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On socle-regularity and some notions of transitivity for Abelian p-groups

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Abstract

In the present work the interconnections between various notions of transitivity for Abelian p-groups and the recently introduced concepts of socle-regular and strongly socle-regular groups are studied.

1 Introduction

Early work in the theory of infinite Abelian p-groups focused on issues such as classification by cardinal invariants. This led initially to the rich theory known now as Ulm's theorem and, in some sense, culminated in deep classification of the class of groups known variously as *simply presented, totally projective* or *Axiom 3* groups. Such groups are, of necessity, somewhat special. On the other hand, there was also interest in properties of groups that were held by "the majority" of Abelian p-groups. Within this latter category, the extensive classes of transitive and fully transitive groups were prominent. Recently, the present authors introduced two new classes of p-groups which, respectively, properly contained the corresponding classes of transitive and fully transitive groups: these are the socle-regular and strongly socle-regular groups developed in [3] and [4]. The present paper looks further at the interconnections between these classes and some other recent notions of transitivity.

Throughout, all groups will be additively written, reduced Abelian p-groups; standard concepts relating to such groups may be found in [6] or [10]. We follow the notation of these texts but write

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mappings on the right. To avoid subsequent need for definitions of fundamental ideas, we mention that the *height* of an element x in the group G (written like $h_G(x)$) is the ordinal α if $x \in p^\alpha G \setminus p^{\alpha+1}G$ with the usual convention that $h(0) = \infty$. The *Ulm sequence* of x with respect to G is the sequence of ordinals or symbols ∞ given by $U_G(x) = (h_G(x), h_G(px), h_G(p^2x), \dots)$; the collection of such sequences may be partially ordered pointwise. Finally we recall an *ad hoc* notion introduced in [3] which continues to be useful here: suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$.

2 Various Notions of Transitivity

The notions of transitivity and full transitivity for Abelian p-groups were introduced by Kaplansky in [9] and became a topic of ongoing interest in Abelian group theory with the publication of Kaplansky's famous "little red book" [10]. Recall that a p-group G is said to be *transitive* (*fully transitive*) if for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$ ($U_G(x) \leq U_G(y)$) there is an automorphism (endomorphism) ϕ of G with $x\phi = y$. In recent times two addition notions of transitivity have been introduced: in [7] a group G is said to be *Krylov transitive* if, for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$, there is an endomorphism ϕ of G with $x\phi = y$. Finally, a group G was said in [7] to be *weakly transitive* if, given $x, y \in G$ and endomorphisms ϕ, ψ of G with $x\phi = y,\ y\psi = x$, there is an automorphism θ of G with $x\theta = y$. Notice in this last concept that although there is no explicit reference to Ulm sequences, the existence of the endomorphisms ϕ, ψ ensures that $U_G(x) = U_G(y)$.

To avoid a great deal of repetition, we find it convenient to use the expression *G is *-transitive* to mean that G has a fixed one of the the four transitivity properties discussed above.

In [2], Corner showed that transitivity and full transitivity of a group G are determined by the action of the endomorphism ring on the first Ulm subgroup $p^\omega G$. Following his example, we say that if Φ is a unital subring of the endomorphism ring $\mathrm{End}(G)$ of G and if H is a Φ-invariant subgroup of G, then

(i) Φ is transitive on H if, for any x, y in H with $U_G(x) = U_G(y)$, there is a unit $\phi \in \Phi$ with $x\phi = y$;

(ii) Φ is Krylov transitive on H if, for any x, y in H with $U_G(x) = U_G(y)$, there is an element $\phi \in \Phi$ with $x\phi = y$;

(iii) Φ is fully transitive on H if, for any x, y in H with $U_G(x) \leq U_G(y)$, there is an element $\phi \in \Phi$ with $x\phi = y$;

(iv) Φ is weakly transitive on H if, for any x, y in H and elements $\phi, \psi \in \Phi$ with $x\phi = y$ and $y\psi = x$, there is a unit $\theta \in \Phi$ with $x\theta = y$.

Our first result follows exactly as in [2, Lemma 2.1] or [7, Proposition 3.8], so we state it without proof:

Proposition 2.1 *The p-group G is *-transitive if, and only if,* $\mathrm{End}(G)$ *acts *-transitively on $p^\omega G$.*

An immediate consequence of Proposition 2.1 is the fact that addition of a separable summand has no influence on the transitivity properties.

Corollary 2.2 *If G is *-transitive and H is separable, then $K = G \oplus H$ is *-transitive.*

Proof. The proof for transitivity, full transitivity and Krylov transitivity follows by an identical argument to that given in [1, Proposition 2.6]. Suppose then that G is weakly transitive. It suffices, by Proposition 2.1, to show that $\mathrm{End}(K)$ acts weakly transitively on $p^\omega K = p^\omega G \oplus 0$. Suppose $(g_0, 0)$, $(g_1, 0) \in p^\omega K$ and there are endomorphisms ϕ, ψ of K with $(g_0, 0)\phi = (g_1, 0)$, $(g_1, 0)\psi = (g_0, 0)$. Representing ϕ, ψ as matrices in the standard way, $\phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ and $\psi = \left(\begin{smallmatrix} \alpha_1 & \gamma_1 \\ \delta_1 & \beta_1 \end{smallmatrix}\right)$, we conclude that $g_0\alpha = g_1$ and $g_1\alpha_1 = g_0$ for endomorphisms α, α_1 of G. Since G is weakly transitive, there is an automorphism θ of G with $g_0\theta = g_1$ and $g_1\theta^{-1} = g_0$. The matrix $\Delta = \left(\begin{smallmatrix} \theta & 0 \\ 0 & 1_H \end{smallmatrix}\right)$ then represents an automorphism of K and it is easy to check that $(g_0, 0)\Delta = (g_1, 0)$. ∎

There are, of course, many interrelations between the various notions of transitivity: for example, it is immediate that either transitivity or full transitivity implies Krylov transitivity. We list a representative sample of these connections:

Proposition 2.3 *(i) A group G is fully transitive if, and only if, its square $G \oplus G$ is transitive;*
(ii) If $p \neq 2$ and G is transitive, then G is fully transitive;
(iii) A direct summand of a transitive group is Krylov transitive;
(iv) If $p \neq 2$, then G is fully transitive if, and only if, G is Krylov transitive if, and only if, G is a summand of a transitive group;
(v) If G is Krylov transitive and weakly transitive, then G is transitive and vice versa;
(vi) If G is fully transitive and weakly transitive, then G is transitive.

Proof. A proof of (i) may be found as Corollary 3 in [5]; (ii) is a fundamental observation of Kaplansky [10, Theorem 26]. For (iii) assume $G = H \oplus K$ and that $x, y \in H$ with $U_H(x) = U_H(y)$. But then $U_G((x, 0)) = U_G((y, 0))$ and so there is an automorphism $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ with $(x, 0)\Phi = (y, 0)$. Hence, $y = x\alpha$ for the endomorphism α of H and H is Krylov transitive.

The equivalence of the first two parts of (iv) may be found in [7], while the final equivalence is Corollary 5 in [5].

Observe that (v) follows easily: if $U_G(x) = U_G(y)$, then Krylov transitivity implies that there are endomorphisms α, β of G with $x\alpha = y$, $y\beta = x$. By weak transitivity, there is the required automorphism ψ of G with $x\psi = y$. Conversely, G being transitive directly ensures that G is both Krylov transitive and weakly transitive. Finally (vi) follows immediately since full transitivity implies Krylov transitivity. ∎

Our next result is an analogue for Krylov transitivity of part of a well-known result of Kaplansky [10, Theorem 26], the other part being contained in (ii) and (iii) above.

Theorem 2.4 *Suppose G is a Krylov transitive reduced p-group and that G has at most two Ulm invariants equal to 1, and if it has exactly two, they correspond to successive ordinals, then G is fully transitive.*

Before embarking on the proof of Theorem 2.4, we establish two simple lemmas; recall from [10] that a group is said to have *property P(α)*, for an ordinal α, if for any element $x \in (p^\alpha G)[p] \setminus p^{\alpha+1}G$ there is an element y such that both y and $x + y$ are also of order p and height α.

Lemma 2.5 *Suppose that $x \in G$ with $U_G(x) = (\alpha_0, \alpha_1, \dots)$ and $y \in G[p]$ with $U_G(y) = (\alpha_0, \infty, \dots)$. Then if G is Krylov transitive and has property P(α_0), there is an endomorphism ϕ of G with $x\phi = y$.*

Proof. If $h(x+y)=\alpha_0$, then $U_G(x+y)=U_G(x)$ and so, by Krylov transitivity, there is an endomorphism ψ of G with $x\psi = x+y$. The mapping $\phi=\psi-1$ then has the desired property. Suppose then that $h(x+y)>\alpha_0$. Since we are assuming $\mathrm{P}(\alpha_0)$, there is an element z of height α_0 and order p such that $y-z$ also has height α_0 and order p. Now $(x+z)=(x+y)-(y-z)$ has height exactly α_0 since $h(x+y)>\alpha_0$, while $h(y-z)=\alpha_0$. It follows that $U_G(x+z)=(\alpha_0,\alpha_1,\alpha_2,\dots)=U_G(x)$. Thus, by Krylov transitivity, there is an endomorphism of G mapping x to $x+z$ and so, of course, there is a mapping $\psi : x\mapsto z$. Moreover, $U_G(z)=U_G(y)$ and so there is an endomorphism θ with $z\theta=y$. The composite $\phi=\psi\theta$ then maps $x\mapsto y$, as required. ∎

Our second lemma has been used previously in [7, Lemma 2.2]; its elementary proof may be found there.

Lemma 2.6 *If G is a p-group such that for all $x,y\in G$ with $y\in G[p]$ and $U_G(x)\le U_G(y)$, there is an endomorphism φ of G mapping x onto y, then G is fully transitive.*

Proof of Theorem 2.4 It suffices to show that the conditions of Lemma 2.6 above are satisfied. So assume that y is a fixed but arbitrary element of $G[p]$ and $x\in G$ with $U_G(x)\le U_G(y)$; clearly we may assume $y\neq 0$. The proof is by induction on the order of the element x. Denote $U_G(x)$ by $(\alpha_0,\alpha_1,\dots)$.

If $o(x)=p$, then $\alpha_1=\infty$ and we have $U_G(y)=(\beta_0,\infty,\dots)$ with $\beta_0\ge\alpha_0$. If $\beta_0=\alpha_0$, then x,y will have equal Ulm sequences and so, by Krylov transitivity, there is a map $\phi : x\mapsto y$. If $\beta_0>\alpha_0$, then the Ulm sequences of x and $x+y$ will be equal and Krylov transitivity yields a map $\psi : x\mapsto x+y$. The mapping $\phi=\psi-1_G$ will then have the desired property.

So now assume that x is of order p^n and that for all elements t with $o(t)<p^n$, if $U_G(t)\le U_G(s)$ with $s\in G[p]$, there is an endomorphism : $t\mapsto s$. Now the Ulm sequence of x has the form $(\alpha_0,\alpha_1,\dots,\alpha_{n-1},\infty,\dots)$; note that by an identical argument to that used in the previous paragraph, we may assume $U_G(y)=(\alpha_0,\infty,\dots)$. It follows from the existence of the gaps in the Ulm sequences for x,y that the Ulm invariants $f_G(\alpha_0),f_G(\alpha_{n-1})$ are both non-zero. Moreover if $h(x+y)=\alpha_0$, we are finished since $U_G(x+y)$ would then be $(\alpha_0,\alpha_1,\dots,\alpha_{n-1},\infty,\dots)=U_G(x)$ and Krylov transitivity would yield the required mapping.

Suppose then that $h(x+y)=\delta_0>\alpha_0$. Then $U_G(x+y)=(\delta_0,\alpha_1,\dots,\alpha_{n-1},\infty,\dots)$ and so $\alpha_0<\delta_0<\alpha_1$ i.e. there is a gap between α_0 and α_1. In particular $\alpha_{n-1}>\alpha_0+1$, so that α_0 and α_{n-1} are not successive ordinals. By our assumption on the Ulm invariants, one of the non-zero cardinals $f_G(\alpha_0),f_G(\alpha_{n-1})$ is not equal to 1. If $f_G(\alpha_0)\neq 1$, then G has the property $\mathrm{P}(\alpha_0)$ and so by Lemma 2.5, there is the desired mapping $x\mapsto y$.

If $f_G(\alpha_{n-1})\neq 1$, then G has property $\mathrm{P}(\alpha_{n-1})$. Furthermore, by [10, Lemma 29], we may write $x=v+w$ in such a way that $o(v)<o(x)$ and w is normal relative to x, has order p^n and $h(p^{n-1}w)=\alpha_{n-1}$. By induction there is an endomorphism mapping $v\mapsto y$. The proof is completed by an appeal to [10, Lemma 31]. ∎

By making use of ideas from [5], we can derive more information about the inter-relationships of the various transitivity properties. Recall that for a reduced group of length τ, the *Ulm support* $\mathrm{supp}(G)$ of G is the set of all ordinals $\sigma<\tau$ for which $f_G(\sigma)$ is non-zero.

Theorem 2.7 *If $G=G_1\oplus G_2$ and $supp(p^\omega G_1)=supp(p^\omega G_2)$, then the following are equivalent:*
(i) G is Krylov transitive;
(ii) G is fully transitive;

(iii) G is transitive.

Proof. Under the hypothesis of the theorem, the equivalence of (ii) and (iii) follows from [5, Theorem 1]. The implication (ii) $\Rightarrow$(i) holds even without the additional hypothesis. Thus it remains only to establish that (i) $\Rightarrow$ (ii).

Suppose then that G is Krylov transitive. Let B denote the standard basic group and set $H = G \oplus B \oplus B$; note that it follows from Corollary 2.2 that H is again Krylov transitive. Moreover a straightforward check shows that no Ulm invariant of H is equal to 1. It follows then from Theorem 2.4 that H is fully transitive. But then G, as a summand of a fully transitive group, is also fully transitive. ∎

Thus we can immediately deduce:

Corollary 2.8 *If $G = A \oplus A$ for some group A, then G is Krylov transitive if, and only if, it is fully transitive if, and only if, it is transitive.*

In [7] it was shown that full transitivity (Krylov transitivity) and weak transitivity are independent notions and Corner's original examples of non-transitive but fully transitive groups, and vice versa, show that Krylov transitivity is independent of the notions of transitivity and full transitivity. It would be interesting to know:

Question 2.1 Does there exist a Krylov transitive group which is neither transitive nor fully transitive? Such a group would necessarily be a 2-group.

The four notions of transitivity also share the property that subgroups of the form $p^\beta G$ are, in some circumstances, the key to determining the *-transitivity of the whole group G. The following generalizes [8, Theorems 3 and 4].

Proposition 2.9 *Suppose that $G/p^\beta G$ is totally projective for some ordinal β. Then G is *-transitive if, and only if, $p^\beta G$ is *-transitive.*

Proof. Let $H = p^\beta G$ and observe that if $h_H(x) = \alpha$, then $h_G(x) = \beta + \alpha$. Consequently, if $x, y \in H$ and $U_H(x) = U_H(y)(U_H(x) \leq U_H(y))$, then $U_G(x) = U_G(y)(U_G(x) \leq U_G(y))$. Thus if G is transitive, Krylov transitive or fully transitive, it follows easily that $p^\beta G$ has the same property. If G is weakly transitive and $x, y \in H$ are such that there are endomorphisms ϕ, ψ of H with $x\phi = y, y\psi = x$, then since ϕ, ψ do not decrease heights computed in G and G/H is totally projective, it follows from a well-known result of Hill (see, e.g., [8]) that ϕ, ψ extend to endomorphisms ϕ', ψ' of G and, of course, $x\phi' = y,\ y\psi' = x$. Since G is, by assumption, weakly transitive, there is an automorphism, θ' say, of G with $x\theta' = y$. Then $\theta = \theta' \restriction H$ is an automorphism of H with $x\theta = y$, as required.

Conversely suppose that $p^\beta G$ is *-transitive and let $x, y \in G$ be elements such that $U_G(x) = U_G(y)(U_G(x) \leq U_G(y))$ [there exist endomorphisms ϕ, ψ of G with $x\phi = y,\ y\psi = x$]; note that in the third case one also has that $U_G(x) = U_G(y)$. Let n, m be the smallest integers such that $p^n x \in p^\beta G, p^m y \in p^\beta G$; observe that $m \leq n$ with equality in the first and third cases. In the case of transitivity or Krylov transitivity of $p^\beta G$, we have an automorphism (endomorphism) ϕ of H with $p^n x\phi = p^n y$ and this extends to an isomorphism (endomorphism) of $\langle H, x\rangle \to \langle H, y\rangle$ by mapping $x \mapsto y$. Since this is height-preserving (not height-decreasing) in G, the aforementioned Hill's result again yields an extension of ϕ to G with the required property.

In the case of weak transitivity, we have a pair of endomorphisms ϕ, ψ of G and their restrictions to H also interchange x and y. Hence there is an automorphism of H mapping x to y and again, by the total-projectivity of G/H, we get the desired automorphism of G sending x to y.

Finally consider the case where H is assumed to be fully transitive. As noted above, $p^n x, p^n y$ both belong to $p^\beta G$ and $U_H(p^n x) \leq U_H(p^n y)$. So there exists an endomorphism of H mapping $p^n x \mapsto p^n y$. This mapping extends to a mapping from $\langle H, x\rangle \to \langle H, y\rangle$ by mapping $x \mapsto y$. Since heights in G are not decreased and the quotient G/H is totally projective, there exists the desired endomorphism of G mapping $x \mapsto y$. ∎

Remark 2.1 In the cases of transitivity, Krylov transitivity and full transitivity, it is not necessary to assume that $G/p^\beta G$ is totally projective in order to deduce that $p^\beta G$ inherits the corresponding transitivity property.

The various notions of transitivity behave somewhat differently in relation to the formation of direct summands: a summand of a fully transitive group is again fully transitive, but this is not true in general for transitive or weakly transitive groups – see, for example, [1] and [7].

Proposition 2.10 *Let $G = H \oplus K$, then if G is Krylov-transitive, H is also Krylov-transitive.*

Proof. Suppose that G is Krylov transitive and let x, y be elements of H with $U_H(x) = U_H(y)$. Then the elements $(x, 0), (y, 0)$ of G have equal Ulm sequences in G and consequently there is an endomorphism of G mapping $(x, 0)$ to $(y, 0)$; this, of course, necessitates the existence of an endomorphism of H mapping x to y. ∎

There are, however, some situations in which summands of transitive (weakly transitive) groups inherit the transitivity property. Recall that a homomorphism $\phi : G \to H$ is said to be *small* if for every natural number k, there is a natural number n such that $(p^n G)[p^k]\phi = 0$. Weakening this definition, we shall say that a homomorphism $\varphi : G \to H$ is *almost small* if $p^\omega G \subseteq ker\varphi$. Clearly, every small homomorphism is almost small, whereas the converse does not hold always. Also, if H is separable, then each homomorphism between G and H is almost small.

Recall - see [1] - that a group G is said to be of *type A* if $U(\mathrm{End}(G) \restriction p^\omega G) = \mathrm{Aut}(G) \restriction p^\omega G$. Before stating our result on summands, we derive the following lemma:

Lemma 2.11 *Suppose that $K = G \oplus H$ and every homomorphism from G to H is almost small. Then if G is of type A and $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ is an automorphism of K, there is an automorphism ϕ of G with $\phi \restriction p^\omega G = \alpha \restriction p^\omega G$.*

Proof. Since Φ is an automorphism of K, its restriction to $p^\omega K$ is an automorphism of $p^\omega K = p^\omega G \oplus p^\omega H$. Letting bars denote restrictions to first Ulm subgroups, we get $\overline{\Phi} = \left(\begin{smallmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\delta} & \overline{\beta} \end{smallmatrix}\right)$ and the assumption of almost smallness forces $\overline{\gamma} = 0$. Since every endomorphism of $p^\omega K$ must have a matrix representation which is lower triangular, we deduce that $\overline{\alpha}$ is a unit of $\mathrm{End}(p^\omega G)$. Since G is of type A, there is an automorphism ϕ of G with $\phi \restriction p^\omega G = \overline{\alpha} = \alpha \restriction p^\omega G$, as required. ∎

Proposition 2.12 *If $K = G \oplus H$ and every homomorphism from G to H is almost small, then if K is transitive (weakly transitive) and G is of type A, then G is also transitive (weakly transitive).*

Proof. We consider first the situation where K is transitive. Suppose that $x, y \in p^\omega G$ with $U_G(x) = U_G(y)$. Then $(x, 0)$ and $(y, 0)$ are elements of $p^\omega K$ having the same Ulm sequences in K. Since K is transitive, there is an automorphism $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ of K with $(x, 0)\Phi = (y, 0)$. By the previous Lemma 2.11, there is an automorphism ϕ of G with $\phi \restriction p^\omega G = \alpha \restriction p^\omega G$, so that $x\phi = x\alpha = y$. Hence $\mathrm{End}(G)$ acts transitively on $p^\omega G$ and by Proposition 2.1 we have that G is transitive, as required.

Finally suppose that K is weakly transitive and x, y are as above with endomorphisms θ, ψ of G such that $x\theta = y, y\psi = x$. Then the endomorphisms of K, given by the matrix representations $\Theta = \left(\begin{smallmatrix} \theta & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\Psi = \left(\begin{smallmatrix} \psi & 0 \\ 0 & 0 \end{smallmatrix}\right)$, have the property that $(x, 0)\Theta = (y, 0)$ and $(y, 0)\Psi = (x, 0)$. Since K is weakly transitive, there is an automorphism $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ of K with $(x, 0)\Phi = (y, 0)$. Appealing again to the previous lemma, yields an automorphism ϕ of G with $\phi \restriction p^\omega G = \alpha \restriction p^\omega G$. Thus $x\phi = x\alpha = y$ and $\mathrm{End}(G)$ acts weakly transitively on $p^\omega G$, as required. ∎

Our next result shows that Krylov transitive groups behave nicely when "squared", provided that the lattice of Ulm sequences of the first Ulm subgroup is a chain.

Theorem 2.13 *Suppose G is a group such that all elements of $p^\omega G$ have comparable Ulm sequences. Then $G \oplus G$ is Krylov transitive if, and only if, G is Krylov transitive. This property may fail if there are elements of $p^\omega G$ with incomparable Ulm sequences.*

Proof. The necessity follows directly from Proposition 2.10; in fact, there is no need for any assumption about $p^\omega G$ for this implication. For the sufficiency, let $H = G \oplus G$ and suppose $\underline{x}, \underline{y} \in p^\omega H$ with $U_H(\underline{x}) = U_H(\underline{y})$, where $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$. By assumption the Ulm sequences of elements of $p^\omega G$ are comparable, so there is no loss in generality in assuming that $U_H(\underline{x}) = U_G(x_1)$, $U_H(\underline{y}) = U_G(y_1)$. Since G is Krylov transitive, there is an endomorphism $\theta : G \to G$ with $x_1\theta = y_1$. Appealing to the comparability hypothesis again, either $U_G(y_1) = U_G(y_2)$ or $U_G(y_1) < U_G(y_2)$.

If the first possibility arises, $U_G(x_1) = U_G(y_2)$ and so there is an endomorphism ψ of G with $x_1\psi = y_2$. If $\Delta = \left(\begin{smallmatrix} \theta & \psi \\ 0 & 0 \end{smallmatrix}\right)$, then $\Delta \in \mathrm{End}(H)$ and $\underline{x}\Delta = \underline{y}$.

In the second situation $U_G(y_1) < U_G(y_2)$ and so we have $U_G(y_1 + y_2) = U_G(y_1)$; again there is an endomorphism of G, ϕ say, with $x_1\phi = y_1 + y_2$. Now set $\Delta = \left(\begin{smallmatrix} \theta & \theta+\phi \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{End}(H)$ and $\underline{x}\Delta = \underline{y}$.

For the second part of the theorem, recall that Corner [2, Section 4] has constructed a transitive, but non fully transitive 2-group C with $2^\omega C = \langle a \rangle \oplus \langle b \rangle$, where a and b have orders 2 and 8 respectively. Note that C is, of course, Krylov transitive since it is even transitive and that the elements $a, 2b$ of $2^\omega C$ have incomparable Ulm sequences. It is shown in [7, Example 3.16] that $C \oplus C$ is weakly transitive. However $C \oplus C$ is not Krylov transitive: if it were, it would follow from Proposition 2.3 (v) that $C \oplus C$ is transitive, which in turn implies by (i) of the same proposition that C is fully transitive – contradiction. ∎

Question 2.2 Does there exist a non fully transitive Krylov transitive group which satisfies the above theorem? Owing to [7, Proposition 2.3] such a group must necessarily be a 2-group.

Remark 2.2 In the second part of of the proof of Theorem 2.13 it is possible to show directly, arguing as in the proof of [7, Example 3.16], that there is no endomorphism of $C \oplus C$ mapping the element $(a + 2b, 0)$ to $(a + 2b, a)$ although both elements have Ulm sequence $(\omega, \omega + 2, \infty, \dots)$. Moreover, by what we have shown above, if G is transitive, then $G \oplus G$ need not be Krylov transitive. So, is it true that G is Krylov transitive non transitive if, and only if, $G \oplus G$ is Krylov transitive non transitive?

Our next result is simply a reworking of Corner's Proposition 2.2 in [2]: observe that in the proof there, it suffices to have Krylov transitivity at each of the key stages.

Proposition 2.14 *Let G be a Krylov transitive group such that $p^\omega G$ is a homocyclic group of finite exponent. Then*

(i) G is fully transitive.

(ii) If there is a direct decomposition $G = G_1 \oplus G_2$ with $p^\omega G_i \neq 0$ $(i = 1, 2)$, then G is transitive.

It is possible to improve somewhat on Theorem 2.7 by using the methods of [5] and replacing full transitivity by Krylov transitivity. Rather than adopt either extreme of leaving the task to the reader or re-writing the proofs in their entirety, we point out the key argument needed to replace the use of full transitivity.

Lemma 2.15 *Suppose that $G = G_1 \oplus G_2$ and $x \in G_1, y \in G_2$ with $U_{G_1}(x) \geq U_{G_2}(y)$. Then if G is Krylov transitive, there is a homomorphism $\delta : G_2 \to G_1$ with $y\delta = x$.*

Proof. Consider the elements (x, y) and $(0, y)$ of G. Since their Ulm sequences are, respectively, the infima $U_{G_1}(x) \wedge U_{G_2}(y)$ and $U_{G_2}(y) \wedge U_{G_1}(0)$, it follows that they are both equal to $U_{G_2}(y)$. By Krylov transitivity, there is a matrix $\Phi = \left(\begin{smallmatrix} \theta & \psi \\ \delta & \beta \end{smallmatrix}\right)$ with $(0, y)\Phi = (x, y)$ which gives $y\delta = x$ as required. ∎

Proposition 2.16 *If $G = G_1 \oplus G_2$ is Krylov transitive, G_2 is transitive and $supp(p^\omega G_1) \subseteq supp(p^\omega G_2)$, then G is transitive.*

Proof. The proof is based on Lemma 2 in [5]. In the proof of Lemma 2, two applications of full transitivity are made. The first such is actually based on elements a_1, b_2 with equal Ulm sequences and it is immediate that Krylov transitivity will suffice for the argument there. The second application of full transitivity occurs at the bottom of [5, p.1607] but it is easily seen to involve the set-up invoked in Lemma 2.15 above. Consequently this too will carry over to the Krylov transitivity situation.

The final stage of the proof is carried out in an identical fashion to Proposition 2 in [5]. However the appeals to full transitivity can be replaced by the argument of Lemma 2.15. ∎

We now give an example that indicates that Question 2.1 may be rather difficult.

Example 2.17 *If $G = C_1 \oplus C_2$ where C_1 (respectively C_2) is a non transitive, fully transitive 2-group (is a transitive, non fully transitive 2-group) as constructed by Corner [2], then G is not fully transitive and it is Krylov transitive if, and only if, it is transitive.*

Proof. That G is not fully transitive is immediate since the summand C_2 is not fully transitive. One implication is trivial. Note then that the group $2^\omega C_1$ is elementary while $2^\omega C_2 \cong \mathbb{Z}(2) \oplus \mathbb{Z}(8)$ and so $\mathrm{supp}(2^\omega C_1) \subseteq \mathrm{supp}(2^\omega C_2)$. If G is Krylov transitive, then since C_2 is transitive, it would follow from Proposition 2.16 that G is transitive. ∎

We remark that it can be shown directly that the group G above is not transitive.

We close this section with a generalization of a problem due to Corner [2].

Question 2.3 Are Krylov transitive groups with finite first Ulm subgroup weakly transitive?

Notice that it follows from [7, Corollary 3.11] this is true for groups of type A (even without the assumption of Krylov transitivity); by a simple modification of the argument in [7, Corollary 3.13], one can show that the converse does not hold.

3 Socle-Regularity and Strong Socle-Regularity

In [3], [4] the notions of socle-regularity and strong socle-regularity were introduced; the question of whether or not a summand of a socle-regular group is again socle-regular, was left unanswered in [3]. We can now answer this in the affirmative. Notice that the same problem was settled in the negative for strongly socle-regular groups in [4]. Recall the definitions: a group G is said to be *socle-regular* (*strongly socle-regular*) if for all fully invariant (characteristic) subgroups F of G, there exists an ordinal α (depending on F) such that $F[p] = (p^\alpha G)[p]$. It is self-evident that strongly socle-regular groups are themselves socle-regular, whereas the converse is not valid (see [4]).

Proposition 3.1 *A summand of a socle-regular group is again socle-regular.*

Proof. Let $G = A \oplus B$ be a socle-regular group; we show A is also socle-regular.

Let F be an arbitrary fully invariant subgroup of A and set $C = \langle x\gamma : x \in F[p],\ \gamma \in \mathrm{Hom}(A, B)\rangle$. Note that C is an elementary group. We claim that (i) $C\delta \leq F[p]$ for all $\delta : B \to A$ and (ii) $C\beta \leq C$ for all $\beta \in \mathrm{End}(B)$.

Assuming for the moment that we have established these claims, consider the subgroup $F[p] \oplus C$ of G. If $\Delta = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ is an arbitrary endomorphism of G (with the usual conventions), then $(F[p] \oplus C)\Delta \leq (F[p]\alpha + C\delta, F[p]\gamma + C\beta)$. Clearly $F[p]\alpha \leq F[p]$ by the full invariance of F in A and $F[p]\gamma \leq C$ by definition, so that the claims above yield $(F[p] \oplus C)\Delta \leq F[p] \oplus C$ i.e. $F[p] \oplus C$ is fully invariant in G. Now G is socle-regular, so there is an ordinal τ such that $F[p] \oplus C = (p^\tau G)[p] = (p^\tau A)[p] \oplus (p^\tau B)[p]$. It follows that $F[p] = (p^\tau A)[p]$ and since F was an arbitrary fully invariant subgroup of A, we have that A is socle-regular, as required.

To establish the first claim, note that if $c \in C$, then $c = \sum x_i\gamma_i$ for some $x_i \in F[p],\ \gamma_i : A \to B$. But then $c\delta = \sum(x_i\gamma_i)\delta = \sum x_i(\gamma_i\delta)$ and $\gamma_i\delta \in \mathrm{End}(A)$. Thus $x_i(\gamma_i\delta) \in F[p]$ since the latter is fully invariant in A.

For the second claim, it suffices to note, using the same notation as above, that $\gamma_i\beta \in \mathrm{Hom}(A, B)$ so that $c\beta = \sum x_i(\gamma_i\beta) \in C$ by definition. ∎

As noted above, it was shown in [4] that summands of strongly-socle-regular groups need not be strongly socle-regular. However we do have the following elementary classification showing that socle-regular groups are precisely the summands of strongly socle-regular groups:

Corollary 3.2 *The following are equivalent for a group G:*
(i) G is a summand of a strongly socle-regular group;
(ii) G is a socle-regular group;
(iii) the square $G \oplus G$ is strongly socle-regular;
(iv) the square $G \oplus G$ is socle-regular.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Proposition 3.1, while $(ii) \Rightarrow (iii)$ was established in [4, Theorem 3.6]. The final implication $(iii) \Rightarrow (i)$ is immediate. The equivalence $(ii) \iff (iv)$ was obtained in [3, Theorem 1.4]. ∎

The following extends [4, Proposition 3.3 (i)].

Corollary 3.3 *Any summand A of a strongly socle-regular group is strongly socle-regular if* $\mathrm{End}(A)$ *is additively generated by* $\mathrm{Aut}(A)$.

Proof. Suppose $G = A \oplus B$ is a strongly socle-regular group. Applying Proposition 3.1, we deduce that A is socle-regular. Since $\mathrm{Aut}(A)$ generates $\mathrm{End}(A)$, in view of [4, Proposition 2.5] every characteristic subgroup C of A is fully invariant in A and hence has the required form $C[p] = (p^\alpha A)[p]$. ∎

Under certain circumstances, socle-regularity and strong socle-regularity do coincide.

Proposition 3.4 *Suppose that $G = A \oplus B$ with $p^n A \cong p^n B$, for some non-negative n. Then G is socle-regular if, and only if, G is strongly socle-regular.*

Proof. One implication is clear and does not depend on the additional hypothesis on G. Conversely suppose that G is socle-regular; note that it is immediate that $p^n G$ is also socle-regular. Then, $p^n G$ is isomorphic to the square of a fixed group, $p^n A$, and consequently its endomorphism ring is additively generated by its automorphism group. Furthermore, it follows from Corollary 3.2 above that $p^n G$ is strongly socle-regular. That G itself is strongly socle regular follows from [4, Proposition 2.6 (iii)]. ∎

It was shown in [3, Theorem 0.3] that fully transitive groups were socle-regular and in [4, Theorem 2.4] that transitive groups were socle-regular (indeed they are even strongly socle-regular). It is, perhaps, not surprising then that Krylov transitive groups share the same property.

Proposition 3.5 *If the group G is Krylov transitive, then G is socle-regular.*

Proof. Let F be a fully invariant subgroup of G and let $\alpha_0 = \min(F[p])$, so that $h(y) \geq \alpha_0$ for all $y \in F[p]$. Clearly $F[p] \leq (p^{\alpha_0} G)[p]$.

Conversely suppose that $x \in (p^{\alpha_0} G)[p]$, so that $U_G(x) = (\alpha, \infty, \dots)$ for some $\alpha \geq \alpha_0$. Choose a fixed, but arbitrary, $z \in F[p]$ such that $h(z) = \alpha_0$. If $\alpha = \alpha_0$, then $U_G(x) = U_G(z)$ and so, by Krylov transitivity, there is an endomorphism ϕ of G with $z\phi = x$. Hence $x \in (F[p])\phi \leq F[p]$, since F is fully invariant in G. If $\alpha > \alpha_0$, then $h(x+z) = \alpha_0$ and so $U_G(x+z) = (\alpha_0, \infty, \dots) = U_G(z)$. Again, by Krylov transitivity, there is an endomorphism ψ with $z\psi = x + z$. But then $\psi - 1_G$ is an endomorphism of G and $z(\psi - 1_G) = (x+z) - z = x$, forcing $x \in F[p]$ since the latter is fully invariant in G. Thus in either case $(p^{\alpha_0} G)[p] \leq F[p]$, as required. ∎

Remark 3.1 There is, however, no possibility of extending the above proposition to strong socle-regularity: in [4, Example 3.5] a fully transitive, and hence Krylov transitive, group is exhibited which is not strongly socle-regular. Moreover, the above proposition cannot be reversed. In fact, there even exists a strongly socle-regular group which is not Krylov transitive. Indeed, the group C, discussed in the second part of the proof of Theorem 2.13, has been shown in [3, Example] to be socle-regular.

Hence its square $C \oplus C$ is also socle-regular, whence by Corollary 3.2 (iii) it is strongly socle-regular, but is not Krylov transitive.

The class of weakly transitive groups is not, however, contained in the class of socle-regular groups:

Proposition 3.6 *There exists a weakly transitive group X which is not socle-regular.*

Proof. Let T be a separable p-group such that $\mathrm{End}(T) = J_p \oplus E_s(T)$ where J_p is the ring of p-adic integers and $E_s(T)$ is the ideal consisting of all small endomorphisms of T; such groups are easy to find, the first example being due to Pierce [11]. Let B be a basic subgroup of T, so that T/B is divisible of rank $\lambda > 1$ say. Now construct a p-group X such that $X/p^\omega X \cong T$ and $p^\omega X$ is elementary of rank λ – for instance, use the pullback construction of Lemma 44.1 in [12].

Then $\mathrm{End}(X/p^\omega X) = J_p \oplus E_s(X/p^\omega X)$ and hence $\mathrm{End}(X) = J_p \oplus E_\theta(X)$, where $E_\theta(X)$ is the ideal of thin endomorphisms of X – see [2]. In this situation every thin endomorphism is actually small – see [2, Lemma 7.2]. Note that if φ is small, $(p^\omega X)\varphi = 0$.

We claim X is weakly transitive (by Proposition 2.1 it is enough to check this on $p^\omega X$): if $x, y \in p^\omega X$ with $x\phi = y$ and $y\psi = x$ for endomorphisms ϕ, ψ, then $\phi = r + \varphi_1$ and $\psi = s + \varphi_2$, where each φ_i is small. This forces r, s to be mutually inverse p-adic integers with $xr = y,\ ys = x$, so X is certainly weakly transitive.

Finally we claim that X is not socle-regular. Consider any proper subgroup F of $p^\omega X$. Since $p^{\omega+1}X = 0$, $F = F[p] \neq (p^\alpha X)[p]$ for any α. However F is fully invariant since endomorphisms of X act on $p^\omega X$ as scalar multiplications. Thus X is not socle-regular, as claimed. ∎

In virtue of Proposition 3.5 and [7] there is an abundance of socle-regular groups that are not weakly transitive. However, it would be interesting to know whether or not there exists a strongly socle-regular group which is not weakly transitive.

In light of Proposition 2.12, the following is not too surprising.

Proposition 3.7 *If $G = K \oplus H$ is strongly socle-regular where K is of type A and each homomorphism between K and H is almost small (in particular, either H is separable or $Hom(K, H) = Small(K, H)$), then K is strongly socle-regular.*

Proof. Suppose that C is an arbitrary characteristic subgroup of K. If $C[p] \not\subseteq p^\omega K$, then applying Proposition 1.1 (ii) from [4] we get that $C[p] = (p^t K)[p]$ for some natural t, and we are done. So, we may assume that $C[p] \leq p^\omega K$. Assume that we have shown that $C[p] \oplus \{0\}$ is characteristic in G. Then, by strong socle-regularity, we will have $C[p] \oplus \{0\} = (p^\delta G)[p] = (p^\delta K)[p] \oplus (p^\delta H)[p]$ for some $\delta \geq \omega$, insuring that $C[p] = (p^\delta K)[p]$ as required.

It suffices then to show that $C[p] \oplus \{0\}$ is characteristic in G. If $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ is an arbitrary automorphism of G, then $C[p]\gamma = 0$ since γ is, by assumption, almost small. Thus $(C[p] \oplus \{0\})\Phi = (C[p]\alpha \oplus \{0\})$. It follows from Lemma 2.11 that there is an automorphism ϕ of K with $\phi \restriction p^\omega K = \alpha \restriction p^\omega K$ and this clearly yields the desired result since $C[p]$ is a characteristic subgroup of K. ∎

It was shown in [3] ([4]) that socle-regularity (strong socle-regularity) of a group G is inherited by the subgroups $p^\alpha G$ for all α. It is clear that the converse cannot hold in general, but it was shown in [4, Proposition 2.6] that strong socle-regularity "lifts" from a subgroup $p^\alpha G$ to G provided that $G/p^\alpha G$ is totally projective and $\alpha < \omega^2$; we do not know if the ordinal restriction is necessary. It is,

of course, possible to modify the argument in [4, Proposition 2.6] to show directly that an analogous result holds for socle-regularity. However with our classification of socle-regularity in terms of strong socle-regularity, we can easily deduce the result. Our first observation has no ordinal restriction.

Proposition 3.8 *Let $G/p^\alpha G$ be totally projective for some ordinal α. Then the implication (b)$\Rightarrow$(a) holds, where*

(a) G is socle-regular if $p^\alpha G$ is socle-regular;

(b) G is strongly socle-regular if $p^\alpha G$ is strongly socle-regular.

Proof. Suppose $p^\alpha G$ is socle-regular. Then, in view of [4, Theorem 3.6], $p^\alpha(G\oplus G) = p^\alpha G\oplus p^\alpha G$ is strongly socle-regular. Observe that $(G\oplus G)/p^\alpha(G\oplus G) = (G\oplus G)/(p^\alpha G\oplus p^\alpha G) \cong (G/p^\alpha G)\oplus(G/p^\alpha G)$ is totally projective, whence by hypothesis $G\oplus G$ is strongly socle-regular. It follows again from [4, Theorem 3.6] that G is socle-regular, as asserted. ∎

As an immediate consequence of Proposition 3.8 and Proposition 2.6 (v) from [4], we have the following strengthening of Theorem 1.7 from [3].

Corollary 3.9 *Suppose $G/p^\alpha G$ is totally projective for some ordinal $\alpha < \omega^2$. Then G is socle-regular if, and only if, $p^\alpha G$ is socle-regular.*

Next, we show that the converse of Proposition 1.8 from [3] does not hold.

Example 3.1 There are socle-regular groups A and B, with each homomorphism between them being small, such that $A\oplus B$ is not socle-regular.

Proof. Let A, B be 2-groups with $2^\omega A \cong 2^\omega B \cong \mathbb{Z}(2)\oplus\mathbb{Z}(8)$ as in Corner's construction [2] of transitive but not fully transitive groups. It is easy to arrange that the groups A, B have the additional property that $\mathrm{Hom}(A,B) = \mathrm{Hom}_s(A,B)$ and $\mathrm{Hom}(B,A) = \mathrm{Hom}_s(B,A)$. Now consider the group $G = A\oplus B$ and its subgroup $H = (2^\omega A)[2]\oplus(2^{\omega+1}B)[2]$. The latter is fully invariant in G because any endomorphism of G acts diagonally on H since the cross homomorphisms, being small, act trivially on the components of H. However, an easy check shows that H cannot be of the form $(2^\alpha G)[2]$ for any α. ∎

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