Dublin Institute of Technology
ARROW@DIT

Articles | School of Mathematics

2007-01-01

Generalised Fourier transform for the Camassa-Holm hierarchy

Adrian Constantin
Trinity College, adrian@maths.tcd.ie

Vladimir Gerdjikov
Bulgarian Academy of Sciences, gerjikov@inrne.bas.bg

Rossen Ivanov
Dublin Institute of Technology, rivanov@dit.ie

Follow this and additional works at: http://arrow.dit.ie/scschmatart

Part of the Dynamic Systems Commons, Mathematics Commons, Non-linear Dynamics Commons, and the Partial Differential Equations Commons

Recommended Citation
A. Constantin, V. Gerdjikov and R. Ivanov, Generalised Fourier transform for the Camassa-Holm hierarchy, Inv. Problems 23 (2007) 1565-1597

This Article is brought to you for free and open access by the School of Mathematics at ARROW@DIT. It has been accepted for inclusion in Articles by an authorized administrator of ARROW@DIT. For more information, please contact yvonne.desmond@dit.ie, arrow.admin@dit.ie.

This work is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 License

Generalised Fourier transform for the Camassa-Holm hierarchy

Adrian Constantin[a,†], **Vladimir S. Gerdjikov**[b,‡] and **Rossen I. Ivanov**[a,*,1]

[a] *School of Mathematics, Trinity College Dublin, Dublin 2, Ireland*
[b] *Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tzarigradsko chaussee, 1784 Sofia, Bulgaria*

[†] *e-mail: adrian@maths.tcd.ie*
[‡] *e-mail: gerjikov@inrne.bas.bg*
[*] *e-mail: ivanovr@maths.tcd.ie*

Abstract

The squared eigenfunctions of the spectral problem associated to the Camassa-Holm equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform for the Camassa-Holm hierarchy as a Generalised Fourier transform. The main result of this work is the derivation of the completeness relation for the squared solutions of the Camassa-Holm spectral problem. We show that all the fundamental properties of the Camassa-Holm equation such as the integrals of motion, the description of the equations of the whole hierarchy and their Hamiltonian structures can be naturally expressed making use of the completeness relation and the recursion operator, whose eigenfunctions are the squared solutions.

PACS: 02.30.Ik, 05.45.Yv, 45.20.Jj, 02.30.Jr

Key Words: Conservation Laws, Lax Pair, Integrable Systems, Solitons.

[1] Present address: School of Mathematical Sciences, DIT Kevin Street, Dublin 8, Ireland

1 Introduction

In this introductory section we shall give a brief account of the basic results about the spectral problem related to the Camassa-Holm equation (CH) [12]. The CH equation

$$u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

where ω is a real constant, arises as a compatibility condition of two linear problems [12]

$$\Psi_{xx} = \Big(\frac{1}{4} + \lambda(m+\omega)\Big)\Psi \tag{2}$$

$$\Psi_t = \Big(\frac{1}{2\lambda} - u\Big)\Psi_x + \frac{u_x}{2}\Psi + \gamma\Psi \tag{3}$$

where $m \equiv u - u_{xx}$ and γ is an arbitrary constant. We will use the freedom provided by the presence of γ for a proper normalization of the eigenfunctions.

The CH equation models just like the Korteweg-de Vries (KdV) equation the propagation of two-dimensional shallow water waves over a flat bed. It also arises in the study of the propagation of axially symmetric waves in hyperelastic rods [32, 30] and its high-frequency limit models nematic liquid crystals [47, 9]; more about the physical applications of this equation can be found e.g. in [12, 52, 53, 34, 35, 45, 49, 75]. Both KdV and CH are integrable systems [68, 5, 14, 26, 16, 61] (see also [12, 39, 40]), but while all smooth data yield solutions of the KdV equation existing for all times, certain smooth initial data for CH lead to global solutions and others to breaking waves: the solution remains bounded but its slope becomes unbounded in finite time (see [18, 15, 10]). The solitary waves of KdV are smooth solitons, while the solitary waves of CH, which are also solitons, are smooth if $\omega > 0$ [12, 53, 70, 71, 72] and peaked (called "peakons" and representing weak solutions) if $\omega = 0$ [12, 19, 5, 6, 27, 63]. Both solitary wave forms for CH are stable [29, 28, 31]. The CH equation arises also as a geodesic equation on the diffeomorphism group (if $\omega = 0$) [15, 23, 24, 60] and on the Bott-Virasoro group (if $\omega > 0$) [67, 22].

The inverse scattering transform (IST) for the CH equation and the related Riemann-Hilbert problem are considered e.g. in [16, 20, 55, 33].

The IST can be realized as a generalized Fourier transform.The complete basis of functions is represented by the squares of the fundamental (Jost) solutions of the corresponding spectral problem. For the famous Zakharov-Shabat spectral problem (and its various generalizations) the problem is studied in detail and the results can be found in several important works, such as [54, 36, 41, 42, 44]. The generalized Fourier transform for the Sturm-Liouville spectral problem, associated to the fundamentally important KdV equation is also well studied and in this relation we can mention the books and articles [3, 11, 57, 36, 48, 58, 59].

The CH spectral problem (2) is of a weighted Sturm-Liouville-type. Our aim will be to construct the generalized Fourier transform which linearizes the CH equation as well as all equations of the whole CH hierarchy of integrable equations. Our main result is the derivation of the completeness relation for the squared solutions of (2). We show that all the fundamental properties of the CH equation such as the integrals of motion, the description of the higher CH-type equations and their Hamiltonian structures can be naturally expressed making

use of the recursion operator $L_\pm$ and the completeness relation. In fact, the squared solutions are eigenfunctions of $L_\pm$ and the completeness of the squared solutions is the spectral decomposition of $L_\pm$.

In Section 2 are given all the necessary mathematical preliminaries about the spectral problem (2). This includes the definition of the Jost solutions and two sets of scattering data, as well as the time evolution of the scattering data.

In Section 3 are presented the asymptotics of the Jost solutions for large values of the spectral parameter ($|k| \to \infty$). The main difference with respect to the standard spectral problem, given by the Schrödinger equation, (e.g. like in the KdV case) lies in the fact that (2) is a weighted spectral problem, which requires different asymptotic expansions.

The completeness relations of the eigenfunctions and the squared eigenfunctions of the spectral problem (2) are presented in Section 4. The last one gives the possibility of expansion of an arbitrary function of the specified class over the complete basis.

The Wronskian relations, derived in Section 5 allow to compute the generalised Fourier coefficients for the potential of our spectral problem and its variation.

The symplectic variables, i.e. the variables, expressed in terms of the squared solutions and satisfying the cannonical relations, with respect to a certain skew-product are given in Section 6.

The recursion operator (whose eigenfunctions are the squared solutions) is computed in Section 7.

In Section 8 the whole Camassa-Holm hierarchy is constructed. The time-evolution of the scattering data for the hierarchy is obtained.

In Section 9 the Hamiltonian structure of the Camassa-Holm hierarchy is explored. The canonical Hamiltonians are expressed both via the potential of the spectral problem and the scattering data. The hierarchy of Poisson structures and action-angle variables with respect to these structures is also obtained in this section.

Finally, in Section 10 the Inverse Scattering Transform for the CH hierarchy is outlined in the framework presented earlier for the CH equation alone [20].

2 Preliminaries

In general, there exists an infinite sequence of conservation laws (multi-Hamiltonian structure) $H_n[m]$, $n = 0, \pm 1, \pm 2, \ldots$, [12, 38, 73, 51, 39, 62] such that

$$H_1 = \frac{1}{2}\int m u \mathrm{d}x, \tag{4}$$

$$H_2 = \frac{1}{2}\int (u^3 + u u_x^2 + 2\omega u^2)\mathrm{d}x, \tag{5}$$

$$(\partial - \partial^3)\frac{\delta H_n[m]}{\delta m} = (2\omega\partial + m\partial + \partial m)\frac{\delta H_{n-1}[m]}{\delta m}. \tag{6}$$

The CH equation can be written as

$$m_t = \{m, H_1\}, \tag{7}$$

where the Poisson bracket is defined as

$$\{A, B\} \equiv -\int \frac{\delta A}{\delta m}(2\omega\partial + m\partial + \partial m)\frac{\delta B}{\delta m}\mathrm{d}x, \tag{8}$$

or in more obvious antisymmetric form.

$$\{A, B\} = -\int (\omega + m)\Big(\frac{\delta A}{\delta m}\partial\frac{\delta B}{\delta m} - \frac{\delta B}{\delta m}\partial\frac{\delta A}{\delta m}\Big)\mathrm{d}x. \tag{9}$$

For simplicity we will consider the case where m is a Schwartz class function, $\omega > 0$ and $m(x, 0)+\omega > 0$. Then $m(x, t)+\omega > 0$ for all t [16]. Let $k^2 = -\frac{1}{4}-\lambda\omega$, i.e.

$$\lambda(k) = -\frac{1}{\omega}\Big(k^2 + \frac{1}{4}\Big). \tag{10}$$

The spectrum of the problem (2) under these conditions is described in [16]. The continuous spectrum in terms of k corresponds to k – real. The discrete spectrum consists of finitely many points $k_n = i\kappa_n$, $n = 1, \ldots, N$ where κ_n is real and $0 < \kappa_n < 1/2$.

A basis in the space of solutions of (2) can be introduced by the analogs of the Jost solutions of eq. (1), $f^+(x, k)$ and $\bar{f}^+(x, \bar{k})$. For all real $k \neq 0$ it is fixed by its asymptotic when $x \to \infty$ [16], see also [68]:

$$\lim_{x\to\infty} e^{-ikx} f^+(x, k) = 1, \tag{11}$$

Another basis can be introduced, $f^-(x, k)$ and$\bar{f}^-(x, \bar{k})$ fixed by its asymptotic when $x \to -\infty$ for all real $k \neq 0$:

$$\lim_{x\to-\infty} e^{ikx} f^-(x, k) = 1, \tag{12}$$

Since $m(x)$ and ω are real one gets that if $f^+(x, k)$ and $f^-(x, k)$ are solutions of (1) then

$$\bar{f}^+(x, \bar{k}) = f^+(x, -k), \qquad \text{and} \qquad \bar{f}^-(x, \bar{k}) = f^-(x, -k), \tag{13}$$

are also solutions of (1). The relations (13) are known as involutions.
In particular, for real $k \neq 0$ we get:

$$\bar{f}^\pm(x, k) = f^\pm(x, -k), \tag{14}$$

and the vectors of the two bases are related [2]:

$$f^-(x, k) = a(k)f^+(x, -k) + b(k)f^+(x, k), \qquad \mathrm{Im}\ k = 0. \tag{15}$$

The Wronskian $W(f_1, f_2) \equiv f_1\partial_x f_2 - f_2\partial_x f_1$ of any pair of solutions of (2) does not depend on x. Therefore

$$W(f^-(x, k), f^-(x, -k)) = W(f^+(x, -k), f^+(x, k)) = 2ik \tag{16}$$

[2] According to the notations used in [20, 21] $f^+(x, k) \equiv \bar{\psi}(x, \bar{k})$, $f^-(x, k) \equiv \varphi(x, k)$.

Computing the Wronskians $W(f^-, f^+)$ and $W(\bar{f}^+, f^-)$ and using (15), (16) we obtain:

$$a(k) = (2ik)^{-1} W(f^-(x,k), f^+(x,k)), \tag{17}$$
$$b(k) = -(2ik)^{-1} W(f^-(x,k), f^+(x,-k)). \tag{18}$$

From (15) and (16) it follows that for real k

$$a(k)a(-k) - b(k)b(-k) = 1. \tag{19}$$

It is well known [16] that $f^+(x,k)e^{-ikx}$ and $f^-(x,k)e^{ikx}$ have analytic extensions in the upper half of the complex k-plane. Likewise $\bar{f}^+(x,\bar{k})e^{i\bar{k}x}$ and $\bar{f}^-(x,\bar{k})e^{-i\bar{k}x}$ allow analytic extension in the lower half of the complex k-plane. An important consequence of these properties is that $a(k)$ also allows analytic extension in the upper half of the complex k-plane and

$$\bar{a}(\bar{k}) = a(-k), \qquad \bar{b}(\bar{k}) = b(-k), \tag{20}$$

As a result (19) acquires the form:

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{21}$$

In other words the relation (17) is valid in the upper half plane Im $k \geq 0$ [21, 20], while (18) makes sense only on the real line Im $k = 0$. In analogy with the spectral problem for the KdV equation, one can introduce the quantities $\mathcal{T}(k) = a^{-1}(k)$ (transmission coefficient) and $\mathcal{R}^{\pm}(k) = b(\pm k)/a(k)$, (reflection coefficients – to the right with superindex $(+)$ and to the left with superindex $(-)$ respectively). From (21) it follows that

$$|\mathcal{T}(k)|^2 + |\mathcal{R}^{\pm}(k)|^2 = 1. \tag{22}$$

It is sufficient to know $\mathcal{R}^{\pm}(k)$ only on the half line $k > 0$, since from (20), $\bar{a}(k) = a(-k)$, $\bar{b}(k) = b(-k)$ and thus $\mathcal{R}^{\pm}(-k) = \bar{\mathcal{R}}^{\pm}(k)$. Also, from (22)

$$|a(k)|^2 = (1 - |\mathcal{R}^{\pm}(k)|^2)^{-1}, \tag{23}$$

One can show that $\mathcal{R}^{\pm}(k)$ uniquely determines $a(k)$ [21].

At the points κ_n of the discrete spectrum, $a(k)$ has simple zeroes [16], i.e.:

$$a(k) = (k - i\kappa_n)\dot{a}_n + \frac{1}{2}(k - i\kappa_n)^2 \ddot{a}_n + \cdots, \tag{24}$$

and the Wronskian $W(f^-, f^+)$, (17) vanishes. Thus f^- and f^+ are linearly dependent:

$$f^-(x, i\kappa_n) = b_n f^+(x, i\kappa_n). \tag{25}$$

In other words, the discrete spectrum is simple, there is only one (real) linearly independent eigenfunction, corresponding to each eigenvalue $i\kappa_n$, say

$$f_n^-(x) \equiv f^-(x, i\kappa_n) \tag{26}$$

From (26) and (11), (12) it follows that $f_n^-(x)$ falls off exponentially for $x \to \pm\infty$, which allows one to show that $f_n(x)$ is square integrable. Moreover, for compactly supported potentials $m(x)$ (cf. (25) and (15))

$$b_n = b(i\kappa_n), \qquad b(-i\kappa_n) = -\frac{1}{b_n}. \tag{27}$$

One can argue [68], that the results from this case can be extended to Schwarz-class potentials by an appropriate limiting procedure.
The asymptotic of f_n^-, according to (14), (11), (25) is

$$f_n^-(x) = e^{\kappa_n x} + o(e^{\kappa_n x}), \qquad x \to -\infty; \tag{28}$$
$$f_n^-(x) = b_n e^{-\kappa_n x} + o(e^{-\kappa_n x}), \qquad x \to \infty. \tag{29}$$

The sign of b_n obviously depends on the number of the zeroes of f_n^-. Suppose that $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_N < 1/2$. Then from the oscillation theorem for the Sturm-Liouville problem [8], f_n^- has exactly $n-1$ zeroes. Therefore

$$b_n = (-1)^{n-1}|b_n|. \tag{30}$$

The sets

$$\mathcal{S}^\pm \equiv \{\mathcal{R}^\pm(k) \quad (k > 0), \quad \kappa_n, \quad C_n^\pm \equiv \frac{b_n^{\pm 1}}{i\dot{a}_n}, \quad n = 1, \ldots N\} \tag{31}$$

are called scattering data. Throughout this work the dot stands for a derivative with respect to k and $\dot{a}_n \equiv \dot{a}(i\kappa_n)$, $\ddot{a}_n \equiv \ddot{a}(i\kappa_n)$, etc. In what follows we will show that each set – $\mathcal{S}^+$ or $\mathcal{S}^-$ of scattering data uniquely determines the potential $m(x)$. The derivation is similar to those for other integrable systems, e.g. [68, 36, 48, 76].

The time evolution of the scattering data can be easily obtained as follows. From (15) with $x \to \infty$ one has

$$\lim_{x\to\infty} \left(f^-(x,k) - a(k)e^{-ikx} - b(k)e^{ikx}\right) = 0. \tag{32}$$

The substitution of $f^-(x,k)$ into (3) with $x \to \infty$ gives

$$\lim_{x\to\infty} (f_t^- - \frac{1}{2\lambda} f_x^- + \gamma f^-) = 0. \tag{33}$$

From (32), (33) with the choice $\gamma = \gamma_- = ik/2\lambda$ we obtain

$$a_t(k,t) = 0, \tag{34}$$

$$b_t(k,t) \quad = \quad \frac{ik}{\lambda} b(k,t). \tag{35}$$

Thus

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)e^{\frac{ik}{\lambda}t}; \tag{36}$$

$$\mathcal{T}(k,t) = \mathcal{T}(k,0), \qquad \mathcal{R}^{\pm}(k,t) = \mathcal{R}^{\pm}(k,0)e^{\pm\frac{ik}{\lambda}t}. \tag{37}$$

Similarly, one can substitute $f^+(x,k)$ as $x \to -\infty$

$$\lim_{x\to\infty} \left(f^+(x,k) - a(k)e^{ikx} + b(-k)e^{-ikx}\right) = 0 \tag{38}$$

into (3). Then the choice of the constant is $\gamma = \gamma_+ = -ik/2\lambda$ and the final result (36) is, of course the same.

In other words, $a(k)$ is independent on t and will serve as a generating function of the conservation laws.

The time evolution of the data on the discrete spectrum is found as follows. $i\kappa_n$ are zeroes of $a(k)$, which does not depend on t, and therefore $\kappa_{n,t} = 0$. From (27) and (35) one can find

$$C_n^{\pm}(t) = C_n^{\pm}(0)\exp\Big(\pm\frac{4\omega\kappa_n}{1-4\kappa_n^2}t\Big). \tag{39}$$

3 Asymptotics of the Jost solutions for $|k| \to \infty$

The analyticity properties of the Jost solutions and of $a(k)$ play an important role in our considerations. We will need also the asymptotics of the Jost solutions for $|k| \to \infty$ which have the form [20, 21]

$$f^+(x,k) = e^{ikx+ik\int_\infty^x(\sqrt{q(y)/\omega}-1)dy}\Big[\Big(\frac{\omega}{q(x)}\Big)^{1/4} + O\Big(\frac{1}{k}\Big)\Big], \tag{40}$$

$$f^-(x,k) = e^{-ikx-ik\int_{-\infty}^x(\sqrt{q(y)/\omega}-1)dy}\Big[\Big(\frac{\omega}{q(x)}\Big)^{1/4} + O\Big(\frac{1}{k}\Big)\Big], \tag{41}$$

where, for simplicity $q(x) \equiv m(x) + \omega$.

An immediate consequence of the above formulae and (17) is:

$$\lim_{k\to\infty} a(k)e^{i\alpha k} = 1, \qquad k \in \mathbb{C}_+, \tag{42}$$

where

$$\alpha = \int_{-\infty}^{\infty}\Big(\sqrt{\frac{q(x)}{\omega}} - 1\Big)\mathrm{d}x. \tag{43}$$

Since $a(k)$ is t-independent, then α, as well as all the coefficients I_k in the asymptotic expansion

$$\ln a(k) = -i\alpha k + \sum_{s=1}^{\infty}\frac{I_s}{k^{2s+1}}, \tag{44}$$

must be integrals of motion. The integral α is the unique Casimir function for the CH, [56]. One can easily check that $\{m, \alpha\} \equiv 0$, for the uniqueness argument and for the geometric interpretation see [56].

The densities, p_s of $I_s = \int_{-\infty}^{\infty} p_s \mathrm{d}x$ can be expressed in terms of $m(x)$ using a set of recurrent relations obtained in [73, 51, 21].

Using the analyticity properties of $a(k)$ one can prove that it satisfies the following dispersion relation ($k \in \mathbb{C}_+$):

$$\ln a(k) = -i\alpha k + \sum_{n=1}^{N} \ln \frac{k - i\kappa_n}{k + i\kappa_n} - \frac{k}{\pi i} \int_0^{\infty} \frac{\ln(1 - |\mathcal{R}^{\pm}(k')|^2)}{k'^2 - k^2} \mathrm{d}k'. \tag{45}$$

where $i\kappa_n$ are the zeroes of $a(k)$. Thus we are able to recover the function $a(k)$ in its whole domain of analyticity $\mathbb{C}_+$ knowing just its modulus $|a(k)|$ or the reflection coefficient $\mathcal{R}^+(k)$, for real k, and the location of its zeroes, see (23).

The dispersion relation (45) allows one to express the integrals of motion also in terms of the scattering data [21]:

$$I_s = \frac{1}{\pi i} \int_0^{\infty} \ln(1 - |\mathcal{R}^{\pm}(k)|^2)\, k^{2s} \mathrm{d}k - \sum_{n=1}^{N} \frac{2i(-1)^s \kappa_n^{2s+1}}{2s+1}. \tag{46}$$

These are known as the trace identities. In addition the integral α is expressed through the scattering data as follows. Note that for $k = i/2$, $\lambda(i/2) = 0$ from (10). In this case therefore the spectral problem (2) does not depend on m, and the eigenfunctions are equal to their asymptotics: $f^{\pm}(x, i/2) = e^{\mp \frac{x}{2}}$. From (17) we obtain $a(i/2) = 1$ and from (45) for $k = i/2$ we have

$$\alpha = \sum_{n=1}^{N} \ln \left(\frac{1 + 2\kappa_n}{1 - 2\kappa_n} \right)^2 + \frac{4}{\pi} \int_0^{\infty} \frac{\ln(1 - |\mathcal{R}^+(\widetilde{k})|^2)}{4\widetilde{k}^2 + 1} \mathrm{d}\widetilde{k}. \tag{47}$$

4 Completeness relations for the Jost solutions

Here we outline the spectral properties of the linear problem (2).

Let us consider the function

$$R_1(x, y, k) \equiv \frac{f^-(x,k) f^+(y,k)}{a(k)} \theta(y - x) + \frac{f^-(y,k) f^+(x,k)}{a(k)} \theta(x - y) \tag{48}$$

where $\theta(x)$ is the step function.

Lemma 1. i) *$R_1(x, y, k)$ is an analytic function of k for $k \in \mathbb{C}_+$;*

ii) *$R_1(x, y, k)$ has simple poles for $k = i\kappa_n$;*

iii) *$R_1(x, y, k)$ is a kernel of a bounded integral operator for $Im\, k > 0$. For $Im\, k = 0$, $R_1(x, y, k)$ is a kernel of a unbounded integral operator.*

iv) *$R_1(x, y, k)$ satisfies the equation:*

$$\frac{d^2 R_1}{dx^2} - \left(\frac{1}{4} + \lambda q(x) \right) R_1(x, y, k) = 4ik\delta(x - y). \tag{49}$$

Proof. i) and ii) are obvious.

iii) For $\operatorname{Im} k > 0$ the statement follows from the definitions (11) and (12) of the Jost solutions, which ensure that $R_1(x,y,k)$ falls off exponentially for all $x, y \to \pm\infty$. The same arguments for $\operatorname{Im} k = 0$ can only ensure that $R_1(x,y,k)$ is a bounded function for $x, y \to \pm\infty$.

iv) is a consequence of (2), (17) and the fact that

$$d\theta(x-y)/dx = \delta(x-y).$$

□

Remark 1. *The function $R_1(x,y,k)/(4ik)$ is a kernel of the resolvent for the linear problem (2).*

The explicit expression for the resolvent (48) can be used to prove the following

Proposition 1. *The Jost solutions of (2) satisfy the completeness relation:*

$$\frac{\omega}{q(x)}\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f^-(x,k)f^+(y,k)}{a(k)}\,dk + \sum_{n=1}^{N} C_n^{\pm} f_n^{\pm}(x) f_n^{\pm}(y) \tag{50}$$

By $\int_{-\infty}^{\infty} dk$ is meant $\lim_{L\to\infty}\int_{-L}^{L} dk$ and $f_n^{\pm}(x) \equiv f^{\pm}(x, i\kappa_n)$.

Proof. Consider the contour integral

$$\mathcal{J}_1(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} R_1(x,y,k). \tag{51}$$

where the contour γ_+ is shown on Fig. 1.
From the Cauchy residue theorem it follows that:

$$\mathcal{J}_1(x,y) = \sum_{n=1}^{N} \operatorname*{Res}_{k=i\kappa_n} R_1(x,y,k) = i\sum_{n=1}^{N} C_n^{\pm} f_n^{\pm}(x) f_n^{\pm}(y). \tag{52}$$

In the evaluation of the residues we made use of (25) and of the fact that $\theta(x-y) + \theta(y-x) = 1$.

Next we evaluate the integral $\mathcal{J}_1(x,y)$ by integrating along the contour. For the integration along the infinite semicircle we need the asymptotic of $R_1(x,y,k)$ for $|k| \to \infty$. From (40) and (41) we get

$$R_1(x,y,k) = e^{ik\int_x^y \sqrt{\frac{q(s)}{\omega}}ds}\Big[\Big(\frac{\omega}{q(x)}\Big)^{\frac{1}{4}}\Big(\frac{\omega}{q(y)}\Big)^{\frac{1}{4}} + o(1/k)\Big]\theta(y-x) + (x \leftrightarrow y) \tag{53}$$

Only the leading terms in (53), which are entire functions of k contribute to the integral. This allows us to deform the infinite semicircle until it coincides with the real k-axis. Then the integration over k is easily performed with the result:

$$\mathcal{J}_{1,\infty}(x,y) = \frac{i\omega}{q(x)}\delta(x-y). \tag{54}$$

To evaluate the integral over the real axis $\mathcal{J}_{1,R}(x,y)$ we will use the the fact that $R_1(x,y,k)$ can be written in the form

$$R_1(x,y,k) \quad = \quad \frac{f^-(x,k)f^+(y,k)}{a(k)} + N_1(x,y,k)\theta(x-y), \tag{55}$$

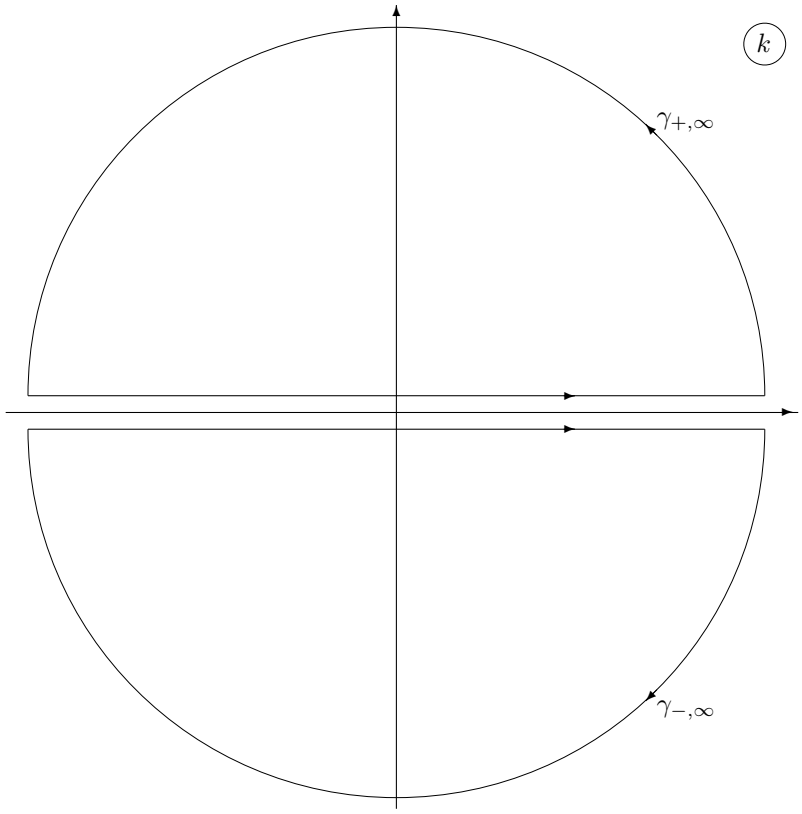

Figure 1: The contours $\gamma_\pm = \mathbb{R} \cup \gamma_{\pm,\infty}$ of integrations.

where

$$N_1(x,y,k) \equiv \frac{1}{a(k)}[f^-(x,k)f^+(y,k) - f^+(x,k)f^-(y,k)] \tag{56}$$

is an odd function of k and does not contribute to $\mathcal{J}_{1,R}(x,y)$. Indeed, from (15) we have

$$\begin{aligned} N_1(x,y,k) &= \frac{f^-(x,k)f^+(y,k)}{a(k)} - f^+(x,k)[f^+(y,-k) + \mathcal{R}(k)f^+(y,k)] \\ &= \frac{f^-(x,k)f^+(y,k)}{a(k)} - f^+(y,-k)\frac{[f^-(x,-k) - b(-k)f^+(x,-k)]}{a(-k)} \\ &- \mathcal{R}(k)f^+(x,k)f^+(y,k) \\ &= \frac{f^-(x,k)f^+(y,k)}{a(k)} - \frac{f^+(y,-k)f^-(x,-k)}{a(-k)} \\ &- \mathcal{R}(-k)f^+(x,-k)f^+(y,-k) - \mathcal{R}(k)f^+(x,k)f^+(y,k). \end{aligned}$$

Now it remains to equate the two expressions for

$$\mathcal{J}_1(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} R_1(x,y,k) = \mathcal{J}_{1,\infty}(x,y) + \mathcal{J}_{1,R}(x,y). \tag{57}$$

to obtain the completeness relation for the Jost solutions. □

Remark 2. *Another way to prove the completeness relations for the Jost solutions is to note that $\bar{R}_1(x,y,\bar{k})$ is a kernel of the resolvent defined for $Im\,k<0$ and then apply the contour integration method to the integral*

$$\mathcal{J}_1'(x,y)=\frac{1}{2\pi i}\oint_{\gamma_+}dk\,R_1(x,y,k)-\frac{1}{2\pi i}\oint_{\gamma_-}dk\,\bar{R}_1(x,y,\bar{k}). \tag{58}$$

5 Wronskian relations and generalized Fourier series expansion

An important tool for the analysis of the mapping between the potential $m(x)$ of eq. (2) and the scattering data, related to this potential, are the so-called *Wronskian relations*.

For what follows, we define the squared eigenfunctions

$$F^{\pm}(x,k)\equiv(f^{\pm}(x,k))^2,\qquad F_n^{\pm}(x)\equiv F^{\pm}(x,i\kappa_n). \tag{59}$$

Proposition 2. *Let $f(k_1,x)$ and $g(k_2,x)$ be two eigenfunctions of the spectral problem (2). Then the following identity (Wronskian relation) holds:*

$$\int_{-\infty}^{\infty}q(x)f(k_1,x)g(k_2,x)dx=\frac{f_x(k_1,x)g(k_2,x)-f(k_1,x)g_x(k_2,x)}{\lambda(k_1)-\lambda(k_2)}\Big|_{x=-\infty}^{\infty} \tag{60}$$

Proof. It follows immediately from the fact that $f(k_1,x)$ and $g(k_2,x)$ satisfy (2).

Corrollary 1. *From (60), (14), (11), (15), and from the formula*

$$\lim_{x\to\infty}\mathrm{P}\frac{e^{ikx}}{k}=\pi i\delta(k),$$

where P *means Principal Value, one can obtain various 'orthogonality' relations for the eigenfunctions, such as (cf. [21])*

$$\int_{-\infty}^{\infty}q(x)f^{+}(k_1,x)f^{-}(k_2,x)dx = -2\pi\omega a(k_1)\delta(k_1-k_2), \tag{61}$$

$$\int_{-\infty}^{\infty}q(x)f_n^{+}(x)f_p^{-}(x)dx = i\omega\dot{a}_n\delta_{np}. \tag{62}$$

□

Let us define a scalar product as usual by

$$(f,g)\equiv\int_{-\infty}^{\infty}f(x)g(x)\mathrm{d}x, \tag{63}$$

provided the corresponding integral exists.

We need also the skew-symmetric product

$$[\![f,g]\!] \equiv \int_{-\infty}^{\infty} q(x)(f_x(x)g(x) - g_x(x)f(x))\mathrm{d}x, \tag{64}$$

related to the Poisson bracket (9) via

$$\{A,B\} = \left[\!\!\left[\frac{\delta A}{\delta m}, \frac{\delta B}{\delta m}\right]\!\!\right]. \tag{65}$$

Some other useful Wronskian relation are formulated in the next two propositions.

Proposition 3. *Let $f(k,x)$ be an eigenfunctions of the spectral problem (2). Then the following identity holds:*

$$(m_x, f^2) = -\frac{1}{\lambda}\Big(f_x^2 - ff_{xx}\Big)\Big|_{x=-\infty}^{\infty} \tag{66}$$

Proof. Since $f(x)$ satisfies (2),

$$(f_x^2 - ff_{xx})\Big|_{x=-\infty}^{\infty} \quad = \quad \int_{-\infty}^{\infty} (f_x^2 - ff_{xx})_x \mathrm{d}x = -\lambda(m_x, f^2). \tag{67}$$

□

The right hand side of (67) can be expressed also through the skew-symmetric product (64):

$$(m_x, f^2(x,k)) \equiv (q_x, f^2(x,k)) = \left[\!\!\left[f^2(x,k), \left(\sqrt{\frac{\omega}{q(x)}} - 1\right)\right]\!\!\right]. \tag{68}$$

Corrollary 2. *From Proposition 3 and equations (32), (38) and (68) it follows*

$$(m_x, F^{\pm}(k)) \equiv \left[\!\!\left[F^{\pm}(x,k), \left(\sqrt{\frac{\omega}{q(x)}} - 1\right)\right]\!\!\right] = -\frac{4k^2}{\lambda} a(k)b(\mp k). \tag{69}$$

and the limiting procedure gives on the discrete spectrum

$$(m_x, F_n^{\pm}) = \frac{1}{2}[\![q(x), F_n^{\pm}(x)]\!] = 0, \qquad (m_x, \dot{F}_n^{\pm}) = \frac{1}{2}[\![q(x), \dot{F}_n^{\pm}(x)]\!] = \pm\frac{4i\kappa_n^2}{\lambda_n C_n^{\pm}}, \tag{70}$$

where $\lambda_n \equiv \lambda(i\kappa_n)$.

Another type of Wronskian relations relate the variations of the potential $m(x)$ with the variation of the scattering data [21]:

$$(f(x,k)\delta f_x - f_x \delta f(x,k))|_{x=-\infty}^{\infty} = \int_{-\infty}^{\infty} \mathrm{d}x\, \lambda \delta q(x) f^2(x,k), \tag{71}$$

where $\delta f(x,k)$ is the variation of the Jost solution $f(x,k)$ corresponding to the variation $\delta q(x)$ of the potential. The right hand side of (71) can also be expressed through the skew symmetric product as follows:

$$(f(x,k)\delta f_x - f_x \delta f(x,k))|_{x=-\infty}^{\infty} = -\lambda [\![f^2(x,k), \delta \mathcal{Q}_\pm(x)]\!], \tag{72}$$

where $\delta \mathcal{Q}_\pm(x) = (1/\sqrt{q(x)} \int_{\pm\infty}^{x} \mathrm{d}y\, \delta \sqrt{q(y)}$. For the derivation of (72) we performed an integration by parts and used the condition $\delta \mathcal{Q}_\pm(\mp\infty) \equiv \pm\delta\alpha = 0$.

From (71) and (72) we obtain:

$$[\![F^{\mp}(x,k), \delta \mathcal{Q}_\pm]\!] = \mp \frac{2ik}{\lambda} a^2(k) \delta \mathcal{R}^{\pm}(k), \tag{73}$$

$$[\![F_n^{\mp}(x), \delta \mathcal{Q}_\pm]\!] = \frac{2\kappa_n}{\lambda_n C_n^{\pm}} \delta \kappa_n, \tag{74}$$

$$[\![\dot{F}_n^{\mp}(x), \delta \mathcal{Q}_\pm]\!] = \frac{2i\kappa_n}{\lambda_n} \dot{a}_n^2 \delta C_n^{\pm} - \left(\frac{4\kappa_n^2 + 1}{4i\omega\lambda_n\kappa_n} - \frac{\ddot{a}_n}{\dot{a}_n} \right) \frac{\omega \delta \lambda_n}{\lambda_n C_n^{\pm}}. \tag{75}$$

Using (71) one can also derive the following relations for the variations of the scattering data, for details see [21]:

$$\frac{\delta a(k)}{\delta m(x)} = -\frac{\lambda}{2ik} f^+(x,k) f^-(x,k) \tag{76}$$

$$\frac{\delta b(k)}{\delta m(x)} = \frac{\lambda}{2ik} f^+(x,-k) f^-(x,k) \tag{77}$$

$$\frac{\delta \ln \lambda_n}{\delta m(x)} = \frac{i F_n^-(x)}{\omega b_n \dot{a}_n} \tag{78}$$

The equations (76) – (78) lead further to

$$\frac{\delta \mathcal{R}^{\pm}(k)}{\delta m(x)} = \pm \frac{\lambda(k)}{2ika^2(k)} F^{\mp}(x,k) \tag{79}$$

$$\frac{\delta C_n^{\pm}}{\delta m(x)} = -\frac{\lambda_n}{2i\kappa_n \dot{a}_n^2} \left[\left(\frac{4\kappa_n^2 + 1}{4i\omega\kappa_n\lambda_n} - \frac{\ddot{a}_n}{\dot{a}_n} \right) F_n^{\mp}(x) + \dot{F}_n^{\mp}(x) \right] \tag{80}$$

$$\frac{\delta \kappa_n}{\delta m(x)} = -\frac{\lambda_n C_n^{\pm}}{2\kappa_n} F_n^{\pm}(x) \tag{81}$$

$$\frac{\delta \lambda_n}{\delta m(x)} = -\frac{1}{\omega} \lambda_n C_n^{\pm} F_n^{\pm}(x) \tag{82}$$

These relations allow us to calculate the skew symmetric products between the squared solutions.

Proposition 4. *Let $f_1(k_1,x)$, $f_2(k_2,x)$, $g_1(k_1,x)$, $g_2(k_2,x)$ be eigenfunctions of the spectral problem (2). Then*

$$[\![f_1(k_1) g_1(k_1), f_2(k_2) g_2(k_2)]\!] = \frac{(f_1 \partial_x f_2 - f_2 \partial_x f_1)(g_1 \partial_x g_2 - g_2 \partial_x g_1))}{\lambda(k_1) - \lambda(k_2)} \Big|_{x=-\infty}^{\infty} \tag{83}$$

Proof. Straightforward to check using the fact that all functions are eigenfunctions of the spectral problem (2) – see [21] for details if necessary. □

Corrollary 3. *From Proposition 4 and (32), (38) it follows*

$$[\![F^{\pm}(k_1), U^{\mp}(k_2)]\!] = \pm 2\pi\omega\lambda(k_1)\delta(k_1 - k_2), \qquad k_{1,2} \in \mathbb{R} \tag{84}$$

where $U^{\pm}(x,k) \equiv \frac{\lambda(k)}{2ika^2(k)} F^{\pm}(x,k)$. *On the discrete spectrum*

$$[\![\dot{F}_n^{\pm}, F_m^{\mp}]\!] = \mp 2i\omega\kappa_n \dot{a}_n^2 \delta_{nm}, \qquad [\![F_n^{+}, F_m^{-}]\!] = 0, \qquad [\![\dot{F}_n^{\pm}, F_m^{\pm}]\!] = 0. \tag{85}$$

Up to now we demonstrated that the mapping between the scattering data $\mathcal{R}^{\pm}(k)$ and the potential $m(x)$ (or, equivalently, $q(x)$) is expressed through the squared solutions of (2). The same squared solutions relate also the variations of the reflection coefficients $\delta\mathcal{R}^{\pm}(k)$ with the corresponding variations of the potential $\delta m(x)$ (or, equivalently, $\delta q(x)$).

In order to ensure that these mappings are one-to-one we have to prove that the squared solutions form a complete set of functions in the space of allowed potentials; in other words we will prove that the functions $F^{\pm}(x,k)$, $F_n^{\pm}(x)$ form a basis in the space of allowed potentials. To this end we consider the function

$$\begin{aligned} R(x,y,k) \equiv & \frac{F^{-}(x,k)F^{+}(y,k)}{ka^2(k)}\theta(y-x) \\ & + \frac{2f^{+}(x,k)f^{-}(x,k)f^{+}(y,k)f^{-}(y,k) - F^{-}(y,k)F^{+}(x,k)}{ka^2(k)}\theta(x-y) \end{aligned} \tag{86}$$

Lemma 2. i) *$R(x,y,k)$ is an analytic function of k for $k \in \mathbb{C}_+$;*

ii) *$R(x,y,k)$ has second order poles at $k = i\kappa_n$;*

iii) *$R(x,y,k)$ is a kernel of bounded integral operator for $\operatorname{Im} k > 0$. For $\operatorname{Im} k = 0$ $R_1(x,y,k)$ is a kernel of a unbounded integral operator.*

Proof. i) and ii) are obvious.

iii) For $\operatorname{Im} k > 0$ the statement follows from the definitions (11) and (12) of the Jost solutions, which ensure that $R(x,y,k)$ falls off exponentially for all $x, y \to \pm\infty$. The same arguments for $\operatorname{Im} k = 0$ can only ensure that $R(x,y,k)$ is bounded function for $x, y \to \pm\infty$. □

Proposition 5. *The following completeness relation holds:*

$$\begin{aligned} \frac{\omega}{\sqrt{q(x)q(y)}}\theta(x-y) = & -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F^{-}(x,k)F^{+}(y,k)}{ka^2(k)}dk \\ & + \sum_{n=1}^{N} \frac{1}{i\kappa_n \dot{a}_n^2}\Big[\dot{F}_n^{-}(x)F_n^{+}(y) + F_n^{-}(x)\dot{F}_n^{+}(y) - \Big(\frac{1}{i\kappa_n} + \frac{\ddot{a}_n}{\dot{a}_n}\Big)F_n^{-}(x)F_n^{+}(y)\Big]. \end{aligned} \tag{87}$$

Proof. The proof is similar to the one of Proposition 1.

Consider the contour integral

$$\mathcal{J}(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} R(x,y,k)\mathrm{d}k. \tag{88}$$

where the contour γ_+ is shown on Fig. 1. From the Cauchy residue theorem it follows that:

$$\begin{aligned}\mathcal{J}(x,y) &= \sum_{n=1}^{N} \operatorname*{Res}_{k=i\kappa_n} R(x,y,k) \qquad (89)\\ &= \sum_{n=1}^{N} \frac{1}{i\kappa_n \dot{a}_n^2}\Big[\dot{F}_n^-(x)F_n^+(y) + F_n^-(x)\dot{F}_n^+(y) - \Big(\frac{1}{i\kappa_n} + \frac{\ddot{a}_n}{\dot{a}_n}\Big)F_n^-(x)F_n^+(y)\Big].\end{aligned}$$

In this evaluation we used (24), (25) and the fact that $\theta(x-y)+\theta(y-x) = 1$.

Next we evaluate the integral $\mathcal{J}(x,y)$ by integrating along the contour. For the integration along the infinite semicircle we need the asymptotic of $R(x,y,k)$ for $|k| \to \infty$. Due to (40) and (41) we get

$$\begin{aligned}R(x,y,k) &= \frac{2}{k}\frac{\omega}{\sqrt{q(x)q(y)}}\theta(x-y)(1+o(1/k)) \qquad (90)\\ &+ \frac{1}{k}\left(R_0(x,y,k)\theta(y-x) - R_0(x,y,-k)\theta(x-y)\right)(1+o(1/k)),\\ R_0(x,y,k) &= \frac{\omega}{\sqrt{q(x)q(y)}}e^{2ik\int_x^y \mathrm{d}y\sqrt{q(y)/\omega}}.\end{aligned}$$

Only the leading terms in (53), which are are entire functions of k contribute to the integral. This allows us to deform the infinite semicircle until it coincides with the real k-axis. Then the integration over k is easily performed with the result:

$$\mathcal{J}_\infty(x,y) = \frac{\omega}{\sqrt{q(x)q(y)}}\theta(x-y). \tag{91}$$

To evaluate the integral over the real axis $\mathcal{J}_R(x,y)$ we will use the the fact that $R(x,y,k)$ can be written in the form

$$R(x,y,k) = \frac{F^-(x,k)F^+(y,k)}{ka(k)} - \frac{N_1^2(x,y,k)}{k}\theta(x-y) \tag{92}$$

where $N_1(x,y,k)$ is defined in (56). The second term in (55) is an odd function of k and does not contribute to $\mathcal{J}_R(x,y)$.

Now it remains to equate the two expressions for

$$\mathcal{J}(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} R(x,y,k)\mathrm{d}k = \mathcal{J}_\infty(x,y) + \mathcal{J}_R(x,y). \tag{93}$$

to obtain the completeness relation for the squared solutions. □

Corrollary 4. *The completeness relation (87) can be rewritten in the following equivalent form:*

$$\theta(x-y) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{F}^-(x,k)\tilde{F}^+(y,k)}{ka^2(k)}dk \tag{94}$$

$$+\sum_{n=1}^{N}\frac{1}{i\kappa_n\dot{a}_n^2}\Big[\dot{\tilde{F}}_n^-(x)\tilde{F}_n^+(y)+\tilde{F}_n^-(x)\dot{\tilde{F}}_n^+(y)-\Big(\frac{1}{i\kappa_n}+\frac{\ddot{a}_n}{\dot{a}_n}\Big)\tilde{F}_n^-(x)\tilde{F}_n^+(y)\Big].$$

where

$$\tilde{F}^{\pm}(x,k)=\sqrt{\frac{q(x)}{\omega}}F^{\pm}(x,k). \tag{95}$$

The completeness relation allows one to expand any function $X(x)$ over the squared solutions $F^+(x,k)$ or $F^-(x,k)$. To this end we multiply both sides of (87) with $\frac{1}{2}q_yX(y)+q(y)X_y$ (resp. with $\frac{1}{2}q_xX(x)+q(x)X_x$) and integrate over $\mathrm{d}y$ (resp. $\mathrm{d}x$). A simple calculation gives:

$$\pm\omega X(x)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{k}\frac{F^{\mp}(x,k)\xi_X(k)}{a^2(k)}+\sum_{n=1}^{N}\frac{1}{i\kappa_n\dot{a}_n^2}\left(\dot{F}^{\mp}\xi_{X,n}^{\pm}+F^{\mp}\dot{\xi}_{X,n}^{\pm}\right), \tag{96}$$

provided $X(x)$ vanishes for $x\to\pm\infty$. The expansion coefficients $\xi_X^{\pm}(k)$ are given by:

$$\xi_X^{\pm}(k)=\int_{-\infty}^{\infty}\mathrm{d}kF^{\pm}(y,k)\left(\frac{1}{2}q_yX(y)+q(y)X(y)\right)=\frac{1}{2}[\![F^{\pm}(x,k),X(x)]\!], \tag{97}$$

where $[\![\cdot\,,\cdot]\!]$ is the skew-symmetric product (64). Similarly for $\xi_{X,n}^{\pm}$ and $\dot{\xi}_{X,n}^{\pm}$ we find:

$$\xi_{X,n}^{\pm}=\frac{1}{2}[\![F_n^{\pm}(x),X(x)]\!],\qquad \dot{\xi}_{X,n}^{\pm}=\frac{1}{2}[\![\dot{F}_n^{\pm}(x),X(x)]\!]-\left(\frac{1}{i\kappa_n}+\frac{\ddot{a}_n}{\dot{a}_n}\right)\xi_{X,n}^{\pm}, \tag{98}$$

Analogously, if we multiply the completeness relation (94) by $\tilde{X}_y$ (resp. by $\tilde{X}_x$) and integrate over $\mathrm{d}y$ (resp. $\mathrm{d}x$) we get:

$$\pm\omega\tilde{X}(x)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{k}\frac{\tilde{F}^{\mp}(x,k)\tilde{\xi}_{\tilde{X}}(k)}{a^2(k)}+\sum_{n=1}^{N}\frac{1}{i\kappa_n\dot{a}_n^2}\left(\dot{\tilde{F}}^{\mp}\tilde{\xi}_{\tilde{X},n}^{\pm}+\tilde{F}^{\mp}\dot{\tilde{\xi}}_{\tilde{X},n}^{\pm}\right), \tag{99}$$

where the expansion coefficients $\tilde{\xi}_{\tilde{X}}^{\pm}(k)$, $\tilde{\xi}_{\tilde{X},n}^{\pm}$ and $\dot{\tilde{\xi}}_{\tilde{X},n}^{\pm}$ are expressed by another skew-symmetric product:

$$\tilde{\xi}_{\tilde{X}}^{\pm}(k)=\int_{-\infty}^{\infty}\mathrm{d}k\tilde{F}^{\pm}(y,k)\tilde{X}_y(y)=\frac{1}{2}[\![\tilde{F}^{\pm}(x,k),\tilde{X}(x)]\!]_{\sim}, \tag{100}$$

$$\tilde{\xi}_{\tilde{X},n}^{\pm}=\frac{1}{2}[\![\tilde{F}_n^{\pm}(x),\tilde{X}(x)]\!]_{\sim},\qquad \dot{\tilde{\xi}}_{\tilde{X},n}^{\pm}=\frac{1}{2}[\![\dot{\tilde{F}}_n^{\pm}(x),\tilde{X}(x)]\!]_{\sim}-\left(\frac{1}{i\kappa_n}+\frac{\ddot{a}_n}{\dot{a}_n}\right)\tilde{\xi}_{\tilde{X},n}^{\pm},$$

Obviously the two skew-symmetric products are related by a gauge-like transformation. Indeed, if $F^{\pm}(x,k)$ and $\tilde{F}^{\pm}(x,k)$ are related by (95) and if $\tilde{X}(x)=\sqrt{q(x)/\omega}X(x)$ then:

$$[\![F^{\pm}(x,k),X(x)]\!]=\omega[\![\tilde{F}^{\pm}(x,k),\tilde{X}(x)]\!]_{\sim}. \tag{101}$$

The following lemma demonstrates that the mapping between $X(x)$ and its set of expansion coefficients is one-to-one.

Lemma 3. i) *A necessary and sufficient condition for $X(x)$ to vanish is that all its expansion coefficients vanish:*

$$\xi_X^{\pm}(k) = 0, \qquad \xi_{X,n}^{\pm} = 0, \qquad \dot{\xi}_{X,n}^{\pm} = 0. \tag{102}$$

ii) *A necessary and sufficient condition for $\tilde{X}(x)$ to vanish is:*

$$\tilde{\xi}_{\tilde{X}}^{\pm}(k) = 0, \qquad \tilde{\xi}_{\tilde{X},n}^{\pm} = 0, \qquad \dot{\tilde{\xi}}_{\tilde{X},n}^{\pm} = 0. \tag{103}$$

Proof. i) Let us assume that $X(x) = 0$ for all x. Substituting it into the right hand side of (97), (98) we immediately get (102). Let us now assume that (102) holds. Inserting it into the right hand side of the expansion (96) we find that $X(x) = 0$ for all x. ii) is proved analogously using (100) and the expansion (99). □

Remark 3. *We introduced two sets of squared solutions $F^{\pm}(x,k)$ and $\tilde{F}^{\pm}(x,k)$ related by the gauge-like transformation (94). Since $q(x)$ is always positive the two sets are obviously equivalent.*

6 Symplectic basis and canonical variables

Here we introduce a third set of squared solutions, known as *symplectic basis* whose special properties will become clear below. The symplectic basis was introduced for the first time in analyzing the nonlinear Schrödinger (NLS) equation in [43].

The canonical (action-angle) variables for the CH equation are known [21]:

$$\rho(k) \equiv \frac{2k}{\pi\omega\lambda(k)^2}\ln|a(k)| = -\frac{k}{\pi\omega\lambda(k)^2}\ln(1-\mathcal{R}^{\pm}(k)\mathcal{R}^{\pm}(-k)), \tag{104}$$

$$\phi(k) \equiv \arg b(k) = \frac{1}{2i}\ln\frac{\mathcal{R}^{+}(k)}{\mathcal{R}^{-}(k)}, \tag{105}$$

$$\rho_n = \lambda_n^{-1}, \qquad \phi_n = \ln|b_n|, \tag{106}$$

satisfying

$$\{\rho(k_1),\phi(k_2)\} = \delta(k_1-k_2) \tag{107}$$

$$\{\rho(k_1),\rho(k_2)\} = \{\phi(k_1),\phi(k_2)\} = 0, \qquad k_{1,2} > 0 \tag{108}$$

$$\{\rho_m,\phi_n\} = \delta_{mn}, \tag{109}$$

$$\{\rho_m,\rho_n\} = \{\phi_m,\phi_n\} = 0. \tag{110}$$

The following proposition gives the symplectic basis in the space of Schwartz-class functions:

Proposition 6. *The quantities*

$$\begin{aligned}\mathcal{P}(x,k) &= \frac{1}{2\pi i\omega\lambda(k)}\Big(\mathcal{R}^{-}(k)F^{-}(x,k)-\mathcal{R}^{-}(-k)F^{-}(x,-k)\Big)\\ &= -\frac{1}{2\pi i\omega\lambda(k)}\Big(\mathcal{R}^{+}(k)F^{+}(x,k)-\mathcal{R}^{+}(-k)F^{+}(x,-k)\Big),\end{aligned} \tag{111}$$

$$\mathcal{Q}(x,k) = -\frac{\lambda(k)}{4kb(k)b(-k)}\Big(\mathcal{R}^-(k)F^-(x,k)+\mathcal{R}^+(k)F^+(x,k)\Big), \quad (112)$$

$$P_n(x) = \frac{1}{\omega\lambda_n}C_n^+F_n^+(x) = \frac{1}{\omega\lambda_n}C_n^-F_n^-(x), \quad (113)$$

$$Q_n(x) = \frac{i\lambda_n}{4\kappa_n}\Big(C_n^+\dot{F}_n^+(x)-C_n^-\dot{F}_n^-(x)\Big). \quad (114)$$

satisfy the following canonical relations:

$$[\![\mathcal{P}(k_1),\mathcal{Q}(k_2)]\!] = \delta(k_1-k_2), \quad (115)$$

$$[\![\mathcal{P}(k_1),\mathcal{P}(k_2)]\!] = [\![\mathcal{Q}(k_1),\mathcal{Q}(k_2)]\!] = 0, \qquad k_{1,2}>0 \quad (116)$$

$$[\![P_m,Q_n]\!] = \delta_{mn}, \quad (117)$$

$$[\![P_m,P_n]\!] = [\![Q_m,Q_n]\!] = 0. \quad (118)$$

Proof. Due to (65),

$$\mathcal{P}(x,k) = \frac{\delta\rho(k)}{\delta m(x)}, \qquad \mathcal{Q}(x,k)=\frac{\delta\phi(k)}{\delta m(x)}, \quad (119)$$

$$P_n(x) = \frac{\delta\rho_n}{\delta m(x)}, \qquad Q_n(x)=\frac{\delta\phi_n}{\delta m(x)} \quad (120)$$

and with (79) – (82) we obtain (111) – (113). From (105), (106) one can see that ϕ_n can be obtained by a limiting procedure $k\to i\kappa_n$ from $\phi(k)$ (up to an overall multiplier i). Therefore, $Q_n(x)$ can be obtained in the same way from $Q(x,k)$, giving (114). Note that (117) is satisfied due to (85), (113) and (114). □

The usefulness of the symplectic basis (111) – (113) is more evident from the following two propositions.

Proposition 7.

$$\frac{1}{2\sqrt{q(x)q(y)}}\varepsilon(x-y) = \int_0^\infty \Big(\mathcal{P}(x,k)\mathcal{Q}(y,k)-\mathcal{Q}(x,k),\mathcal{P}(y,k)\Big)dk + \sum_{n=1}^N\Big(P_n(x)Q_n(y)-Q_n(x)P_n(y)\Big). \quad (121)$$

where $\varepsilon(x-y)=\Big(\theta(x-y)-\theta(y-x)\Big)/2$.

Proof. From (111) we have the identity

$$\mathcal{R}^-(k)F^-(x,k)+\mathcal{R}^+(k)F^+(x,k) = \mathcal{R}^-(-k)F^-(x,-k) + \mathcal{R}^+(-k)F^+(x,-k). \quad (122)$$

Applying several times (122) one can check further that

$$4\omega\Big(P(x,k)Q(y,k)-Q(x,k)P(y,k)\Big) = -\frac{1}{2\pi i}\Big(\frac{F^-(x,k)F^+(y,k)-F^+(x,k)F^-(y,k)}{ka^2(k)}+(k\to -k)\Big) \quad (123)$$

With (113) and (114) it is straightforward to obtain

$$4\omega \sum_{n=1}^{N} \Big(P_n(x)Q_n(y) - Q_n(x)P_n(y) \Big) = \sum_{n=1}^{N} \Big(\frac{\dot{F}_n^-(x)F_n^+(y) - \dot{F}_n^+(x)F_n^-(y)}{i\kappa_n \dot{a}_n^2} + (x \leftrightarrow y) \Big). \tag{124}$$

Now, (121) follows from (123), (124) and the representation (87). □

The completeness relation (119) allows one to expand any smooth function $X(x)$ vanishing for $x \to \pm\infty$ over the symplectic basis. To this end we multiply both sides of (119) by $\frac{1}{2}q_y X(y) + q(y)X_y$ and integrate over $\mathrm{d}y$. The result is:

$$X(x) = \int_0^\infty \mathrm{d}k \; (\mathcal{P}(x,k)\phi_X(k) - \mathcal{Q}(x,k)\rho_X(k)) + \sum_{n=1}^{N} \left(P_n(x)\phi_{n,X} - Q_n(x)\rho_{n,X} \right), \tag{125}$$

where

$$\begin{aligned} \phi_X(k) &= [\![\mathcal{Q}(y,k), X(y)]\!], \qquad \rho_X(k) = [\![\mathcal{P}(y,k), X(y)]\!], \\ \phi_{n,X} &= [\![Q_n(y), X(y)]\!], \qquad \rho_{n,X} = [\![P_n(y), X(y)]\!]. \end{aligned} \tag{126}$$

Corrollary 5. *The completeness relation (121) can be cast in the following equivalent form:*

$$\begin{aligned} \frac{1}{2}\varepsilon(x-y) &= \int_0^\infty \Big(\tilde{\mathcal{P}}(x,k)\tilde{\mathcal{Q}}(y,k) - \tilde{\mathcal{Q}}(x,k), \tilde{\mathcal{P}}(y,k) \Big) dk \\ + \quad & \sum_{n=1}^{N} \Big(\tilde{P}_n(x)\tilde{Q}_n(y) - \tilde{Q}_n(x)\tilde{P}_n(y) \Big), \end{aligned} \tag{127}$$

where the elements of the new symplectic basis $\tilde{\mathcal{P}}(x,k)$, $\tilde{\mathcal{Q}}(x,k)$ are related to the old ones by:

$$\tilde{\mathcal{P}}(x,k) = \sqrt{q(x)}\mathcal{P}(x,k), \qquad \tilde{\mathcal{Q}}(x,k) = \sqrt{q(x)}\mathcal{Q}(x,k), \tag{128}$$

for all $k \in \mathbb{R} \cup \{i\kappa_n\}$.

With (127) the analogue of the expansion (125) for any smooth function $\tilde{X}(x)$ vanishing for $x \to \pm\infty$ is:

$$\begin{aligned} \tilde{X}(x) &= \int_{-\infty}^{\infty} \mathrm{d}k \Big(\tilde{\mathcal{P}}(x,k)\tilde{\phi}_X(k) - \tilde{\mathcal{Q}}(x,k)\tilde{\rho}_X(k) \Big) \\ + \quad & \sum_{n=1}^{N} \Big(\tilde{P}_n\tilde{\phi}_{X,n} - \tilde{Q}_n\tilde{\rho}_{X,n} \Big), \end{aligned} \tag{129}$$

where the expansion coefficients $\tilde{\rho}_X(k)$, $\tilde{\rho}_{X,n}$ and $\tilde{\phi}_X(k)$, $\tilde{\phi}_{X,n}$ are given by:

$$\tilde{\rho}_X(k) = [\![\tilde{\mathcal{Q}}(k), \tilde{X}]\!]_{\sim}, \qquad \tilde{\phi}_X(k) = [\![\tilde{\mathcal{P}}(k), \tilde{X}]\!]_{\sim}, \tag{130}$$

$$\tilde{\rho}_{X,n} = [\![\tilde{Q}_n, \tilde{X}]\!]_{\sim}, \qquad \tilde{\phi}_X(k) = [\![\tilde{P}_n, \tilde{X}]\!]_{\sim},$$

where $[\![\,\cdot,\,\cdot\,]\!]_{\sim}$ is the new skew-symmetric product (101).

The following lemma demonstrates that the mapping between $\tilde{X}(x)$ and its set of expansion coefficients is one-to-one.

Lemma 4. *A necessary and sufficient condition for $\tilde{X}(x)$ to vanish identically is that all its expansion coefficients vanish:*

$$\tilde{\rho}_{\tilde{X}}(k) = 0, \qquad \tilde{\phi}_{\tilde{X}}(k), \qquad \tilde{\rho}_{\tilde{X},n} = 0, \qquad \tilde{\phi}_{\tilde{X},n} = 0. \tag{131}$$

The proof is analogous to the one of Lemma 3.

7 Recursion operator for the Camassa-Holm hierarchy

As in Section 5 we can view the completeness relations (87) and (94) as spectral decompositions for the recursion operators $L_\pm$ and $\tilde{L}_\pm$ defined below.

Proposition 8. *Let us define the recursion operators $L_\pm$ and $\tilde{L}_\pm$ and their inverse $\hat{L}_\pm$ and $\hat{\tilde{L}}_\pm$ as follows:*

$$L_\pm = \mathcal{D}_x^{-1}\Big[4q(x) - 2\int_{\pm\infty}^{x} dy\, m'(y)\Big]. \qquad \tilde{L}_\pm = \frac{4\omega}{\sqrt{q}}\mathcal{D}_x^{-1}\int_{\pm\infty}^{x} dy\,\sqrt{q(y)}\partial_x\cdot, \tag{132}$$

$$\hat{L}_\pm = \frac{1}{4\sqrt{q}}\int_{\pm\infty}^{x} dy\,\frac{1}{\sqrt{q}}\mathcal{D}_y\partial_y\,\cdot, \qquad \hat{\tilde{L}}_\pm = \int_{\pm\infty}^{x} dy\,\frac{1}{\sqrt{q}}\partial_y\mathcal{D}_y\frac{\sqrt{q}}{4\omega}\cdot. \tag{133}$$

where $\mathcal{D}_x = \partial_x^2 - 1$. Then the following relations hold:

$$L_\pm F^\pm(x,k) = \frac{1}{\lambda}F^\pm(x,k), \qquad L_\pm F_n^\pm(x) = \frac{1}{\lambda_n}F_n^\pm(x), \tag{134}$$

$$L_\pm \dot{F}_n^\pm(x) = \frac{1}{\lambda_n}\dot{F}_n^\pm(x) + \frac{2ik_n}{\omega\lambda_n^2}F_n^\pm(x).$$

$$\tilde{L}_\pm \tilde{F}^\pm(x,k) = \frac{1}{\lambda}\tilde{F}^\pm(x,k), \qquad \tilde{L}_\pm \tilde{F}_n^\pm(x) = \frac{1}{\lambda_n}\tilde{F}_n^\pm(x), \tag{135}$$

$$\tilde{L}_\pm \dot{\tilde{F}}_n^\pm(x) = \frac{1}{\lambda_n}\dot{\tilde{F}}_n^\pm(x) + \frac{2ik_n}{\omega\lambda_n^2}\tilde{F}_n^\pm(x).$$

$$\hat{L}_\pm F^\pm(x,k) = \lambda F^\pm(x,k), \qquad \hat{L}_\pm F_n^\pm(x) = \lambda_n F_n^\pm(x), \tag{136}$$

$$\hat{L}_\pm \dot{F}_n^\pm(x) = \lambda_n \dot{F}_n^\pm(x) - \frac{2ik_n}{\omega}F_n(x)^\pm.$$

$$\hat{\tilde{L}}_\pm \tilde{F}^\pm(x,k) = \lambda \tilde{F}^\pm(x,k), \qquad \hat{\tilde{L}}_\pm \tilde{F}_n^\pm(x) = \lambda_n \tilde{F}_n^\pm(x), \tag{137}$$

$$\hat{\tilde{L}}_\pm \dot{\tilde{F}}_n^\pm(x) = \lambda_n \dot{\tilde{F}}_n^\pm(x) - \frac{2ik_n}{\omega}\tilde{F}_n^\pm(x).$$

Proof. It is not difficult to prove that $L_\pm\hat{L}_\pm$ and $\tilde{L}_\pm\hat{\tilde{L}}_\pm$ act as an identity operator on any function $X(x)$. In order to prove (134) one can make use of the

fact that it can be reformulated as follows:

$$\Big[\partial_x^2 - 4\lambda q(x) + 2\lambda \int_{\pm\infty}^{x} \mathrm{d}y\, m'(y)\, \bullet \Big](f^{\pm})^2 = (f^{\pm})^2, \tag{138}$$

This follows from the fact that $f^{\pm}(x,k)$ are eigenfunctions of the spectral problem (2) with the asymptotics (11) and (12). The rest of the relations are proved analogously. □

Corrollary 6. *The recursion operators $L_{\pm}$, $\tilde{L}_{\pm}$ and their inverse $\hat{L}_{\pm}$, $\hat{\tilde{L}}_{\pm}$ satisfy the following relations:*

$$[\![L_+X, Y]\!] = [\![X, L_-Y]\!], \qquad [\![\hat{L}_+X, Y]\!] = [\![X, \hat{L}_-Y]\!], \tag{139}$$

$$[\![\tilde{L}_+X, Y]\!]_{\sim} = [\![X, \tilde{L}_-Y]\!]_{\sim}, \qquad [\![\hat{\tilde{L}}_+X, Y]\!]_{\sim} = [\![X, \hat{\tilde{L}}_-Y]\!]_{\sim}, \tag{140}$$

for any pair of functions $X(x)$ and $Y(x)$.

Proof. It follows easily from the definitions of the skew symmetric product and of the recursion operators using integration by parts.

□

Let us now define the operator $L \equiv \frac{1}{2}(L_+ + L_-)$ cf. (132), and $\tilde{L} \equiv \frac{1}{2}(\tilde{L}_+ + \tilde{L}_-)$. A simple direct computation shows that a kernel of L for $\omega \neq 0$ is empty, therefore it is possible to define the inverse operator L^{-1}. It is clear from (111), (113), (134) that

$$L\mathcal{P}(x,k) = \lambda^{-1}\mathcal{P}(x,k), \qquad LP_n(x) = \lambda_n^{-1}P_n(x). \tag{141}$$

One can compute the action of L to the remaining part of the symplectic basis, making use of the following proposition:

Proposition 9. *The following relations hold:*

$$L_{\mp}F^{\pm}(x,k) = \frac{1}{\lambda}F^{\pm}(x,k) \pm \frac{8k^2}{\lambda}a(k)b(\mp k). \tag{142}$$

Proof. One can write (142) as

$$\Big[\partial_x^2 - 1 - 4\lambda q(x) + 2\lambda \int_{\mp\infty}^{x} \mathrm{d}y\ m'(y)\, \bullet \Big](f^{\pm})^2 \mp \frac{8k^2}{\lambda}a(k)b(\mp k) = 0,$$

which is fulfilled due to the fact that $f^{\pm}$ are eigenfunctions of the spectral problem (2) with asymptotics (11), (14), (32), (38). □

Corrollary 7. *From the above Proposition and (112) it follows*

$$L_{\pm}\mathcal{Q}(x,k) = \frac{1}{\lambda}\mathcal{Q}(x,k) \pm 2k. \tag{143}$$

Now it is clear that the eigenfunctions for the operators L and $\tilde{L}$ are the elements of the symplectic bases $\mathcal{P}(x,k)$, $\mathcal{Q}(x,k)$, $P_n(x)$, $Q_n(x)$ and $\tilde{\mathcal{P}}(x,k)$, $\tilde{\mathcal{Q}}(x,k)$, $\tilde{P}_n(x)$, $\tilde{Q}_n(x)$, e.g.

$$L\mathcal{P}(x,k) = \frac{1}{\lambda}\mathcal{P}(x,k), \qquad L\mathcal{Q}(x,k) = \frac{1}{\lambda}\mathcal{Q}(x,k), \tag{144}$$

$$LP_n(x) = \frac{1}{\lambda_n}P_n(x), \qquad LQ_n(x) = \frac{1}{\lambda_n}Q_n(x). \tag{145}$$

Proposition 10. *The inverse of the recursion operators L and $\tilde{L}$ are given by:*

$$L^{-1} \equiv \hat{L} = \frac{1}{2}(\hat{L}_+ + \hat{L}_-), \qquad \tilde{L}^{-1} \equiv \hat{\tilde{L}} = \frac{1}{2}(\hat{\tilde{L}}_+ + \hat{\tilde{L}}_-).$$

Proof. Checked by direct calculation. □

Corrollary 8. *The recursion operators L, $\tilde{L}$ and their inverse $\hat{L}$, $\hat{\tilde{L}}$ are 'self-adjoint' with respect to the skew symmetric product:*

$$[\![LX, Y]\!] = [\![X, LY]\!], \qquad [\![\hat{L}X, Y]\!] = [\![X, \hat{L}Y]\!], \tag{146}$$

$$[\![\tilde{L}X, Y]\!]_{\sim} = [\![X, \tilde{L}Y]\!]_{\sim}, \qquad [\![\hat{\tilde{L}}X, Y]\!]_{\sim} = [\![X, \hat{\tilde{L}}Y]\!]_{\sim}, \tag{147}$$

for any functions $X(x)$ and $Y(x)$.

8 Expansions over the squared solutions and the CH hierarchy

It has been demonstrated that the squared solutions satisfy the completeness relation and therefore can be considered as generalized exponents. In this section we will derive the expansions of three important functions and demonstrate how they can be used for establishing the fundamental properties of the Camassa-Holm hierarchy.

The first of these functions is the 'potential' $m(x)$, or, rather one of the functions $\sqrt{\omega/q(x)} - 1$ or $\sqrt{q(x)/\omega} - 1$, which are completely determined by $m(x)$ and vice-versa. Its generalized Fourier coefficients are determined by the scattering data.

Proposition 11.

$$\omega\Big(\sqrt{\frac{\omega}{q(x)}} - 1\Big) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2k\mathcal{R}^{\pm}(k)}{\lambda(k)}F^{\pm}(x,k)\,dk + \sum_{n=1}^{N}\frac{2\kappa_n}{\lambda_n}C_n^{\pm}F_n^{\pm}(x), \tag{148}$$

$$\omega\Big(1 - \sqrt{\frac{q(x)}{\omega}}\Big) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2k\mathcal{R}^{\pm}(k)}{\lambda(k)}\tilde{F}^{\pm}(x,k)\,dk + \sum_{n=1}^{N}\frac{2\kappa_n}{\lambda_n}C_n^{\pm}\tilde{F}_n^{\pm}(x), \tag{149}$$

$$\sqrt{\frac{\omega}{q(x)}} - 1 = -2\int_0^{\infty}k\mathcal{P}(x,k)\,dk + 2\sum_{n=1}^{N}\kappa_n P_n(x), \tag{150}$$

$$1 - \sqrt{\frac{q(x)}{\omega}} = -2\int_0^{\infty}k\tilde{\mathcal{P}}(x,k)\,dk + 2\sum_{n=1}^{N}\kappa_n \tilde{P}_n(x). \tag{151}$$

Proof. The first expansion (148) is obtained by multiplying both sides of (87) by m_y, integrating with respect to y and using (69), (70). The second one (149) follows from the first one and from (95). The expansion coefficients of the third expansion (150) can be calculated using (126), the definition of the symplectic basis (111)–(114) and (69), (70). The fourth expansion (151) is an immediate consequence of the third (150) and (128). □

The generalized Fourier expansion for the variation of the 'potential' reads as follows:

Proposition 12.

$$\frac{\omega}{\sqrt{q(x)}}\int_{\pm\infty}^{x}\delta\sqrt{q(y)}\,dy = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{\lambda(k)}\delta\mathcal{R}^{\pm}(k)F^{\pm}(x,k)\,dk \pm \sum_{n=1}^{N}\Big[\frac{1}{\lambda_n}(\delta C_n^{\pm}-C_n^{\pm}\delta\lambda_n)F_n^{\pm}(x)+\frac{C_n^{\pm}}{i\lambda_n}\delta\kappa_n\dot{F}_n^{\pm}(x)\Big] \tag{152}$$

$$\int_{\pm\infty}^{x}\delta\sqrt{\frac{q(y)}{\omega}}\,dy = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{\lambda(k)}\delta\mathcal{R}^{\pm}(k)\tilde{F}^{\pm}(x,k)\,dk \pm \sum_{n=1}^{N}\Big[\frac{1}{\lambda_n}(\delta C_n^{\pm}-C_n^{\pm}\delta\lambda_n)\tilde{F}_n^{\pm}(x)+\frac{C_n^{\pm}}{i\lambda_n}\delta\kappa_n\dot{\tilde{F}}_n^{\pm}(x)\Big] \tag{153}$$

$$\int_{\pm\infty}^{x}\delta\sqrt{\frac{q(y)}{\omega}}\,dy = \int_{0}^{\infty}dk\Big(\tilde{\mathcal{P}}(x,k)\delta\phi(k)-\tilde{\mathcal{Q}}(x,k)\delta\rho(k)\Big) + \sum_{n=1}^{N}\Big(\tilde{P}_n(x)\delta\phi_n-\tilde{Q}_n(x)\delta\rho_n\Big). \tag{154}$$

Proof. The expansion (152) follows from (96) choosing $X(x)=\delta\mathcal{Q}_{\pm}(x)$. The corresponding expansion coefficients (97), (98) are expressed in terms of the scattering data variations using (73)–(75). Eq. (153) follows from (152) and (95). The expansion (154) follows from (129) with $\tilde{X}(x)=\int_{\pm\infty}^{x}\delta\sqrt{\frac{q(y)}{\omega}}\mathrm{d}y$. Note that condition $\delta\alpha=0$ ensures that the left hand side of (154) is independent of the choice of the lower limit of the integration. The corresponding expansion coefficients are evaluated using the definition of the symplectic basis (111)–(114), the Wronskian relations (73)–(75) and (128).

□

The expansions (152)–(154) are valid for all variations of the potential $\delta m(x)$ preserving the value of the integral α. An important subclass of these variations are due to the evolution of $m(x,t)$.

Effectively we consider a one-parameter family of spectral problems, allowing a dependence on the additional parameter t, such that $m(x,t)$ is a Schwartz class function for all values of t. The variation of the potential with respect to t is given by:

$$\delta m(x,t)\equiv m(x,t+\delta t)-m(x,t)\simeq m_t\delta t+\mathcal{O}((\delta t)^2). \tag{155}$$

For such potentials the corresponding scattering data, e.g. $\mathcal{R}^{\pm}(k,t)$, $C_n^{\pm}(t)$, κ_n generically will depend also on t. Keeping only the first order terms with respect

to δt we find that the corresponding variations of the scattering data are given by:

$$\delta\mathcal{R}^{\pm}(k) = \mathcal{R}_t^{\pm}(k)\delta t + \mathcal{O}((\delta t)^2), \qquad \delta C_n^{\pm} = C_{n,t}^{\pm}\delta t + \mathcal{O}((\delta t)^2). \tag{156}$$

With all these explanations from Proposition 12 one easily proves the following

Corrollary 9.

$$\begin{aligned}
\frac{\omega}{\sqrt{q(x)}}\int_{\pm\infty}^{x}(\sqrt{q(y)})_t\,dy &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{\lambda(k)}\mathcal{R}_t^{\pm}(k)F^{\pm}(x,k)\,dk \qquad (157)\\
&\pm \sum_{n=1}^{N}\Big[\frac{1}{\lambda_n}(C_{n,t}^{\pm}-C_n^{\pm}\lambda_{n,t})F_n^{\pm}(x)+\frac{C_n^{\pm}}{i\lambda_n}\kappa_{n,t}\dot{F}_n^{\pm}(x)\Big]\\
\int_{\pm\infty}^{x}\left(\sqrt{\frac{q(y)}{\omega}}\right)_t dy &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i}{\lambda(k)}\mathcal{R}_t^{\pm}(k)\tilde{F}^{\pm}(x,k)\,dk \qquad (158)\\
&\pm \sum_{n=1}^{N}\Big[\frac{1}{\lambda_n}(C_{n,t}^{\pm}-C_n^{\pm}\lambda_{n,t})\tilde{F}_n^{\pm}(x)+\frac{C_n^{\pm}}{i\lambda_n}\kappa_{n,t}\dot{\tilde{F}}_n^{\pm}(x)\Big]\\
\int_{\pm\infty}^{x}\left(\sqrt{\frac{q(y)}{\omega}}\right)_t dy &= \int_0^{\infty}dk\Big(\tilde{\mathcal{P}}(x,k)\phi_t(k)-\tilde{\mathcal{Q}}(x,k)\rho_t(k)\Big)\\
&+ \sum_{n=1}^{N}\Big(\tilde{P}_n(x)\phi_{n,t}-\tilde{Q}_n(x)\rho_{n,t}\Big). \qquad (159)
\end{aligned}$$

Proposition 13. *Let $\Omega(z)$ be a rational function such that its poles lie outside the spectrum $\mathbb{R}\cup\bigcup_{n=1}^{N}\{i\kappa_n,-i\kappa_n\}$. Then:*

$$\begin{aligned}
\Omega(L_{\pm})\Big(\sqrt{\frac{\omega}{q(x)}}-1\Big) &= \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2k\mathcal{R}^{\pm}(k)}{\omega\lambda(k)}\Omega(\lambda^{-1})F^{\pm}(x,k)\,dk\\
&+ \sum_{n=1}^{N}\frac{2\kappa_n}{\omega\lambda_n}C_n^{\pm}\Omega(\lambda_n^{-1})F_n^{\pm}(x), \qquad (160)
\end{aligned}$$

$$\begin{aligned}
\Omega(\tilde{L}_{\pm})\Big(1-\sqrt{\frac{q(x)}{\omega}}\Big) &= \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{2k\mathcal{R}^{\pm}(k)}{\omega\lambda(k)}\Omega(\lambda^{-1})\tilde{F}^{\pm}(x,k)\,dk\\
&+ \sum_{n=1}^{N}\frac{2\kappa_n}{\omega\lambda_n}C_n^{\pm}\Omega(\lambda_n^{-1})\tilde{F}_n^{\pm}(x), \qquad (161)
\end{aligned}$$

$$\Omega(L)\left(\sqrt{\frac{\omega}{q(x)}}-1\right) = -2\int_0^{\infty}k\Omega(\lambda^{-1})\mathcal{P}(x,k)\,dk + 2\sum_{n=1}^{N}\kappa_n\Omega(\lambda_n^{-1})P_n(x), \tag{162}$$

$$\Omega(\tilde{L})\left(1-\sqrt{\frac{q(x)}{\omega}}\right) = -2\int_0^{\infty}k\Omega(\lambda^{-1})\tilde{\mathcal{P}}(x,k)\,dk + 2\sum_{n=1}^{N}\kappa_n\Omega(\lambda_n^{-1})\tilde{P}_n(x). \tag{163}$$

Proof. Eq. (160) is obtained by acting on the expansion (148) with the operator $\Omega(L_\pm)$ and using (134). The condition imposed on the poles of $\Omega(z)$ ensures that $\Omega(\lambda^{-1})$ and $\Omega(\lambda_n^{-1})$ are all finite, so the right hand side of (160) is well defined. Eq. (161) follows from (149) and (135). Eqs. (162) and (163) are derived analogously using the expansions (150) and (151) and (144).

□

Corrollary 10. *If in Proposition 13 we use the operators $\Omega(\hat{L}_\pm)$, $\Omega(\hat{\tilde{L}}_\pm)$, $\Omega(\hat{L})$ and $\Omega(\hat{\tilde{L}})$ respectively, then in the right hand sides of (160)–(163) the factors $\Omega(\lambda^{-1})$ and $\Omega(\lambda_n^{-1})$ will be replaced by $\Omega(\lambda)$ and $\Omega(\lambda_n)$ respectively.*

Proof. Follows easily from the arguments used in Proposition 13 and (136) and (137). □

Now it is easy to describe the hierarchy of Camassa-Holm equations. To every choice of the function $\Omega(z)$, known also as the dispersion law we can put into correspondence the Nonlinear Evolution Equation (NLEE):

$$\frac{2}{\sqrt{q}}\int_{\pm\infty}^{x}(\sqrt{q})_t \mathrm{d}y + \Omega(\mathcal{L})\Big(\sqrt{\frac{\omega}{q}}-1\Big) = 0, \tag{164}$$

$$\int_{\pm\infty}^{x}(\sqrt{q})_t \mathrm{d}y + \Omega(\tilde{\mathcal{L}})\Big(1-\sqrt{\frac{q}{\omega}}\Big) = 0, \tag{165}$$

where $\mathcal{L}$ (resp. $\tilde{\mathcal{L}}$) is any of the operators L_+, L_- or L (resp. $\tilde{L}_+$, $\tilde{L}_-$ or $\tilde{L}$).

What will be demonstrated below is that the hierarchy (164) (resp. (165)) can be generated by each of the recursion operators $L_\pm$, L, (resp. $\tilde{L}_\pm$, $\tilde{L}$) and their inverse. The fact that the expansions over the squared solutions provide the spectral decompositions of the recursion operators makes evident the interpretation of the ISM as a generalized Fourier transform. Using these expansions we will show that each CH-type equation is equivalent to a linear evolution equation for the scattering data.

Proposition 14. i) *Each of the NLEE (164) and (165) is equivalent to the following linear evolution equations for the scattering data:*

$$\mathcal{R}_t^\pm(k) \mp ik\Omega(\lambda^{-1})\mathcal{R}^\pm(k) = 0, \tag{166}$$

$$C_{n,t}^\pm \pm \kappa_n\Omega(\lambda_n^{-1})C_n^\pm = 0, \tag{167}$$

$$\kappa_{n,t} = 0. \tag{168}$$

ii) *Each of the NLEE (164) and (165) is equivalent to the following linear evolution of the action-angle variables:*

$$\phi_t(k,t) - k\Omega(\lambda^{-1}) = 0, \qquad \rho_t(k) = 0, \tag{169}$$

$$\phi_{n,t} + \kappa_n\Omega(\lambda_n^{-1}) = 0, \qquad \kappa_{n,t} = 0. \tag{170}$$

Proof. i) Let us consider the NLEE (164) fixing up $\mathcal{L} = L_\pm$ and let us expand the left hand side of (164) over the squared solutions $F^\pm(x,k,t)$ using the expansions (157) and (160). It is easy to check that the corresponding expansion coefficients coincide with the left hand sides of (166)–(168). It remains to make use of i) in Lemma 3. To prove the equivalence of (165) to the linear eqs. (166)–(168) we

fix up $\tilde{\mathcal{L}} = \tilde{L}_\pm$ and make use of the expansions (158), (161) and of ii) in Lemma 3.

ii) Now we choose $\tilde{\mathcal{L}} = \tilde{L}$ and expand the left hand side of (165) over the symplectic basis using (169), (170). It remains to apply Lemma 4. □

The NLEE (164) and (165) can be simplified as follows:

$$q_t + \sqrt{q}\Big[\sqrt{q}\Omega(\mathcal{L})\Big(\sqrt{\frac{\omega}{q}} - 1\Big)\Big]_x = 0, \tag{171}$$

$$q_t + \Big[\Omega(\tilde{\mathcal{L}})\Big(1 - \sqrt{\frac{q}{\omega}}\Big)\Big]_x = 0, \tag{172}$$

Example 1. *With $\Omega(z) = z$ one can easily check that*

$$L_\pm\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = 2u \tag{173}$$

and thus the equation (171) becomes the Camassa-Holm equation (1).

The higher degree polynomials in (171) produce the other members of the Camassa-Holm hierarchy. All members of the hierarchy share the same spectral problem (2), and thus their solutions have the same x-dependence. The only difference is the time-evolution of the scattering data of the members of the hierarchy.

Now it is clear how to extend the dispersion law $\Omega(z)$ for the case of ratio of two polynomials:

Corrollary 11. *Let*

$$\Omega(z) = \frac{\Omega_2(z)}{\Omega_1(z)}. \tag{174}$$

where $\Omega_1(z)$ and $\Omega_2(z)$ are two polynomials. The corresponding NLEE can be written in the form:

$$\Omega_1(L_\pm)\frac{2}{\sqrt{q}}\int_{\pm\infty}^{x}(\sqrt{q})_t\,dy + \Omega_2(L_\pm)\Big(\sqrt{\frac{\omega}{q}} - 1\Big) = 0 \tag{175}$$

is also equivalent to (166) – (168) where $\Omega(z)$ is given by (174).

Example 2. *$\Omega_1(z) = z$ and $\Omega_2(z) = 1$, i.e. $\Omega(z) = 1/z$. The equation (175) due to (132) has the form*

$$q_t + \partial_x(\partial_x^2 - 1)\sqrt{\frac{\omega}{q}} = 0, \tag{176}$$

which is exactly the extended Dym equation [12, 40, 69].

Of course, further generalizations are possible e.g. by introducing another time-like variable, see [11].

Corrollary 12. *The functional derivative* $\frac{\delta f}{\delta m(x)}$ *can be expanded over the symplectic basis as follows:*

$$\frac{\delta f}{\delta m(x)} = \int_0^\infty \Big(\{f,\phi(k)\}\mathcal{P}(x,k) - \{f,\rho(k)\}\mathcal{Q}(x,k)\Big)dk + \sum_{n=1}^N \Big(\{f,\phi_n\}P_n(x) - \{f,\rho_n\}Q_n(x)\Big). \tag{177}$$

Proof. Insert $X(x) = \frac{\delta f}{\delta m(x)}$ into eq. (125) and (126). In view of (119) and (120) and (65) all expansion coefficients turn into the corresponding Poisson brackets. □

Corrollary 13. *The Poisson bracket* $\{f,g\}$ *can be expressed as:*

$$\{f,g\} = \int_0^\infty \Big(\{f,\rho(k)\}\{g,\phi(k)\} - \{f,\phi(k)\}\{g,\rho(k)\}\Big)dk + \sum_{n=1}^N \Big(\{f,\rho_n\}\{g,\phi_n\} - \{f,\phi_n\}\{g,\rho_n\}\Big). \tag{178}$$

Proof. Multiply both sides of (178) by $\frac{\delta g}{\delta m(x)}$ and take the skew-symmetric product. Due to (65) the left hand side becomes $\{f,g\}$. The right hand side follows from (126) and (119), (120).

□

Corrollary 14. *From (177) with* $f = H_1 = \frac{1}{2}\int mudx$ *and the relations* $\{\phi, H_1\} = k/\lambda(k)$, $\{\phi_n, H_1\} = -\kappa_n/\lambda_n$, *see [21] we obtain*

$$u(x) = -\int_0^\infty \frac{k}{\lambda(k)}\mathcal{P}(x,k)dk + \sum_{n=1}^N \frac{\kappa_n}{\lambda_n}P_n(x). \tag{179}$$

Since the potential $m(x)$ has the meaning of 'momentum' [67, 60], (179) gives expansion of $u = (1-\partial_x)^{-1}m$ over the 'modes' $\mathcal{P}(x,k)$, $P_n(x)$, representing the 'momentum' part of the symplectic basis. Actually, (179) is equivalent to (148) due to (173) and the fact that

$$L_\pm\mathcal{P}(x,k) = \lambda^{-1}(k)\mathcal{P}(x,k), \qquad L_\pm P_n(x) = \lambda_n^{-1}P_n(x), \tag{180}$$

see (111), (113), (134).

9 Hamiltonian formulation for the CH hierarhy

Let us start from the following observation. From the identities (173), (179) we have

$$u = \frac{1}{2}L\Big(\sqrt{\frac{\omega}{q}} - 1\Big), \tag{181}$$

$$(182)$$

and therefore, taking into consideration (180), (141),

$$\Omega(L_\pm)\Big(\sqrt{\frac{\omega}{q}}-1\Big)=\Omega(L)\Big(\sqrt{\frac{\omega}{q}}-1\Big).$$

Thus, the equation (171) can be written as

$$m_t=[\![\delta(\bullet),\frac{1}{2}\Omega(L)\Big(\sqrt{\frac{\omega}{q}}-1\Big)]\!], \tag{183}$$

where $\delta(\bullet)$ denotes the delta function.

Due to (65) we can write this equation in Hamiltonian form

$$m_t=\{m,H^\Omega\}, \tag{184}$$

with Hamiltonian H^Ω such that

$$\frac{\delta H^\Omega}{\delta m}=\frac{1}{2}\Omega(L)\Big(\sqrt{\frac{\omega}{q}}-1\Big). \tag{185}$$

From (162) we have further

$$\begin{aligned}\frac{1}{2}\Omega(L)\Big(\sqrt{\frac{\omega}{q}}-1\Big)&=-\int_0^\infty k\Omega(\lambda^{-1})\mathcal{P}(x,k)\mathrm{d}k+\sum_{n=1}^N\kappa_n\Omega(\lambda_n^{-1})P_n(x)\\&=-\int_0^\infty k\Omega(\lambda^{-1})\frac{\delta\rho(k)}{\delta m(x)}\mathrm{d}k+\sum_{n=1}^N\kappa_n\Omega(\lambda_n^{-1})\frac{\delta\rho_n}{\delta m(x)}.\end{aligned} \tag{186}$$

Therefore, in view of (185) and (186)

$$\frac{\delta H^\Omega}{\delta\rho(k)}=-k\Omega(\lambda^{-1}),\qquad\frac{\partial H^\Omega}{\partial\rho_n}=\kappa_n\Omega(\lambda_n^{-1}), \tag{187}$$

and finally

$$H^\Omega=-\int_0^\infty k\Omega(\lambda^{-1})\rho(k)\mathrm{d}k-\frac{2}{\omega}\sum_{n=1}^N\int\frac{\kappa_n^2}{\lambda_n^2}\Omega(\lambda_n^{-1})\mathrm{d}\kappa_n. \tag{188}$$

Example: As an example we can point out the CH equation ($\Omega(z)=z$). The expression (188) gives

$$H^\Omega\equiv H_1=-\int_0^\infty\frac{k}{\lambda}\rho(k)\mathrm{d}k+\omega^2\sum_{n=1}^N\Big(\ln\frac{1-2\kappa_n}{1+2\kappa_n}+\frac{4\kappa_n(1+4\kappa_n^2)}{(1-4\kappa_n^2)^2}\Big).$$

The last expression was obtained in a different way in [21].□

Noticing that $\int_{\pm\infty}^{x} = \partial_x^{-1}$ we can rewrite (6) as

$$\frac{\delta H_n[m]}{\delta m} = -\frac{1}{2} L_{\pm} \frac{\delta H_{n-1}[m]}{\delta m} \tag{189}$$

due to (132), or as

$$\frac{\delta H_n[m]}{\delta m} = -\frac{1}{2} L \frac{\delta H_{n-1}[m]}{\delta m}. \tag{190}$$

If $H^{\Omega} \equiv H_1^{\Omega}$ is the Hamiltonian with respect to the Poisson bracket (8), the other conservation laws can be generated according to

$$\frac{\delta H_n^{\Omega}[m]}{\delta m} = -\frac{1}{2} L \frac{\delta H_{n-1}^{\Omega}[m]}{\delta m}. \tag{191}$$

For example

$$\begin{aligned}
\frac{\delta H_2^{\Omega}}{\delta m} &= -\frac{1}{2} L \frac{\delta H_1^{\Omega}}{\delta m} \\
&= -\frac{1}{2} L \Big(\int_0^{\infty} k \Omega(\lambda^{-1}) \mathcal{P}(x,k) \mathrm{d}k + \sum_{n=1}^{N} \kappa_n \Omega(\lambda_n^{-1}) P_n(x) \Big) \\
&= -\int_0^{\infty} \frac{k}{(-2\lambda)} \Omega(\lambda^{-1}) \mathcal{P}(x,k) \mathrm{d}k + \sum_{n=1}^{N} \frac{\kappa_n}{(-2\lambda_n)} \Omega(\lambda_n^{-1}) P_n(x),
\end{aligned}$$

or in general

$$\begin{aligned}
\frac{\delta H_j^{\Omega}}{\delta m} &= -\int_0^{\infty} (-2\lambda)^{1-j} k \Omega(\lambda^{-1}) \mathcal{P}(x,k) \mathrm{d}k \\
&\quad + \sum_{n=1}^{N} \kappa_n (-2\lambda_n)^{1-j} \Omega(\lambda_n^{-1}) P_n(x);
\end{aligned}$$

$$\frac{\delta H_j^{\Omega}}{\delta \rho(k)} = -k(-2\lambda)^{1-j} \Omega(\lambda^{-1}), \qquad \frac{\partial H_j^{\Omega}}{\partial \rho_n} = \kappa_n (-2\lambda_n)^{1-j} \Omega(\lambda_n^{-1}), \tag{192}$$

for $j \geq 1$, or

$$H_j^{\Omega} = -\int_0^{\infty} k(-2\lambda)^{1-j} \Omega(\lambda^{-1}) \rho(k) \mathrm{d}k - \frac{2}{\omega} \sum_{n=1}^{N} \int \frac{\kappa_n^2}{\lambda_n^2} (-2\lambda_n)^{1-j} \Omega(\lambda_n^{-1}) \mathrm{d}\kappa_n. \tag{193}$$

Example: For the CH equation ($\Omega(z) = z$) the expression (193) gives[3]

[3] This is also the quantity given in formula (81) of [21], however with a technical error in the contribution from the continuous spectrum. The correct expression should be

$$H_2 = \omega^3 \sum_{n=1}^{N} \Big(\ln \frac{1-2\kappa_n}{1+2\kappa_n} + \frac{4\kappa_n(3+32\kappa_n^2-48\kappa_n^4)}{3(1-4\kappa_n^2)^3} \Big) + \frac{2^8 \omega^3}{\pi} \int_0^{\infty} \frac{k^2 \ln|a(k)|}{(4k^2+1)^4} \mathrm{d}k.$$

$$H_2^\Omega \equiv H_2 = \int_0^\infty \frac{k}{2\lambda^2}\rho(k)\mathrm{d}k + \omega^3 \sum_{n=1}^N \Big(\ln\frac{1-2\kappa_n}{1+2\kappa_n} + \frac{4\kappa_n(3+32\kappa_n^2-48\kappa_n^4)}{3(1-4\kappa_n^2)^3}\Big). \tag{194}$$

□

Since L^{-1} is well defined, it is possible to consider the formal expression (193) for $j \leq 0$.

Example: For the CH equation (193) gives ($j = 0$, $\Omega(z) = z$):

$$H_0 = 2\int_0^\infty k\rho(k)\mathrm{d}k + 2\omega \sum_{n=1}^N \Big(\ln\frac{1-2\kappa_n}{1+2\kappa_n} + \frac{4\kappa_n}{1-4\kappa_n^2}\Big). \tag{195}$$

In terms of $q(x)$ we have [4]

$$H_0 \equiv \int_{-\infty}^\infty (\sqrt{q}-\sqrt{\omega})^2 \mathrm{d}x. \tag{196}$$

Indeed, since $\delta H_0/\delta m = 1-\sqrt{\omega/q}$, $\delta H_1/\delta m = u$, H_0 and H_1 are related through (189), see (173).□

Example: With (189), (132) one can check that for the CH equation

$$H_{-1} = \frac{1}{2}\int_{-\infty}^\infty \Big[\Big(\sqrt[4]{\frac{\omega}{q}} - \sqrt[4]{\frac{q}{\omega}}\Big)^2 + \frac{\sqrt{\omega}q_x^2}{4q^{5/2}}\Big]\mathrm{d}x. \tag{197}$$

On the other hand, for $j = -1$ and $\Omega(z) = z$, (193) gives

$$H_{-1} = -4\int_0^\infty k\lambda\rho(k)\mathrm{d}k + 2\sum_{n=1}^N \Big(\ln\frac{1+2\kappa_n}{1-2\kappa_n} - 4\kappa_n\Big). \tag{198}$$

□

Corrollary 15. *Obviously, $H_1^{\Omega(z)=1/z} = \frac{1}{4}H_{-1}^{\Omega(z)=z}$, therefore the expressions on the right hand side of (197), (198) give the Hamiltonian of (176), (up to a constant factor) with respect to the Poisson bracket (8).*

Here we notice that for the CH equation ($\Omega(z) = z$) the integral α, (43), (47)

$$\alpha \equiv \int_{-\infty}^\infty \Big(\sqrt{\frac{q}{\omega}} - 1\Big)\mathrm{d}x = \sum_{n=1}^N \ln\Big(\frac{1+2\kappa_n}{1-2\kappa_n}\Big)^2 + \int_0^\infty \frac{\lambda}{k}\rho(k)\mathrm{d}k \tag{199}$$

is not of the form (193). However, the hierarchy is generated by $a(k)$ and $\alpha = \lim_{k\to\infty}\frac{d}{dk}\ln a(k)$. Thus, α from (199) does not give rise to a separate hierarchy $\{\alpha_n\}$ of conservation laws. Another way of seeing this is to assume the contrary, that there exists an independent hierarchy, such that

$$\frac{\delta\alpha_n}{\delta m} = L^n\frac{\delta\alpha}{\delta m}, \qquad n = \pm1, \pm2, \ldots.$$

[4]Note the difference in comparison with the definitions of the integrals in [21].

However, one can check that $L\frac{\delta\alpha}{\delta m}$ and $\frac{\delta\alpha}{\delta m}$ are related as follows:

$$L\frac{\delta\alpha}{\delta m} = -4\omega\frac{\delta\alpha}{\delta m} - 2\frac{\delta H_0}{\delta m},$$

and therefore no other independent integrals arise.

Example: The following integrals are related to α and an integral from (193):

$$\begin{aligned} I_0 &\equiv \int_{-\infty}^{\infty} m\mathrm{d}x = H_0 + 2\omega\alpha \\ &= -\int_0^{\infty}\frac{1}{2k}\rho(k)\mathrm{d}k + 2\omega\sum_{n=1}^{N}\Big(\ln\frac{1+2\kappa_n}{1-2\kappa_n} + \frac{4\kappa_n}{1-4\kappa_n^2}\Big), \\ I_{-1} &\equiv \frac{\sqrt{\omega}}{2}\int_{-\infty}^{\infty}\Big(\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{\omega}} + \frac{q_x^2}{4q^{5/2}}\Big)\mathrm{d}x = H_{-1} - \frac{1}{2}\alpha, \end{aligned}$$

etc. □

The equation (190) can be written as

$$\frac{\delta H_1^{\Omega}}{\delta m} = (-2L^{-1})^{n-1}\frac{\delta H_n^{\Omega}}{\delta m} \tag{200}$$

Therefore, defining a Poisson bracket

$$\{A, B\}_{(n)} \equiv \left[\!\!\left[\frac{\delta A}{\delta m}, (-2L^{-1})^{n-1}\frac{\delta B}{\delta m}\right]\!\!\right], \tag{201}$$

we can write the equation (171) as

$$m_t = \{m, H_n^{\Omega}\}_{(n)}. \tag{202}$$

The following statement gives the canonical variables for (171) with respect to (201).

Proposition 15. *Let us define*

$$\rho(k)_{(n)} \equiv \rho(k), \qquad \phi(k)_{(n)} \equiv (-2\lambda)^{1-n}\phi(k) \tag{203}$$

$$\rho_{l,(n)} \equiv \rho_l, \qquad \phi_{l,(n)} \equiv (-2\lambda_l)^{1-n}\phi_l. \tag{204}$$

These variables satisfy the following canonical relations with respect to the bracket (201):

$$\begin{aligned} \{\rho(k_1)_{(n)}, \phi(k_2)_{(n)}\}_{(n)} &= \delta(k_1 - k_2), \\ \{\rho(k_1)_{(n)}, \rho(k_2)_{(n)}\}_{(n)} &= \{\phi(k_1)_{(n)}, \phi(k_2)_{(n)}\}_{(n)} = 0, \qquad k_{1,2} > 0 \\ \{\rho_{l,(n)}, \phi_{p,(n)}\}_{(n)} &= \delta_{lp}, \\ \{\rho_{l,(n)}, \rho_{p,(n)}\}_{(n)} &= \{\phi_{l,(n)}, \phi_{p,(n)}\}_{(n)} = 0. \end{aligned}$$

Proof. One can perform the verification making use of (144), (145). For the quantities on the continuous spectrum the computations are particularly simple, e.g.

$$\begin{aligned}
\{\rho(k_1)_{(n)}, \phi(k_2)_{(n)}\}_{(n)} &= (-2\lambda(k_2))^{1-n} \left[\!\!\left[\frac{\delta\rho(k_1)}{\delta m}, (-2L^{-1})^{n-1} \frac{\delta\phi(k_2)}{\delta m} \right]\!\!\right] \\
&= (-2\lambda(k_2))^{1-n} \left[\!\!\left[\mathcal{P}(k_1), (-2L^{-1})^{n-1} \mathcal{Q}(k_2) \right]\!\!\right] \\
&= \left[\!\!\left[\mathcal{P}(k_1), \mathcal{Q}(k_2) \right]\!\!\right] = \delta(k_1 - k_2).
\end{aligned}$$

For the quantities on the discrete spectrum one can notice first that

$$\begin{aligned}
\frac{\delta\phi_{l,(n)}}{\delta m} &= \frac{\delta(-2\lambda_l)^{1-n}}{\delta m}\phi_l + (-2\lambda_l)\frac{\delta\phi_l}{\delta m} \\
&= (-2\lambda_l)^{1-n}(n-1)\lambda_l\phi_l P_l + (-2\lambda_l)^{1-n} Q_l. \qquad (205)
\end{aligned}$$

Then

$$\begin{aligned}
&\{\rho_{l,(n)}, \phi_{p,(n)}\}_{(n)} = \\
&= \left[\!\!\left[\frac{\delta\rho_l}{\delta m}, (-2L^{-1})^{n-1} \frac{\delta\phi_{p,(n)}}{\delta m} \right]\!\!\right] \\
&= \left[\!\!\left[P_l, (-2L^{-1})^{n-1}(-2\lambda_p)^{1-n}\Big((n-1)\lambda_p\phi_p P_p + Q_p\Big) \right]\!\!\right] \\
&= (n-1)\lambda_p\phi_p \left[\!\!\left[P_l, P_p \right]\!\!\right] + \left[\!\!\left[P_l, Q_p \right]\!\!\right] = \delta_{lp},
\end{aligned}$$

$$\begin{aligned}
\{\rho_{l,(n)}, \rho_{p,(n)}\}_{(n)} &= \left[\!\!\left[\frac{\delta\rho_l}{\delta m}, (-2L^{-1})^{n-1} \frac{\delta\rho_p}{\delta m} \right]\!\!\right] \\
&= (-2\lambda_p)^{n-1} \left[\!\!\left[P_l, P_p \right]\!\!\right] = 0,
\end{aligned}$$

$$\begin{aligned}
&\{\phi_{l,(n)}, \phi_{p,(n)}\}_{(n)} = \\
&= (-2\lambda_l)^{1-n} \left[\!\!\left[(n-1)\lambda_l\phi_l P_l + Q_l, (n-1)\lambda_p\phi_p P_p + Q_p \right]\!\!\right] \\
&= (-2\lambda_l)^{1-n}\Big((n-1)\lambda_l\phi_l \left[\!\!\left[P_l, Q_p \right]\!\!\right] + (n-1)\lambda_p\phi_p \left[\!\!\left[Q_l, P_p \right]\!\!\right]\Big) \\
&= (-2\lambda_l)^{1-n}(n-1)(\lambda_l\phi_l - \lambda_p\phi_p)\delta_{lp} = 0.
\end{aligned}$$

□

10 Inverse scattering transform

Inverse scattering method for the hierarchy (171) is the same as the one for the CH equation [20]. The only difference is the time-dependence of the scattering

data (166) – (168). For example, the inverse scattering is simplified in the important case of the so-called reflectionless potentials, when the scattering data is confined to the case $\mathcal{R}^{\pm}(k) = 0$ for all real k. This class of potentials corresponds to the N-soliton solutions of the CH hierarchy. In this case $b(k) = 0$ and $|a(k)| = 1$ and $i\dot{a}_p$ is real:

$$i\dot{a}_p = \frac{1}{2\kappa_p} e^{\alpha\kappa_p} \prod_{n\neq p} \frac{\kappa_p - \kappa_n}{\kappa_p + \kappa_n}, \quad \text{where} \quad \alpha = \sum_{n=1}^{N} \ln\Big(\frac{1+2\kappa_n}{1-2\kappa_n}\Big)^2.$$

Thus, $i\dot{a}_p$ has the same sign as b_n, and therefore $C_n^+ \equiv b_n/(i\dot{a}_p) > 0$. The time evolution of C_n^+ is (170)

$$C_n^+(t) = C_n^+(0) \exp\Big(- \kappa_n \Omega(\lambda_n^{-1}) t\Big). \tag{206}$$

The N-soliton solution is [20]

$$u(x,t) = \frac{\omega}{2} \int_0^\infty \exp\Big(- |x - g(\xi,t)|\Big) \xi^{-2} g_\xi^{-1}(\xi,t) \mathrm{d}\xi - \omega, \tag{207}$$

where $g(\xi,t)$ can be expressed through the scattering data as

$$g(\xi,t) \quad \equiv \quad \ln \int_0^\xi \Big(1 - \sum_{n,p} \frac{C_n^+(t)\underline{\xi}^{-2\kappa_n}}{\kappa_n + 1/2} A_{np}^{-1}[\underline{\xi},t]\Big)^{-2} \mathrm{d}\underline{\xi}, \tag{208}$$

with

$$A_{pn}[\xi,t] \quad \equiv \quad \delta_{pn} + \frac{C_n^+(t)\xi^{-2\kappa_n}}{\kappa_p + \kappa_n}.$$

For the peakon solutions ($\omega = 0$) the dependence on the scattering data is also known [5, 6].

The CH multi-soliton solutions also appear in several works [13, 16, 53, 64, 65, 70, 71, 72, 66]. The Darboux transform for the CH equation is obtained in [74]. The construction of multi-soliton and multi-positon solutions using the Darboux/Bäcklund transform is presented in [46, 50].

11 Conclusions

In this paper the Inverse Scattering Transform for the CH hierarchy is interpreted as a Generalized Fourier Transform. The generalized exponents are the squares of the eigenfunctions of the associated spectral problem. Apparently the CH hierarchy is well defined only if $q(x,0) \equiv m(x,0) + \omega > 0$. The only exception is the CH equation itself. The situation for CH when the condition $q(x,0) > 0$ on the initial data does not hold is more complicated and requires separate analysis [55, 7, 16, 18, 10]. Throughout this work the solutions $u(x,t)$ are confined to be functions in the Schwartz class, $\omega > 0$. The inverse scattering is outlined in detail in [20].

The spectral problem (2) is gauge equivalent to a standard Sturm-Liouville problem, well known from the KdV hierarchy

$$-\phi_{yy} + U(y)\phi \quad = \quad \mu\phi, \qquad \mu = -\frac{1}{4\omega} - \lambda,$$

$$\phi(y) = q^{1/4}\Psi, \qquad \frac{dy}{dx} = \sqrt{q}, \tag{209}$$

$$U(y) = \frac{1}{4q(y)} + \frac{q_{yy}(y)}{4q(y)} - \frac{3q_y^2(y)}{16q^2(y)} - \frac{1}{4\omega}. \tag{210}$$

Note that (209) leads to two possible expressions for the change of the variables in the Liouville transformation:

$$y = \sqrt{\omega}x + \int_{-\infty}^{x} (\sqrt{q(x')} - \sqrt{\omega})\mathrm{d}x' + \mathrm{const}, \tag{211}$$

$$y = \sqrt{\omega}x + \int_{\infty}^{x} (\sqrt{q(x')} - \sqrt{\omega})\mathrm{d}x' + \mathrm{const}. \tag{212}$$

These two possibilities, (211), (212) are only consistent iff

$$\int_{-\infty}^{\infty} (\sqrt{q(x)} - \sqrt{\omega})\mathrm{d}x = \mathrm{constant},$$

which is always the case, since the integral under question is (up to a multiplier) the Casimir function α (43); see some details in [61].

The matching of the CH hierarchy to KdV hierarchy requires solving the Ermakov-Pinney equation (210) [16, 25], which is not straightforward. One can eventually obtain a solution in parametric form [53, 64], see also [70, 71, 72, 66]. The analytic properties of the eigenfunctions and especially their asymptotics for $k \to \infty$ in these two cases are substantially different [20], e.g. compare (40), (41) with the well known results for the standard Stourm Liouville problem $e^{ikx}(1 + ...)$. This influences also the dispersion relation for the transmission coefficient. Thus the matching of the IST for these two cases is not automatic. This alternative approach relies on several implicit equations and is considerably less transparent than the approach adopted here.

We have also excluded the possibility of 'creation' or 'death' of solitons, i.e. an appearance of a new discrete eigenvalue as a result of an infinitesimal change δm in m. For the KdV equation this problem is addressed e.g. in [36].

The behavior of the scattering data at $k = 0$ is also an important question. In our analysis we implicitly assumed that the Wronskian

$$W(f^+(x,k), f^-(x,k))|_{x=0} \neq 0.$$

Then $a(k)$ at $k = 0$ has a singularity of type k^{-1}, cf. (17). However it is possible that $W(f^+(x,k), f^-(x,k))|_{x=0} = 0$, and then $a(k)$ is not singular at $k = 0$. To investigate the behavior of the quantities at $k = 0$ one can proceed as in [36]. There is a basis in the space of eigenfunctions of the spectral problem, which can be chosen as $f_1(x) = f^+(x,0)$ and $f_2(x) = -i\dot{f}^+(x,0)$. The asymptotics are

$$f_1(x) \to 1, \quad x \to \infty; \qquad f_1(x) \to Ax + B, \quad x \to -\infty;$$
$$f_2(x) \to x, \quad x \to \infty; \qquad f_2(x) \to Cx + D, \quad x \to -\infty,$$

where A, B, C, D do not depend on x, e.g.

$$
\begin{aligned}
A &= \frac{1}{4\omega}\int_{-\infty}^{\infty} m(x) f_1(x)\mathrm{d}x, \\
B &= 1-\frac{1}{4\omega}\int_{-\infty}^{\infty} x m(x) f_1(x)\mathrm{d}x \\
C &= 1+\frac{1}{4\omega}\int_{-\infty}^{\infty} m(x) f_2(x)\mathrm{d}x. \qquad (213)
\end{aligned}
$$

The Wronskian has the same value at $-\infty$ and $+\infty$, therefore

$$BC - AD = 1. \qquad (214)$$

The behavior of the scattering data is

$$a(k) = \frac{A}{2ik} + \frac{B+C}{2} + o(1), \qquad (215)$$

$$b(k) = -\frac{A}{2ik} + \frac{C-B}{2} + o(1). \qquad (216)$$

Note that A is an integral of motion (cf. [1, 2, 4]) as well as $B+C$ since $a(k)$ does not depend on t. If $A = 0$, the singularity disappears. Then, from (214) it follows that $BC = 1$ (and B and C are integrals of motion by themselves in this case – since both $B+C$ and BC are), i.e. if $C = B^{-1} = \sigma e^{\beta}$, $\sigma = \pm 1$, then

$$a(0) = \sigma \cosh\beta, \qquad b(0) = \sigma \sinh\beta, \qquad \mathcal{R}(0) = \tanh\beta.$$

This situation, although exceptional is the one in which the purely solitonic case is allowed: when $\beta = 0$, $\mathcal{R}(0) = 0$ (if $A \neq 0$, $\mathcal{R}(0) = -1$, see (215), (216)).

The Poisson bracket (9) is defined through variations with respect to m. We established that among these variations there are some that do not vanish in the asymptotic limit $|x| \to \infty$, such as for example the variations of the scattering data. This fact (which is not related to the smoothness or rate of decay of $m(x)$) is related to the presence (in general) of poles of $a(k)$ and $b(k)$ at $k = 0$. More careful analysis [1, 2, 37] leads to a modification in the definition of the Poisson bracket by additional terms, when the behavior of variations like $\delta a(k)/\delta m$, $\delta b(k)/\delta m$ is considered at $k = 0$.

Acknowledgements

A.C. acknowledges funding from the Science Foundation Ireland, Grant 04 BRG/M0042. V.S.G. acknowledges funding from the Bulgarian National Science Foundation, Grant 1410, R.I.I. acknowledges funding from the Irish Research Council for Science, Engineering and Technology. The authors are grateful to both referees for very helpful suggestions.

References

[1] Arkad'ev V. A., Pogrebkov A. K., Polivanov M. K.: Expansions with respect to squares, symplectic and Poisson structures associated with the

Sturm-Liouville problem. I. *Teoret. Mat. Fiz.* **72** (1987) 323–339 (Russian) [*Theor. Math. Phys.* **72** (1987), 909–920 (English)]

[2] Arkad'ev V. A., Pogrebkov A. K., Polivanov M. K.: Expansions with respect to squares, symplectic and Poisson structures associated with the Sturm-Liouville problem. II. *Teoret. Mat. Fiz.* **75** (1988) 170–186 (Russian) [*Theor. Math. Phys.* **75** (1988), 448–460 (English)]

[3] Barcilon V. Iterative solution of the inverse Sturm-Liouville problem, *J. Math. Phys.* **15** (1974), 429–436.

[4] Basu Mallick B. and Kundu A.: Levinson-type theorem for the Korteweg-de Vries system and its consequences, *Phys. Lett. A* **135** (1989), 113–116.

[5] Beals R., Sattinger D. and Szmigielski J. Acoustic scattering and the extended Korteweg-de Vries hierarchy, *Adv. Math.* **140** (1998), 190–206; arXiv: solv-int/9901007v1.

[6] Beals R., Sattinger D. and Szmigielski J. Multi-peakons and a theorem of Stieltjes, *Inv. Problems* **15** (1999), L1–L4; arXiv: solv-int/9903011v1.

[7] Bennewitz, C. On the spectral problem associated with the Camassa-Holm equation, *J. Nonlinear Math. Phys.* **11** (2004), 422–434.

[8] Birkhoff G. and Rota G-C. *Ordinary Differential equations*, (Waltham: Blaisdell Publishing Company, 1969)

[9] Bressan A. and Constantin A. Global solutions of the Hunter-Saxton equation, *SIAM J. Math. Anal.* **37** (2005), 996–1026; arXiv: math/0502059v1 [math.AP].

[10] Bressan A. and Constantin A. Global conservative solutions of the Camassa-Holm equation, *Arch. Rat. Mech. Anal.* **183** (2007), 215–239.

[11] Calogero F. Generalized Wronskian relations, one-dimensional Schrödinger equation and nonlinear partial differential equations solvable by the inverse-scattering method, *Nuovo Cimento B (11)* **31** (1976), 229–249.

[12] Camassa R. and Holm D. An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[13] Camassa R. and Zenchuk A. On the initial value problem for a completely integrable shallow water wave equation, *Phys. Lett. A* **281** (2001), 26–33.

[14] Constantin A. On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.* **155** (1998), 352–363.

[15] Constantin A. Existence of permanent and breaking waves for a shallow water equation: a geometric approach *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321–362.

[16] Constantin A. On the scattering problem for the Camassa-Holm equation, *Proc. Roy. Soc. London A* **457** (2001), 953–970.

[17] Constantin A. Finite propagation speed for the Camassa-Holm equation, *J. Math. Phys.* **46** (2005): Art. No. 023506.

[18] Constantin A. and Escher J. Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematica* **181** (1998), 229–243.

[19] Constantin A. and Escher J. Global weak solutions for a shallow water equation, *Indiana Univ. Math. J.* **47** (1998), 1527–1545.

[20] Constantin A., Gerdjikov V. and Ivanov R. Inverse scattering transform for the Camassa-Holm equation, *Inv. Problems* **22** (2006), 2197–2207; arXiv: nlin/0603019v2 [nlin.SI].

[21] Constantin A. and Ivanov R.: Poisson structure and Action-Angle variables for the Camassa-Holm equation, *Lett. Math. Phys.* **76** (2006), 93–108; arXiv: nlin/0602049v1 [nlin.SI].

[22] Constantin A., Kappeler T., Kolev B. and Topalov P. On geodesic exponential maps of the Virasoro group, *Ann. Glob. Anal. Geom* **31** (2007), 155-180.

[23] Constantin A. and Kolev B. Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** (2003), 787–804; arXiv: math-ph/0305013v1.

[24] Constantin A. and Kolev B. Integrability of invariant metrics on the diffeomorphism group of the circle, *J. Nonlin. Sci.* **16** (2006), 109–122.

[25] Constantin A. and Lenells J. On the inverse scattering approach to the Camassa-Holm equation, *J. Nonlinear Math. Phys.* **10** (2003), 252–255; arXiv: math-ph/0403039v1.

[26] Constantin A. and McKean, H. P. A shallow water equation on the circle, *Commun. Pure Appl. Math.* **52** (1999), 949–982.

[27] Constantin A. and Molinet L. Global weak solutions for a shallow water equation, *Comm. Math. Phys.* **211** (2000), 45–61.

[28] Constantin A. and Molinet L. Orbital stability of solitary waves for a shallow water equation, *Physica D* **157** (2001), 75–89.

[29] Constantin A. and Strauss W. Stability of peakons, *Commun. Pure Appl. Math.* **53** (2000), 603–610.

[30] Constantin A. and Strauss W. Stability of a class of solitary waves in compressible elastic rods, *Phys. Lett. A* **270** (2000), 140–148.

[31] Constantin A. and Strauss W. Stability of the Camassa-Holm solitons, *J. Nonlin. Sci.* **12** (2002), 415–422.

[32] Dai H.-H. Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mechanica* **127** (1998), 193–207.

[33] de Monvel A. and Shepelsky D. Riemann-Hilbert approach for the Camassa-Holm equation on the line, *C.R. Math. Acad. Sci. Paris* **343** (2006) 627–632

[34] Dullin H. R., Gottwald G. A. & Holm D. D. Camassa-Holm, Korteweg-de Vries-5 and other asymptotically equivalent equations for shallow water waves, *Fluid Dynam. Res.* **33** (2003), 73–95.

[35] Dullin H. R., Gottwald G. A. & Holm D. D. On asymptotically equivalent shallow water wave equations, *Physica* **190D** (2004), 1–14.

[36] Eilenberger G. 1981 Solitons. Mathematical Methods for Physicists. Springer Series in Solid-State Sciences. vol. **19** Springer-Verlag: Berlin.

[37] Faddeev L., Takhtajan L. Poisson Structure for the KdV equation, *Lett. Math. Phys* **10** (1985), 183–188.

[38] Fisher M. and Shiff J. The Camassa Holm equation: conserved quantities and the initial value problem, *Phys. Lett. A* **259** (1999), 371–376.

[39] Fokas A. and Fuchssteiner B., On the structure of symplectic operators and hereditary symmetries, *Lett. Nuovo Cimento* **28** (1980), 299–303.

[40] Fokas A. S., Olver P. J. and Rosenau P. 1996 A plethora of integrable bi-Hamiltonian equations, in: Algebraic Aspects of Integrable Systems: In Memory of Irene Dorfman, A. S. Fokas and I. M. Gel'fand, eds., Progress in Nonlinear Differential Equations, vol. 26, Birkhauser, Boston, pp. 93–101.

[41] Gerdjikov V. S. Generalised Fourier transforms for the soliton equations. Gauge-covariant formulation, *Inv. Problems* **2** (1986), 51–74.

[42] Gerdjikov V. S., Ivanov M. I. Expansions over the "squared" solutions and the inhomogeneous nonlinear Schrdinger equation, *Inv. Problems* **8** (1992), 831–847.

[43] Gerdjikov V. S. and Khristov E. Kh. Evolution equations solvable by the inverse-scattering method. I. Spectral theory. *Bulgarian J. Phys.* **7** No.1, 28–41, (1980). (In Russian): On evolution equations solvable by the inverse scattering method. II. Hamiltonian structure and Bäcklund transformations. *Bulgarian J. Phys.* **7** No.2, 119–133, (1980) (In Russian).

[44] Gerdjikov V. S., Yanovski A. B. Completeness of the eigenfunctions for the Caudrey-Beals-Coifman system, *J. Math. Phys.* **35** (1994), 3687–3725.

[45] Gesztesy F. and Holden H. *Soliton Equations and Their Algebro-Geometric Solutions, Volume I: (1+1)-Dimensional Continuous Models*, Cambridge studies in advanced mathematics, volume 79, 2003, (Cambridge: Cambridge University Press).

[46] Hone A. The associated Camassa-Holm equation and the KdV equation, *J. Phys. A: Math. Gen.* **32** (1999), L307–L314.

[47] Hunter J. K. and Saxton R. Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991), 1498–1521.

[48] Iliev I., Khristov E. and Kirchev K. Spectral Methods in Soliton Equations. Pitman Monographs and Surveys in Pure and Appl. Math. vol. **73**, London: Pitman (1994).

[49] Ivanov R., Water waves and integrability, *Philos. Trans. Roy. Soc.: Ser. A*, Theme Issue *Water waves* (2007) doi:10.1098/rsta.2007.2007; arXiv:0707.1839v1 [nlin.SI].

[50] Ivanov R. I. Conformal Properties and Bäcklund Transform for the Associated Camassa-Holm Equation, *Phys. Lett. A* **345** (2005), 235–243; arXiv: nlin/0507005v1 [nlin.SI].

[51] Ivanov R. I. Extended Camassa-Holm hierarchy and conserved quantities, *Zeitschrift für Naturforschung* **61a** (2006), 133–138; arXiv:nlin/0601066v1 [nlin.SI].

[52] Johnson R. S. Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid. Mech.* **457** (2002), 63–82.

[53] Johnson R. S. On solutions of the Camassa-Holm equation, *Proc. Roy. Soc. London A* **459** (2003), 1687–1708.

[54] Kaup D. J. Closure of the squared Zakharov-Shabat eigenstates, *J. Math. Anal. Appl.* **54** (1976), 849–864.

[55] Kaup D. J. Evolution of the scattering data of the Camassa-Holm equation for general initial data, *Stud. Appl. Math.* **117** (2006), 149–164.

[56] Khesin B. and Misiolek G.: Euler equations on homogeneous spaces and Virasoro orbits, *Adv. Math.* **176** (2003), 116-144.

[57] Khristov E. Expansion in products of the solutions of two Sturm-Liouville problems on the semiaxis, *Differentsial'nye Uravneniya* **16** (1980), 2023–2029, 2109. (Russian)

[58] Khristov E. On the Λ-operators associated with two Sturm-Liouville problems on the semi-axis, *Inv. Problems* **14** (1998), 647–660.

[59] Khristov E. On the transformations of symplectic expansions and the respective Bäcklund transformation for the KdV equation, *Serdica Math. J* **29** (2003), 75-94.

[60] Kolev B. Lie groups and mechanics: an introduction, *J. Nonlinear Math. Phys.* **11** (2004), 480–498; arXiv: math-ph/0402052v2.

[61] Lenells J. The scattering approach for the Camassa-Holm equation *J. Nonlin. Math. Phys.* **9** (2002), 389–393; arXiv: nlin/0306021v1 [nlin.SI].

[62] Lenells J. Conservation laws of the Camassa-Holm equation, *J. Phys. A: Math. Gen.* **38** (2005), 869–880.

[63] Lenells J. Traveling wave solutions of the Camassa-Holm equation, *J. Differential Equations* **217** (2005), 393–430.

[64] Li Y. and Zhang, J. The multiple-soliton solutions of the Camassa-Holm equation, *Proc. Roy. Soc. London A* **460** (2004), 2617–2627.

[65] Li Y. Some water wave equations and integrability, *J. Nonlinear Math. Phys.* **12** (2005), 466–481.

[66] Matsuno Y. Parametric representation for the multisoliton solution of the Camassa-Holm equation, *J. Phys. Soc. Japan* **74** (2005), 1983–1987; arXiv: nlin/0504055v1 [nlin.SI].

[67] Misiolek G. A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.* **24** (1998), 203–208.

[68] Novikov S. P., Manakov S. V., Pitaevskii L. P. and Zakharov V. E., *Theory of solitons: the inverse scattering method*, (New York: Plenum, 1984).

[69] Olver P. J. and Rosenau P. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E* **53** (1996), 1900–1906.

[70] Parker A. On the Camassa-Holm equation and a direct method of solution I. Bilinear form and solitary waves. *Proc. Roy. Soc. London A* **460** (2004), 2929–2957.

[71] Parker A. On the Camassa-Holm equation and a direct method of solution II. Soliton solutions, *Proc. Roy. Soc. London A* **461** (2005), 3611–3632.

[72] Parker A. On the Camassa-Holm equation and a direct method of solution III. N-soliton solutions, *Proc. Roy. Soc. London A* **461** (2005), 3893–3911.

[73] Reyes E. Geometric integrability of the Camassa-Holm equation, *Lett. Math. Phys.* **59** (2002), 117–131.

[74] Schiff J. The Camassa-Holm equation: a loopgroup approach, *Physica D* **121** (1998), 24–43.

[75] Stanislavova M. and Stefanov A. Attractors for the viscous Camassa-Holm equation *Discrete and Continuous Dynamical Systems*, **18** (2007), 159-186; math.DS/0612321.

[76] Zakharov V. and Faddeev L. Korteweg-de Vries equation is a completely integrable Hamiltonian system, *Funkz. Anal. Priloz.* **5** (1971) 18–27 (Russian) [*Func. Anal. Appl.* **5** (1971) 280-287 (English)].