\infty} p_1^2 dx = -2H_1 - 2\omega H_0 - \int_{-\infty}^{\infty} \omega^2 dx. \tag{74}$$

From (66) and (67) the equation for p_3 is

$$p_3 + p_3' + 2p_1 p_2 = 0, \tag{75}$$

and the next conserved quantity $\int_{-\infty}^{\infty} p_3 dx = -2\int_{-\infty}^{\infty} p_1 p_2 dx$ is (using (70), (73), (5) – see the technicalities described in [23])

$$\int_{-\infty}^{\infty} p_3 dx = 4H_2 + 4\omega H_1 + 6\omega^2 H_0 + 2\int_{-\infty}^{\infty} \omega^3 dx. \tag{76}$$

Now, let us expand $\ln a(k)$ about the point $k = i/2$. To this end, for simplicity, we define a new parameter, l, such that:

$$k \equiv \frac{i}{2}(1+4l)^{1/2}, \qquad \lambda \equiv \frac{l}{\omega}. \tag{77}$$

The expansion is now about $l = 0$. Using (77) and (67) in (45), and the expressions (72), (74), (76) we finally obtain

$$\ln a(k(l)) = \frac{l}{\omega} H_0 - \left(\frac{l}{\omega}\right)^2 (2H_1 + 2\omega H_0) + \left(\frac{l}{\omega}\right)^3 (4H_2 + 4\omega H_1 + 6\omega^2 H_0) + o(l^3). \tag{78}$$

Now the expansion of (65) in l (77), taking into account (68), (78) gives the Hamiltonians in terms of the scattering data:

$$H_0 = 2\omega \sum_{n=1}^{N} \Big(\ln \frac{1+2\kappa_n}{1-2\kappa_n} + \frac{4\kappa_n}{1-4\kappa_n^2} \Big) - \frac{16\omega}{\pi} \int_0^\infty \frac{\ln|a(\widetilde{k})|}{(4\widetilde{k}^2+1)^2} d\widetilde{k}, \tag{79}$$

$$H_1 = \omega^2 \sum_{n=1}^{N} \Big(\ln \frac{1-2\kappa_n}{1+2\kappa_n} + \frac{4\kappa_n(1+4\kappa_n^2)}{(1-4\kappa_n^2)^2} \Big) + \frac{128\omega^2}{\pi} \int_0^\infty \frac{\widetilde{k}^2 \ln|a(\widetilde{k})|}{(4\widetilde{k}^2+1)^3} d\widetilde{k}, \tag{80}$$

$$H_2 = \omega^3 \sum_{n=1}^{N} \Big(\ln \frac{1-2\kappa_n}{1+2\kappa_n} + \frac{4\kappa_n(3+32\kappa_n^2-48\kappa_n^4)}{3(1-4\kappa_n^2)^3} \Big) - \frac{8\omega^3}{\pi} \int_0^\infty \frac{(-5+28\widetilde{k}^2+80\widetilde{k}^4+64\widetilde{k}^6)\ln|a(\widetilde{k})|}{(4\widetilde{k}^2+1)^4} d\widetilde{k}. \tag{81}$$

In the same fashion the higher conservation laws (which are nonlocal) can be expressed through the scattering data.

3 Poisson brackets of the scattering data

In this section our aim will be to compute the Poisson brackets between the elements of the scattering matrix (20). Let us consider, for example, $\{a(k_1), b(k_2)\}$:

$$\{a(k_1), b(k_2)\} = -\int_{-\infty}^{\infty} q(x) \Big(\frac{\delta a(k_1)}{\delta m(x)} \frac{\partial}{\partial x} \frac{\delta b(k_2)}{\delta m(x)} - \frac{\delta b(k_2)}{\delta m(x)} \frac{\partial}{\partial x} \frac{\delta a(k_1)}{\delta m(x)} \Big) dx. \tag{82}$$

For the computation of $\delta a(k)/\delta m(x)$ and $\delta b(k)/\delta m(x)$ we use (24) and (25):

$$\frac{\delta a(k)}{\delta m(x)} = (2ik)^{-1} \Big(\frac{\delta \varphi(y,k)}{\delta m(x)} \frac{\partial}{\partial y} \bar{\psi}(y,k) - \frac{\delta \bar{\psi}(y,k)}{\delta m(x)} \frac{\partial}{\partial y} \varphi(y,k) + \varphi(y,k) \frac{\partial}{\partial y} \frac{\delta \bar{\psi}(y,k)}{\delta m(x)} - \bar{\psi}(y,k) \frac{\partial}{\partial y} \frac{\delta \varphi(y,k)}{\delta m(x)} \Big). \tag{83}$$

The function $G(x,y,k) \equiv \delta\varphi(y,k)/\delta m(x)$ satisfies the equation, obtained as a variational derivative of (10):

$$\Big(\partial_y^2 - \lambda m(y) + k^2 \Big) G(x,y,k) = \lambda \delta(x-y) \varphi(y,k). \tag{84}$$

Since the source on the right hand side of (84) is zero for $y < x$ (due to the delta-function) and since $\varphi(y,k)$ is defined by its asymptotic when $y \to -\infty$,

i.e. for $y \to -\infty$, $\varphi(y,k)$ does not depend on $m(x)$, the solution of (84) must satisfy

$$G(x,y,k) = 0, \qquad y < x. \tag{85}$$

$G(x,y,k)$, considered as a solution of (84) is a continuous function of y for all y, however due to the source on the right hand side (proportional to a delta function) $\partial G(x,y,k)/\partial y$ has a finite jump at $y = x$. The value of $\partial G(x,y,k)/\partial y$ for $y \to x+0$ can be found by integrating both sides of (84) from $x-\varepsilon$ to $x+\varepsilon$ and then taking $\varepsilon \to +0$:

$$\frac{\partial G(x,y,k)}{\partial y}\Big|_{y=x+0} = \lambda\varphi(x,k). \tag{86}$$

Now we can make use of the fact, that the left hand side of (83) does not depend on y. We take $y = x+\varepsilon$, $\varepsilon > 0$ and then we take $\varepsilon \to 0$. Since $\psi(y,k)$ is defined by its asymptotic when $y \to \infty$, i.e. for $y \to \infty$, $\psi(y,k)$ does not depend on $m(x)$, by an analogous arguments we conclude that $\delta\psi(y,k)/\delta m(x) = 0$ for $y > x$. Then finally from (83) it follows:

$$\frac{\delta a(k)}{\delta m(x)} = -\frac{\lambda}{2ik}\bar{\psi}(x,k)\varphi(x,k). \tag{87}$$

Similarly, we find

$$\frac{\delta b(k)}{\delta m(x)} = \frac{\lambda}{2ik}\psi(x,k)\varphi(x,k). \tag{88}$$

Substituting (87), (88) in (83), we have

$$\begin{aligned}\{a(k_1), b(k_2)\} = \frac{\lambda(k_1)\lambda(k_2)}{(2i)^2 k_1 k_2} \times \\ \int_{-\infty}^{\infty} q(x)\Big(\bar{\psi}(x,k_1)\varphi(x,k_1)\frac{\partial}{\partial x}(\psi(x,k_2)\varphi(x,k_2)) \\ -\psi(x,k_2)\varphi(x,k_2)\frac{\partial}{\partial x}(\bar{\psi}(x,k_1)\varphi(x,k_1))\Big)dx.\end{aligned} \tag{89}$$

The expression under the integral in (89) is a total derivative. Indeed, let f_1, g_1, f_2, g_2 be two pairs of solutions of (10) with spectral parameters k_1 and k_2 correspondingly, i.e.

$$\begin{aligned}\partial_x^2 f_{1,2} &= \Big(\frac{1}{4} + \lambda(k_{1,2})q(x)\Big)f_{1,2}, \\ \partial_x^2 g_{1,2} &= \Big(\frac{1}{4} + \lambda(k_{1,2})q(x)\Big)g_{1,2}.\end{aligned} \tag{90}$$

Then with (90) one can check easily the following identity:

$$q(x)\Big(f_1g_1\frac{\partial}{\partial x}(f_2g_2)-f_2g_2\frac{\partial}{\partial x}(f_1g_1)\Big)=$$
$$\frac{1}{\lambda(k_2)-\lambda(k_1)}\Big((g_1\partial_xg_2-g_2\partial_xg_1)(f_1\partial_xf_2-f_2\partial_xf_1)\Big)_x. \tag{91}$$

Now we can take $f_1=\bar\psi(x,k_1)$, $f_2=\psi(x,k_2)$, $g_1=\varphi(x,k_1)$, $g_2=\varphi(x,k_2)$ and substitute in (89). Then with (91) and with the asymptotic representations for $x\to\infty$

$$\psi(x,k)\to e^{-ikx},\qquad \varphi(x,k)\to a(k)e^{-ikx}+b(k)e^{ikx}, \tag{92}$$

and for $x\to-\infty$

$$\varphi(x,k)\to e^{-ikx},\qquad \psi(x,k)\to \bar a(k)e^{-ikx}-b(k)e^{ikx} \tag{93}$$

we obtain

$$\{a(k_1),b(k_2)\}=\frac{\lambda(k_1)\lambda(k_2)}{(2i)^2k_1k_2(\lambda(k_2)-\lambda(k_1))}\times$$
$$\Big[\lim_{x\to\infty}\Big((k_1^2-k_2^2)a(k_1)a(k_2)e^{-2ik_2x}-(k_1^2-k_2^2)b(k_1)b(k_2)e^{2ik_1x}$$
$$+(k_1+k_2)^2a(k_1)b(k_2)-(k_1+k_2)^2b(k_1)a(k_2)e^{2i(k_1-k_2)x}\Big)$$
$$-\lim_{x\to-\infty}\Big((k_1^2-k_2^2)a(k_1)\bar a(k_2)e^{-2ik_2x}-(k_1^2-k_2^2)\bar b(k_1)b(k_2)e^{-2ik_1x}$$
$$-(k_1-k_2)^2a(k_1)b(k_2)+(k_1-k_2)^2\bar b(k_1)\bar a(k_2)e^{-2i(k_1+k_2)x}\Big)\Big]. \tag{94}$$

The expression on the right hand side in (94) is defined only as a distribution. Then applying the formula $\lim_{x\to\infty}\mathrm{P}\frac{e^{ikx}}{k}=\pi i\delta(k)$ and assuming $k_{1,2}>0$ we have

$$\{\ln a(k_1),\ln b(k_2)\}=\omega\lambda(k_1)\lambda(k_2)\Big(-\frac{k_1^2+k_2^2}{2k_1k_2(k_1^2-k_2^2)}+\frac{\pi i}{2k_1}\delta(k_1-k_2)\Big). \tag{95}$$

In the same fashion the rest of the Poisson brackets between the scattering data can be computed. The result can be expressed in terms of the quantities

$$\rho(k)\equiv-\frac{2k}{\pi\omega\lambda(k)^2}\ln|a(k)|,\qquad \phi(k)\equiv\arg b(k),\qquad k>0: \tag{96}$$

Their Poisson brackets have the canonical form

$$\{\phi(k_1), \rho(k_2)\} = \delta(k_1 - k_2), \quad \{\phi(k_1), \phi(k_2)\} = \{\rho(k_1), \rho(k_2)\} = 0 \tag{97}$$

and thus (96) are the action-angle variables for the CH equation, related to the continuous spectrum. Note that from (97) and (80) we have

$$\dot{\phi} = \{\phi, H_1\} = \frac{k}{\lambda(k)}, \tag{98}$$

which agrees with (41).

Let us now concentrate on the discrete spectrum. Let us denote, for simplicity, $\lambda_n \equiv \lambda(i\kappa_n)$. We will need the variational derivatives $\delta\lambda_n/\delta m(x)$ and $\delta b_n/\delta m(x)$. Due to (32), for $\delta b_n/\delta m(x)$ the expression (88) will be formally used followed by taking the limit $k \to i\kappa_n$. In order to find $\delta\lambda_n/\delta m(x)$ we proceed as follows. Differentiating the equation

$$\varphi^{(n)}_{xx} = \Big(\frac{1}{4} + \lambda_n q(x)\Big)\varphi^{(n)}, \tag{99}$$

we obtain ($\delta q = \delta m$):

$$\delta\varphi^{(n)}_{xx} = (\delta\lambda_n) q\varphi^{(n)} + \lambda_n(\delta m)\varphi^{(n)} + \Big(\frac{1}{4} + \lambda_n q\Big)\delta\varphi^{(n)}. \tag{100}$$

From (99) and (100) it follows

$$(\varphi^{(n)}\delta\varphi^{(n)}_x - \varphi^{(n)}_x\delta\varphi^{(n)})_x = (\delta\lambda_n) q[\varphi^{(n)}]^2 + \lambda_n(\delta m)[\varphi^{(n)}]^2. \tag{101}$$

The integration of (101) gives:

$$(\delta\lambda_n)\int_{-\infty}^{\infty} q(x)[\varphi^{(n)}(x)]^2 dx = -\lambda_n \int_{-\infty}^{\infty} (\delta m(x))[\varphi^{(n)}(x)]^2 dx. \tag{102}$$

or

$$\frac{\delta \ln \lambda_n}{\delta m(x)} = -\frac{[\varphi^{(n)}(x)]^2}{\int_{-\infty}^{\infty} q(y)[\varphi^{(n)}(y)]^2 dy}. \tag{103}$$

From (10) it is not difficult to obtain

$$(\varphi\varphi_{x\lambda} - \varphi_x\varphi_\lambda)_x = q\varphi^2. \tag{104}$$

We will integrate (104) and then take the limit $k \to i\kappa_n$, i.e. $\lambda \to \lambda_n$. We take into account that with this limit, clearly

$$\varphi(x,k) \quad \to \quad b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \qquad x \to \infty; \tag{105}$$

$$\varphi_\lambda(x,k) \quad \to \quad \frac{a'(i\kappa_n)}{\lambda'(i\kappa_n)} e^{\kappa_n x} + o(e^{\kappa_n x}), \qquad x \to \infty. \tag{106}$$

Therefore

$$i\omega b_n a'(i\kappa_n) = \int_{-\infty}^{\infty} q(y)[\varphi^{(n)}(y)]^2 dy \tag{107}$$

and finally

$$\frac{\delta \ln \lambda_n}{\delta m(x)} = \frac{i[\varphi^{(n)}(x)]^2}{\omega b_n a'(i\kappa_n)}. \tag{108}$$

We compute the expression

$$\{\ln \lambda_n, b_l\} = \frac{i\lambda_l}{2\omega b_n \kappa_l a'(i\kappa_n)} \times \int_{-\infty}^{\infty} q(x) \lim_{k_j \to i\kappa_j} \Big(\varphi^2(x,k_n)(\psi(x,k_l)\varphi(x,k_l))_x - \psi(x,k_l)\varphi(x,k_l)(\varphi^2(x,k_n))_x \Big) dx. \tag{109}$$

Taking $f_1 = g_1 = \varphi(x,k_n)$, $f_2 = \psi(x,k_l)$, $g_2 = \varphi(x,k_l)$ in (91) and using the asymptotic representations (92), (93) for $x \to \pm\infty$ we obtain

$$\{\ln \lambda_n, b_l\} = \frac{i\lambda_l}{2 b_n \kappa_l a'(i\kappa_n)} \times \lim_{x\to\infty} \lim_{k_j \to i\kappa_j} \frac{(k_n + k_l)\Big(a(k_n)b(k_n)b(k_l) - a(k_l)b^2(k_n)e^{2i(k_n-k_l)x} \Big)}{k_n - k_l}. \tag{110}$$

Clearly, the right hand side of (110) is zero if $\kappa_n \neq \kappa_l$, since $a(i\kappa_n) = 0$. However, if $\kappa_n = \kappa_l$, the l'Hospital's rule for the limit $\kappa_l \to \kappa_n$ gives

$$\{\ln \lambda_n, b_l\} = -\lambda_n b_n \delta_{nl}. \tag{111}$$

If we define the quantities

$$\rho_n \equiv \lambda_n^{-1}, \qquad \phi_n \equiv -\ln|b_n|, \qquad n = 1, 2, \ldots, N, \tag{112}$$

their Poisson brackets have the canonical form

$$\{\phi_n, \rho_l\} = \delta_{nl}, \qquad \{\phi_n, \phi_l\} = \{\rho_n, \rho_l\} = 0 \tag{113}$$

and thus (113) are the action-angle variables for the CH equation, related to the discrete spectrum. They also commute with the variables on the continuous spectrum (96). Note that from (113) and (80) we have

$$\dot{\phi}_n = \{\phi_n, H_1\} = \{\phi_n, \kappa_n\}\frac{\partial H_1}{\partial \kappa_n} = -\frac{4\omega\kappa_n}{1 - 4\kappa_n^2}, \tag{114}$$

which agrees with (43).

4 Conclusions

In this paper the action-angle variables for the CH equation are computed. They are expressed in terms of the scattering data for this integrable system, when the solutions are confined to be functions in the Schwartz class. The important question about the behavior of the scattering data at $k = 0$ deserves further investigation. It is possible that in the case of singular behavior the Poisson bracket has to be modified in a way, similar to the KdV case, as described in [20, 1, 6]. The case $\omega = 0$ (in which the spectrum is only discrete, cf. [16]) is presented in [34]. The situation when the condition $m(x, 0)+\omega > 0$ does not hold requires separate analysis [27, 4] (if $m(x, 0)+\omega$ changes sign there are infinitely many positive eigenvalues accumulating at infinity, cf. [11]). The action-angle variables for the periodic CH equation and $\omega = 0$ are reported in [32].

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