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# On the socles of characteristic subgroups of Abelian $p$ -groups

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## Abstract

Fully invariant subgroups of an Abelian  $p$ -group have been the object of a good deal of study, while characteristic subgroups have received somewhat less attention. Recently the socles of fully invariant subgroups have been studied and this led to the notion of a socle-regular group. The present work replaces the fully invariant subgroups with characteristic ones and leads in a natural way to the notion of a strongly socle-regular group. A surprising relationship, mirroring that between transitive and fully transitive groups, is obtained.

## 1 Introduction

While attempts have been made to classify all the fully invariant subgroups of an Abelian  $p$ -group - see, for example, the important work of Kaplansky in §18 of [9] - very little investigation of the corresponding problem for characteristic subgroups has been undertaken except in the situations where, essentially, all characteristic subgroups are in fact fully invariant. Given the difficulty of the fully invariant subgroup problem this is not too surprising. In a recent work the authors [4] investigated the somewhat simpler problem of determining the socles of fully invariant subgroups and the present work builds on that approach for characteristic subgroups. It is perhaps worth remarking that although Kaplansky's notions of transitive and fully transitive groups do not involve explicit reference to characteristic or fully invariant subgroups, these latter were clearly motivating concepts for his transitivity notions. As we shall see shortly, our notions of socle-regularity and strong socle-regularity may be interpreted as generalizations of the notions of full transitivity and transitivity respectively.

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Our notation is standard and follows [6, 9], an exception being that maps are written on the right. Recall a key definition from [4]:

A  $p$ -group  $G$  is said to be *socle-regular* if for all fully invariant subgroups  $F$  of  $G$ , there exists an ordinal  $\alpha$  (depending on  $F$ ) such that  $F[p] = (p^\alpha G)[p]$ . (Recall also that in [4], it has been established that the class of socle-regular groups strictly contains the class of fully transitive groups.)

An obvious strengthening of this concept is:

A  $p$ -group  $G$  is said to be *strongly socle-regular* if for all characteristic subgroups  $C$  of  $G$ , there exists an ordinal  $\alpha$  (depending on  $C$ ) such that  $C[p] = (p^\alpha G)[p]$ .

Clearly a strongly socle-regular group is socle-regular but we shall see shortly the reverse does not hold. The primary purpose of the present work is the investigation of strongly socle-regular groups and we shall give a characterization of them in terms of socle-regular groups; see Theorem 3.6 below. There is a clear family resemblance between this characterization and the corresponding relation between transitive and fully transitive groups found by Files and the second author [5].

In the determination of fully invariant subgroups of a  $p$ -group, the presence of a divisible subgroup creates little difficulty but the same is not true for characteristic subgroups. Hence we shall briefly look at the situation when our groups have a non-zero divisible part. We begin with a simple lemma:

**Lemma 1.1** (i) *If  $D$  is a divisible  $p$ -group, then its characteristic subgroups are of the form  $D[p^n]$ , where  $n$  is a natural number and are all fully invariant; in particular  $D$  is strongly socle-regular.*

(ii) *If  $C$  is a characteristic subgroup of the reduced group  $G$  and  $D$  is a divisible group, then  $D \oplus C$  is characteristic in  $A = D \oplus G$ . If  $C$  is not fully invariant in  $G$  then  $D \oplus C$  is not fully invariant in  $A$ .*

*Proof.* The first statement is well known; see, for example Exercise 68 in [9]. The first statement of part (ii) follows from the fact that every endomorphism of  $A$  must have a representation as a lower triangular matrix which ensures that any automorphism must have diagonal entries which are themselves automorphisms. For the final statement in part (ii) observe that if  $C$  is not fully invariant then there is an endomorphism  $\phi$  of  $G$  with  $C\phi \not\subseteq C$ . This mapping  $\phi$  extends to an endomorphism  $\psi$  of  $A$  by mapping  $D$  to 0 and clearly  $D \oplus C$  is mapped outside of  $D \oplus C$  by  $\psi$ . ■

The difficulty arising from the possibility that the prime  $p = 2$  is highlighted in our next result; we do not know if the restriction to odd primes is necessary.

**Theorem 1.2** *Let  $D$  be a divisible  $p$ -group and  $G$  a reduced  $p$ -group. If the group  $A = D \oplus G$  is strongly socle-regular, then both  $D, G$  are strongly socle-regular. Conversely if  $p \neq 2$  and  $G$  is strongly socle-regular, then  $A = D \oplus G$  is also strongly socle-regular.*

*Proof.* It follows from Lemma 1.1 above that  $D$  is strongly socle-regular. While if  $C$  is characteristic in  $G$ , it again follows from Lemma 1.1 that  $D \oplus C$  is characteristic in  $A$ , implying that  $(D \oplus C)[p] = (p^\alpha A)[p]$  for some  $\alpha$ . Thus  $C[p] = (p^\alpha G)[p]$  as required.

For the converse argument note since  $p \neq 2$ , 2 is a unit in  $\text{End}(D)$  and so if  $C$  is a characteristic subgroup of  $A$ , it follows from Proposition 1.3 below that  $C = (C \cap D) \oplus (C \cap G)$  and that  $C \cap D, C \cap G$  are characteristic in  $D, G$  respectively. Thus  $(C \cap D)[p] = D[p]$  by the previous lemma and  $(C \cap G)[p] = (p^\alpha G)[p]$  since  $G$  is, by hypothesis, strongly socle-regular. Thus  $C[p] = D[p] \oplus (p^\alpha G)[p] = (p^\alpha A)[p]$  as required. ■

**Proposition 1.3** *Suppose  $G = A \oplus B$  and 2 is a unit in  $\text{End}(A)$ . Then if  $C$  is a characteristic subgroup of  $G$ ,  $C = (C \cap A) \oplus (C \cap B)$  and  $C \cap A$ ,  $C \cap B$  are characteristic in  $A$ ,  $B$  respectively.*

*Proof.* Clearly  $(C \cap A) \oplus (C \cap B) \leq C$  always. Now suppose that 2 is a unit in  $\text{End}(A)$  and let  $x = (a, b)$  be an arbitrary element of  $C$ . Now the diagonal matrix  $\Delta = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$  represents an automorphism of  $G$  since 2 is a unit and so  $x\Delta = (2a, b) \in C$ . This implies that  $(a, 0) \in C$  and hence that  $(0, b) \in C$ . Thus  $x = (a, 0) + (0, b) \in (C \cap A) \oplus (C \cap B)$ . Since  $x$  was arbitrary, this shows that  $C = (C \cap A) \oplus (C \cap B)$ , as required.

If  $\phi$  is an arbitrary automorphism of  $A$ , then the matrix  $\Phi = \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$  is an automorphism of  $G$  and so  $C\Phi \leq C$ . This implies immediately that  $(C \cap A)\phi \leq (C \cap A)$ , so that  $(C \cap A)$  is characteristic in  $A$ . The proof for  $C \cap B$  is similar. ■

**Remark:** Proposition 1.3 fails if 2 is not a unit in  $\text{End}(A)$ . For example, if  $A = \langle a \rangle$ ,  $B = \langle b \rangle$  cyclic of orders 2, 8 respectively, then it is well known that  $G = A \oplus B$  has a characteristic subgroup  $K = \{0, a \pm 2b, 4b\}$ . However  $K \cap A = \{0\}$ ,  $K \cap B = \langle 4b \rangle$ , so that  $(K \cap A) \oplus (K \cap B) < K$ .

*So we shall assume in the sequel that our groups are always reduced Abelian  $p$ -groups, for some prime  $p$ , but we will not assume that  $p$  is odd.*

## 2 Elementary Properties

The following rather *ad hoc* notation was introduced in [4] but it shall also be useful here: Suppose that  $H$  is an arbitrary subgroup of the group  $G$ . Set  $\alpha = \min\{h_G(y) : y \in H[p]\}$  and write  $\alpha = \min(H[p])$ ; clearly  $H[p] \leq (p^\alpha G)[p]$ .

If  $K$  is also a subgroup of  $G$  containing  $H$ , then of course there may be two different values of  $\min$  associated to  $H$ , depending on where the heights of elements are calculated. We will distinguish these if necessary by writing  $\min^G(H[p])$  and  $\min^K(H[p])$ ; note that if  $K$  is an isotype subgroup of  $G$  then the respective values of  $\min$  coincide. However if  $K$  is not an isotype subgroup of  $G$  then all that one can say is that  $\min^K(H[p]) \leq \min^G(H[p])$ . Our first result collects some elementary facts about the function  $\min$ .

**Proposition 2.1** (i) *If  $F$  is a subgroup of the group  $G$  and  $(p^n G)[p] \leq F[p]$  for some integer  $n$ , then  $\min(F[p])$  is finite.*

(ii) *If  $F$  is a characteristic subgroup of the group  $G$  and  $\min(F[p]) = n$ , a finite integer, then  $F[p] = (p^n G)[p]$ .*

*Proof.* (i) Suppose that  $\alpha = \min(F[p])$ , so that  $\alpha \leq \min\{h_G(x) : x \in (p^n G)[p]\}$ . Now if  $\alpha \geq \omega$ , then  $(p^n G)[p] \leq p^\omega G = p^\omega(p^n G)$ , so that writing  $X = p^n G$ , one has  $X[p] \leq p^\omega X$ . Then either  $p^n G = 0$  or  $X$  is non-zero and divisible; the latter is contrary to the assumption that  $G$  is reduced. Hence, in either case,  $\min(F[p])$  is finite as required.

(ii) As observed above, one inclusion holds always. Conversely, suppose that  $x \in F[p]$  and  $h_G(x) = n$ . Then  $x = p^n y$  and the subgroup generated by  $y$  is a direct summand of  $G$  - see e.g. Corollary 27.2 in [6]. Thus  $G = \langle y \rangle \oplus G_1$  for some subgroup  $G_1$ . Now if  $0 \neq z$  is an arbitrary element of  $(p^n G)[p] \setminus (p^{n+1} G)[p]$ , then  $z = p^n w$  for some  $w \in G$  of height zero and so  $G = \langle w \rangle \oplus G_2$ ; note

that  $G_1, G_2$  are isomorphic since  $\langle y \rangle \cong \langle w \rangle$ , both being cyclic of order  $p^{n+1}$ , and finite cyclic groups have the cancellation property – see, for example, [12]. We may define a homomorphism  $\phi : G \rightarrow G$  by sending  $y \mapsto w$  and mapping  $G_1$  to  $G_2$  via the isomorphism previously noted; note that  $x\phi = z$  and that  $\phi$  is an automorphism. Since  $F[p]$  is characteristic in  $G$ , it follows that  $z \in F[p]$  and so  $(p^n G)[p] \setminus (p^{n+1} G)[p] \subseteq F[p]$ . However if  $v \in (p^{n+1} G)[p]$ , then  $z + v$  has height exactly  $n$  and order  $p$ , so that  $z + v \in F[p]$  by the previous argument. It follows immediately that  $v \in F[p]$  and so  $(p^n G)[p] \subseteq F[p]$  as required. ■

**Corollary 2.2** *If  $G$  is a separable group, then  $G$  is strongly socle-regular.*

*Proof.* This is immediate since the hypothesis of separability implies that for any characteristic subgroup  $F$  of  $G$ ,  $\min(F[p])$  is finite. ■

It was shown in [4] that fully transitive groups were socle-regular, but our next result shows that such groups need not be strongly socle-regular.

**Theorem 2.3** *The class of strongly socle-regular groups is properly contained in the class of socle-regular groups; in particular, there exists a fully transitive group which is not strongly socle-regular.*

*Proof.* Our result is based on an example constructed by Corner in [3]. Let  $K$  denote the Galois field of order  $p$  and let  $R$  be the  $K$ -algebra freely generated by non-commuting indeterminates  $a_0, a_1, \dots$ . Then, using the realization theorem from [2], one may construct a group  $G$  such that  $p^\omega G$  is an elementary group of infinite rank,  $H$  say, such that  $\text{End}(G)$  acts on  $H$  as  $R$  and  $\text{Aut}(G)$  as the group of units of  $R$ ; note that this latter group of units is precisely the set of non-zero elements of  $K$ . Full details of the construction may be found in Section 3 of [3]; in particular the group  $G$  is fully transitive and hence is socle-regular by Theorem 0.3 in [4]. However, it is immediate that every subgroup of  $H$  is characteristic in  $G$  and hence  $G$  cannot be strongly socle-regular. ■

On the other hand, it is rather easy to show that transitive groups are strongly socle-regular, whence socle-regular, thus answering in the negative Question (1) of [4]. We note that this result, unlike Kaplansky’s Theorem 26 in [9], does not require the prime  $p$  to be odd.

**Theorem 2.4** *If  $G$  is a transitive group, then  $G$  is strongly socle-regular. In particular, totally projective groups are strongly socle-regular.*

*Proof.* Suppose  $G$  is a transitive group and let  $C$  be any characteristic subgroup of  $G$ . If  $\min(C[p]) = \alpha$ , then it is clear that  $C[p] \leq (p^\alpha G)[p]$ . We show that the reverse inequality holds. Let  $x$  be an element of  $C[p]$  of height exactly  $\alpha$  and let  $y$  be an arbitrary element of  $(p^\alpha G)[p]$ ; note that the Ulm sequence of  $y$  is  $U_G(y) = (\beta, \infty, \dots)$ , for some  $\beta \geq \alpha$ . If  $\beta = \alpha$ , then since  $G$  is transitive, there is an automorphism  $\theta$  of  $G$  such that  $y = x\theta$ . Since  $C[p]$  is characteristic in  $G$ ,  $y \in C[p]$ . If, however,  $\beta > \alpha$ , then  $h_G(x+y) = h_G(x)$  and it follows that  $U_G(x+y) = U_G(x)$ . Again by transitivity of  $G$ , there is an automorphism  $\phi$  of  $G$  with  $x\phi = x+y$  and so  $y = x\phi - x \in C[p]$ , the last claim coming from the fact that  $C[p]$  is characteristic in  $G$ . Thus we have established that  $(p^\alpha G)[p] \subseteq C[p]$ , as required. The final comment comes from a well-known property of the class of totally projective groups – see e.g. [8]. ■

One knows that, in general, characteristic subgroups need not be fully invariant but there is a rather elementary way of ensuring that they always are.

**Proposition 2.5** *If  $G$  is a group with the property that its automorphism group generates (additively) its endomorphism ring, then every characteristic subgroup of  $G$  is fully invariant. In particular if  $G$  is of the form  $G = H^{(\kappa)}$  for some  $H$  and some cardinal  $\kappa > 1$ , then every characteristic subgroup of  $G$  is fully invariant and hence in this case,  $G$  is strongly socle-regular if, and only if, it is socle-regular.*

*Proof.* The first statement is immediate. If  $G$  has the form  $G = H^{(\kappa)}$  with  $\kappa = n$ , a finite integer  $> 1$ , then  $\text{End}(G)$  may be identified with the ring of  $n \times n$  matrices over  $\text{End}(H)$ . Such a matrix ring has the property that every element is the sum of at most three units - this is essentially due to Kaplansky but appeared in [7] - and so we have the desired result. However if  $\kappa$  is infinite then  $G \cong G \oplus G$  and the argument of the preceding line again gives that every characteristic subgroup is fully invariant. The final claim on the equivalence of socle-regularity and strong socle-regularity is then immediate. ■

We remark that the converse of the above proposition fails: let  $G$  be a separable  $p$ -group ( $p \neq 2$ ) such that  $\text{End}(G) = A \oplus \text{End}_s(G)$  where  $A$  is the completion of the polynomial ring in one variable over the  $p$ -adic integers,  $J_p$ , and  $\text{End}_s(G)$  is the ideal of small endomorphisms. Then it is well known and easy to show that the automorphism group  $\text{Aut}(G)$  does not generate the full endomorphism ring. However, since  $G$  is separable, it is transitive and then Kaplansky's result [9, Theorem 26] implies that all characteristic subgroups of  $G$  are fully invariant.

The property of being strongly socle-regular is inherited by certain subgroups and may be obtained from a subgroup and its quotient in suitable circumstances.

**Proposition 2.6** (i) *If  $A$  is strongly socle-regular, then so also is  $p^\alpha A$  for all ordinals  $\alpha$ .*

(ii) *If  $A$  is strongly socle-regular and  $L$  is a characteristic subgroup of  $A$  such that  $p^\omega L = p^\omega A$ , then  $L$  is strongly socle-regular. In particular, large subgroups of strongly socle-regular groups are again strongly socle-regular.*

(iii)  *$A$  is strongly socle-regular if, and only if,  $p^n A$  is strongly socle-regular for a positive integer  $n$ . In particular, if  $G$  is a subgroup of  $A$  and either  $A/G$  is finite or  $A = G \oplus B$ , where  $B$  is bounded, then  $A$  is strongly socle-regular if, and only if,  $G$  is strongly socle-regular.*

(iv) *If  $p^\omega A$  is strongly socle-regular and  $A/p^\omega A$  is a direct sum of cyclic groups, then  $A$  is strongly socle-regular.*

(v) *Suppose that  $\alpha$  is an ordinal strictly less than the  $\omega^2$ . If  $p^\alpha A$  is strongly socle-regular and  $A/p^\alpha A$  is totally projective, then  $A$  is strongly socle-regular.*

*Proof.* Part (i) follows immediately from the fact that a characteristic subgroup of a characteristic subgroup is again a characteristic subgroup. To establish part (ii), let  $C$  be a characteristic subgroup of  $L$ . It follows immediately that  $C$  is characteristic in  $A$  and hence, as  $A$  is strongly socle-regular,  $C[p] = (p^\alpha A)[p]$  for some ordinal  $\alpha$ . If  $\alpha \geq \omega$ , then an easy induction gives that  $p^\alpha L = p^\alpha A$  and so  $C[p] = (p^\alpha L)[p]$  as required. However, if  $C[p] = (p^n A)[p]$  for some integer  $n$ , then  $C[p] \geq (p^n L)[p]$ , and then it follows from Proposition 2.1 (i) that  $\min^L(C[p])$  is finite. An application of part (ii) of the same Proposition then yields that  $C[p] = (p^m L)[p]$  for some integer  $m$ . The claim in relation to

large subgroups follows immediately from the well-known fact that if  $L$  is a large subgroup of  $A$ , then  $p^\omega L = p^\omega A$  – see e.g. §46.1 in [11].

To establish part (iii), note that if  $C$  is a characteristic subgroup of  $A$  and  $C[p] \not\leq p^n A$ , then  $\min(C[p])$  is finite,  $k$  say and then it follows from Proposition 2.1 (ii) that  $C[p] = (p^k A)[p]$ . If  $C[p] \leq p^n A$ , then  $C[p]$  is actually characteristic in  $p^n A$  and so  $C[p] = (p^\alpha(p^n A))[p] = (p^{n+\alpha} A)[p]$  for some  $\alpha$ ; this follows immediately from the well-known consequence of Zippin's theorem that every automorphism of  $p^n A$  is induced from an automorphism of  $A$ ; for an alternative argument using basic subgroups see [6, Proposition 113.3]. To deduce the particular cases mentioned, note that in either situation there exists an integer  $n$  such that  $p^n A = p^n G$ .

The fourth part (iv) follows from Hill's work [8] on totally projective groups and is essentially identical to the proof of Theorem 1.7 in [4].

The proof of (v) is by transfinite induction, the initial cases following from (iii) and (iv) above. So suppose that we have established the result for all ordinals less than some  $\alpha$ . We now establish the result for  $\alpha$ . There are two possibilities: either  $\alpha$  is a successor or  $\alpha$  is a limit ordinal of the form  $\omega \cdot n$ . In the first case  $\alpha = \beta + 1$  for some  $\beta$ . Let  $X = p^\beta A$  and note that  $pX = p^\alpha A$  is strongly socle-regular. Hence by (ii) above,  $X = p^\beta A$  is strongly socle-regular. Moreover, as  $\beta < \alpha$ , it is easy to show that  $A/p^\beta A$  is totally projective. Hence it follows from our inductive hypothesis that  $A$  is strongly socle-regular. In the second case  $\alpha = \beta + \omega$  for some  $\beta$ . Set  $X = p^\beta A$  so that  $p^\omega X = p^\alpha A$  is strongly socle-regular. Now  $X/p^\omega X \cong p^\beta A/p^\alpha A$  and this is easily seen to be totally projective; hence it is a direct sum of cyclic groups. It now follows from (iii) above that  $X = p^\beta A$  is strongly socle-regular. However, as noted previously,  $A/p^\beta A$  is totally projective and so it follows from the inductive hypothesis that  $A$  is strongly socle-regular. ■

Our final result in this section shows, *inter alia*, that some condition on the quotient  $A/p^\omega A$  is necessary in Proposition 2.6 (iii); the proposition is based on Theorem 1.6 in [4].

**Proposition 2.7** (i) *A group  $G$  with  $p^\omega G$  cyclic, is strongly socle-regular.*

(ii) *There exists a group  $A$  such that  $p^\omega A$  is strongly socle-regular, but  $A$  is not; indeed such a group exists with  $p^\omega A \cong \mathbb{Z}(p) \oplus \mathbb{Z}(p)$ .*

(iii) *The direct sum of two strongly socle-regular groups need not be strongly socle-regular.*

*Proof.* If  $G$  is a group with  $p^\omega G$  cyclic and  $C$  is a characteristic subgroup of  $G$  then, either  $\min(C[p])$  is finite or  $C[p] \leq p^\omega G$ . In the first case  $C[p] = (p^n G)[p]$  for some finite  $n$  by Proposition 2.1 (ii), while in the second case  $C[p] = (p^\omega G)[p]$ .

Let  $A = G \oplus H$ , where  $p^\omega G \cong p^\omega H \cong \mathbb{Z}(p)$ ,  $G/p^\omega G$  is a direct sum of cyclic groups and  $H/p^\omega H$  is torsion-complete; note that  $A/p^\omega A$  is not a direct sum of cyclic groups, but that  $p^\omega A$ , being finite, is strongly socle-regular. However, as shown in [4], the group  $A$  is not even socle-regular and hence, *a fortiori*, is not strongly socle-regular.

The same example of the group  $A$  above suffices for (iii): by part (i), each of  $G$  and  $H$  is strongly socle-regular. ■

### 3 The Class of Strongly Socle-Regular Groups

In this section we investigate some of the elementary properties of the class of strongly socle-regular groups and obtain a characterization of strongly socle-regular groups in terms of socle-regular groups.

We begin with the elementary:

**Proposition 3.1** *If  $G$  is strongly socle-regular, then so also is the direct sum  $A = G^{(\kappa)}$  for any cardinal  $\kappa$ .*

*Proof.* Observe firstly that if  $C$  is an arbitrary characteristic subgroup of  $A$ , then  $C$  is fully invariant in  $A$ ; this follows directly from Proposition 2.5 above. However, it follows from Theorem 1.4 in [4] that  $A$  is socle-regular and so  $C[p] = (p^\alpha A)[p]$  for some ordinal  $\alpha$ . Thus  $A$  is strongly socle-regular. ■

Given that separable groups are always strongly socle-regular, one would expect that the addition of a separable summand would have no effect on strong socle-regularity. Indeed this type of property and its converse hold for socle-regular groups – see Theorem 1.2 in [4]. However the best we can achieve is:

**Proposition 3.2** *Suppose that  $G$  is a strongly socle-regular group. Then if  $H$  is any separable group, the direct sum  $A = G \oplus H$  is strongly socle-regular.*

*Proof.* Let  $C$  be a characteristic subgroup of  $A$ . If  $\min(C[p])$  is finite then by Proposition 2.1 (ii)  $C[p] = (p^n A)[p]$  for some finite integer  $n$ . Otherwise  $C[p] \leq p^\omega A = p^\omega G$ , so that  $C[p] \leq G$ . Now let  $\alpha$  be an arbitrary automorphism of  $G$ . Then  $\alpha$  extends to an automorphism  $\theta$  of  $A$  by setting  $\theta = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ . Since  $C[p]$  is characteristic in  $A$ ,  $C[p]\theta \leq C[p]$ . Hence  $C[p]\alpha \leq C[p]$  and  $C[p]$  is characteristic in  $G$  also. Now the latter is strongly socle-regular, so  $C[p] = (p^\beta G)[p]$  for some ordinal  $\beta$ ; note that  $\beta \geq \omega$  since  $\min(C[p])$  is infinite. However if  $\beta$  is infinite, then  $p^\beta A = p^\beta G$  and hence  $C[p] = (p^\beta A)[p]$ . Thus  $A$  is strongly socle-regular as required. ■

The converse of this proposition is not easy to establish and it seems likely that it may fail but we have not been able to construct an explicit example. This is in marked contrast to the situation for socle-regularity, where a comparable result is easily obtained – see Theorem 1.2 in [4]. The difficulty here arises from the fact that  $G \oplus H$  may have automorphisms which, in the standard matrix representation, do not necessarily have as first entry an automorphism of  $G$ . The best we can achieve is the following – recall (see [1]) that a group  $G$  is said to be *of type A* if  $(\text{Aut } G) \upharpoonright p^\omega G$  is precisely the group of units of  $(\text{End } G) \upharpoonright p^\omega G$ :

**Proposition 3.3** *Suppose that  $H$  is any separable group and  $A = G \oplus H$  is strongly socle-regular. Then  $G$  is strongly socle-regular if any of the following holds:*

- (i)  $\text{End } G$  is additively generated by  $\text{Aut } G$ ;
- (ii)  $G/p^\omega G$  is a direct sum of cyclic groups;
- (iii)  $G$  is of type A.

*Proof.* Since  $A$  is strongly socle-regular, it is socle-regular and case (i) then follows immediately from [4, Theorem 1.2] and Proposition 2.5. For case (ii), note that the separability of  $H$  implies that  $p^\omega A = p^\omega G$  and hence, by Proposition 2.6 (i),  $p^\omega G$  is strongly socle-regular. An application of part (iv) of the same proposition now yields the result.

Finally we consider the situation where  $G$  is assumed to be of type A. Let  $C$  be a characteristic subgroup of  $G$ . If  $C[p] \not\leq p^\omega G$ , then an application of Proposition 2.1 (ii) yields that  $C[p] = (p^n G)[p]$  for

some finite  $n$ . So suppose that  $C[p] \leq p^\omega G$ . We claim that  $C[p] \oplus 0$  is characteristic in  $A$ . For suppose that the matrix  $\Delta = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$  is an automorphism of  $A$ . Then, writing bars to denote restrictions to the first Ulm subgroup of the domain of a map, we have that  $\bar{\Delta} = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\delta} & \bar{\beta} \end{pmatrix}$  is still invertible. Note, however, that as  $H$  is separable, the entry  $\bar{\gamma}$  must be 0, so that  $\bar{\Delta}$  is a lower triangular matrix. This, of course, forces  $\bar{\alpha}$  to be a unit of  $(\text{End } G) \upharpoonright p^\omega G$ . Since  $G$  is of type A, we conclude that  $\bar{\alpha} \in (\text{Aut } G) \upharpoonright p^\omega G$  and hence there is an automorphism  $\phi$  of  $G$  such that  $\bar{\alpha} = \bar{\phi}$ . But now  $(C[p] \oplus 0)\Delta = C[p]\alpha \oplus 0$  since  $C[p] \leq p^\omega G$ . Hence  $C[p]\alpha = C[p]\bar{\alpha} = C[p]\bar{\phi} = C[p]\phi \leq C[p]$ , since  $C[p]$  is characteristic and  $\phi$  is an automorphism. Thus  $C[p] \oplus 0$  is characteristic in  $A$ , as claimed. Since  $A$  is strongly socle-regular,  $C[p] = (C[p] \oplus 0) = (p^\tau A)[p]$  for some ordinal  $\tau$ ; clearly the choice of  $C[p]$  ensures that  $\tau \geq \omega$ . Hence  $C[p] = (p^\tau A)[p] = (p^\tau G)[p]$  and  $G$  is strongly socle-regular as required. ■

**Corollary 3.4** *Let  $A = G + H$ , where  $H$  is separable. If there exists an integer  $t$  such that  $p^t G \cap p^t H = 0$  (in particular if  $G \cap H$  is finite), then  $G$  strongly socle-regular implies that  $A$  is also strongly socle-regular.*

*Proof.* Clearly  $p^t A = p^t G + p^t H$ . However, the hypothesis that  $p^t G \cap p^t H = 0$ , implies that the previous sum is direct:  $p^t A = p^t G \oplus p^t H$ . Now in view of Proposition 2.6 (i),  $p^t G$  is strongly socle-regular since  $G$  is, and one also has that  $p^t H$  is separable. Applying Proposition 3.2, one sees that  $p^t A$  is strongly socle-regular and hence it follows from Proposition 2.6 (iii) that  $A$  is strongly socle-regular. ■

It was observed in [4] that the converse of Proposition 3.1 above, holds for socle-regular groups. However, this is not the case for strongly socle-regular groups.

**Example 3.5** *There exists a group  $G$  with the property that  $G$  is not strongly socle-regular, but  $G \oplus G$  is strongly socle-regular.*

*Proof.* Let  $G$  be the non-transitive, fully transitive group discussed in Theorem 2.3 above; as noted  $G$  is not strongly socle-regular. However the group  $G \oplus G$  is transitive since  $G$  is fully transitive - see [5, Corollary 3] - and thus it follows from Theorem 2.4 that  $G \oplus G$  is strongly socle-regular. ■

We are now in a position to establish the promised characterization of strongly socle-regular groups; there is a clear family resemblance between this result and the characterization of fully transitive groups given by Files and the second author in [5].

**Theorem 3.6** *A group  $G$  is socle-regular if, and only if, the direct sum  $G \oplus G$  is strongly socle-regular.*

*Proof.* If  $G \oplus G$  is strongly socle-regular, then it follows immediately from Theorem 1.4 in [4] that  $G$  is socle-regular.

Conversely suppose that  $G$  is socle-regular and let  $C$  be an arbitrary characteristic subgroup of  $G \oplus G$ . It follows from Proposition 2.5 that  $C$  is actually fully invariant in  $G \oplus G$ . Thus  $C = (C \cap G_1) \oplus (C \cap G_2)$  where each  $C \cap G_i$  is fully invariant in  $G_i \cong G$ ,  $i = 1, 2$ . Since each  $G_i$  is socle-regular, there exist ordinals  $\alpha_1, \alpha_2$  such that

$$C[p] = (C \cap G_1)[p] \oplus (C \cap G_2)[p] = (p^{\alpha_1} G_1)[p] \oplus (p^{\alpha_2} G_2)[p].$$

Now  $C[p]$  is itself a fully invariant subgroup of  $G_1 \oplus G_2$ . We claim that  $\alpha_1 = \alpha_2$ ; if not, then without loss we may assume that  $\alpha_1 < \alpha_2$ . Now let  $f$  denote a fixed isomorphism from  $G_1$  onto  $G_2$ . Then the matrix  $\Phi = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$  represents an endomorphism of  $G_1 \oplus G_2$  and hence the image of  $C[p]$  under  $\Phi$  should be contained in  $C[p]$ . However a simple calculation shows that this image is actually  $0 \oplus (p^{\alpha_1}G_2)[p] \not\subseteq C[p]$  – contradiction. Thus  $\alpha_1 = \alpha_2 = \alpha$ , say and  $C[p] = (p^\alpha(G \oplus G))[p]$ . Since  $C$  was an arbitrary characteristic subgroup of  $G \oplus G$ , we have the desired result. ■

In fact one can easily extend the above characterization to obtain:

**Corollary 3.7** *For a group  $G$  the following are equivalent:*

- (i) *for all cardinals  $\lambda$ ,  $G^{(\lambda)}$  is socle-regular;*
- (ii) *for some cardinal  $\lambda > 0$ ,  $G^{(\lambda)}$  is socle-regular;*
- (iii) *for all cardinals  $\lambda > 1$ ,  $G^{(\lambda)}$  is strongly socle-regular;*
- (iv) *for some cardinal  $\lambda > 1$ ,  $G^{(\lambda)}$  is strongly socle-regular.*

*Proof.* The equivalence of (i) and (ii) follows from Theorem 1.4 in [4], both statements being equivalent to the statement that  $G$  is socle-regular. Clearly (iii) implies (iv), while (iv) implies (ii) since strongly socle-regular groups are socle-regular. Thus it remains only to establish that (ii) implies (iii). The argument is essentially identical to that in Theorem 3.6 above: since  $G$  is socle-regular, so also is  $G^{(\lambda)}$  for any  $\lambda > 1$ . However, as  $\lambda > 1$ , it follows from Proposition 2.5 that the automorphism group of  $G^{(\lambda)}$  generates the full endomorphism ring and hence, every characteristic subgroup  $C$  of  $G^{(\lambda)}$  is fully invariant. The socle-regularity of  $G^{(\lambda)}$  now gives that  $C[p] = (p^\alpha(G^{(\lambda)}))[p]$  for some ordinal  $\alpha$ , as required. ■

Despite the interconnections between strong socle-regularity and transitivity, we can exhibit a strongly socle-regular group which is neither transitive nor fully transitive.

**Corollary 3.8** *There exists a strongly socle-regular group which is neither transitive nor fully transitive.*

*Proof.* Let  $G$  be the transitive 2-group which is not fully transitive, constructed by Corner in Section 4 of [3]. Let  $A = G \oplus G$  and note that  $A$  cannot be fully transitive since its direct summand  $G$  is not fully transitive. Moreover,  $A$  cannot be transitive since by [5, Corollary 3], this would force  $G$  to be fully transitive. However, it was shown in [4] that  $G$  is actually socle-regular and so by the theorem above,  $A$  is strongly socle-regular. ■

We finish off by raising a question, the solution of which would give an interesting insight into the structure of  $p$ -groups: If  $G$  is a socle-regular group and  $p^\omega G$  is finite, is  $G$  necessarily strongly socle-regular?

This question seems quite difficult and we note that an answer to the long-standing question of Corner on the existence of a non-transitive fully transitive group with finite Ulm subgroup – see [3] – would yield some insight. We note that Paras and Strüngmann [10] have answered Corner’s problem in the negative for groups of type A.

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