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Determination of the phase of a transfer function: Theory versus Practice

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Abstract – **In control systems, a way to evaluate the stability of a system is to determine the magnitude and phase of the open loop system transfer function in the frequency domain. But due to the nature of the phase expression, practice and theory are not aligned. Some problems arise when the phase is determined, caused by the properties of the function tan^{-1}. This paper analyses these phase problems for a third order lag system and a third order lag with time delay system. It also presents modifications to obtain the analytically correct phase values from the computed values.**

Keywords – **Phase computation, Delay approximation**

I INTRODUCTION

In control systems, a way to evaluate the stability of a system is to determine the magnitude and phase of the open loop system transfer function in the frequency domain. From the frequency response of the system, a Bode plot may be obtained [1]. This plot graphs the magnitude of the transfer function in decibels and the phase in degrees, both versus frequency ω [2]. A general procedure to find the frequency response of any system, represented by its transform-domain transfer function g(s), is as follows [3]:

1- Substitute jω for s in the transfer function expression to obtain the corresponding frequency-response transfer function, g(jω).
2- Rationalise g(jω) to obtain a Cartesian form:
$$g(j\omega) = \mathrm{Re}(\omega) + j\,\mathrm{Im}(\omega)$$
Re(ω) and Im(ω) are the real and imaginary parts, respectively.
3- Compute the magnitude and the phase using:
$$|g| = \sqrt{\mathrm{Re}(\omega)^2 + \mathrm{Im}(\omega)^2}$$
In decibels: $|g| = 20\log\left(\sqrt{\mathrm{Re}(\omega)^2 + \mathrm{Im}(\omega)^2}\right)$ (1),
$$\phi = \tan^{-1}\left[\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right] \quad (2)$$

Some problems arise when the phase is calculated, caused by the properties of the function tan^{-1}.

This paper analyses these phase problems for two sample processes: a third order lag process and a third order lag with time delay process. It also presents eventual modifications to obtain the correct phase values from the calculated values.

II THIRD ORDER LAG PROCESS

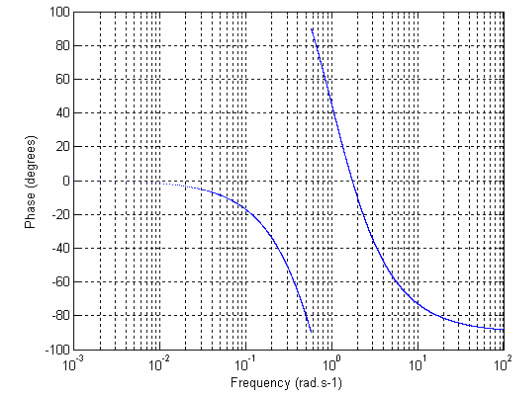
The transfer function of a sample third order lag process is expressed by:
$$g = \frac{1}{(s+1)^3} = \frac{1}{s^3 + 3s^2 + 3s + 1} \quad (3),$$
which becomes (4) in the frequency domain
$$g = \frac{1}{(j\omega+1)^3} = \frac{1}{-j\omega^3 - 3\omega^2 + 3j\omega + 1} \quad (4)$$

The correct expression of the phase for a third order lag is given by:
$$\phi_c = -3\tan^{-1}[\omega] \quad (5)$$
This uses the result which states that the phase of a transfer function of the form $\frac{1}{(Ts+a)^n}$ ($\frac{1}{(Tj\omega+a)^n}$ in the frequency domain) is $-n\tan^{-1}\left[\frac{\omega T}{a}\right]$.

However, using (4), the transfer function phase is computed (for example, using a computer package such as MATLAB or MATHEMATICA) to be:
$$\phi = -\tan^{-1}\left[\frac{-\omega^3 + 3\omega}{-3\omega^2 + 1}\right] \quad (6)$$
This is due to the fact that computer packages use the right hand side form of equation (4) to calculate the Cartesian form, as they do not have subroutines to evaluate the correct phase using the left hand side form of equation (4).

It can be noticed that there is a discontinuity in equation (6) at $-3\omega^2+1=0$ i.e. when $\omega=\pm\sqrt{\frac{1}{3}}rad.s^{-1}$ as the function $\tan^{-1}$ is not defined when its denominator equals zero. Figure 1 represents a graph of phase versus frequency (Equation (6)).

Figure 1: Phase versus frequency for Equation (6)



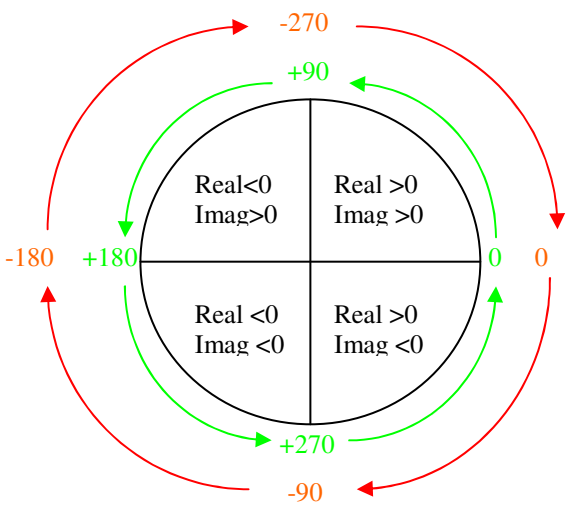
It is known that this graph does not correspond to the actual plot. To analyse the problem, first phase values are calculated at small frequencies:

At $\omega=0$, $\phi=-\tan^{-1}[0]=0°$

At $\omega=1$, $\phi=-\tan^{-1}\left[\frac{-1+3}{-3+1}\right]=-\tan^{-1}\left[\frac{2}{-2}\right]=45°$

The last phase is wrong, as it is known that the phase should be in the bottom right quadrant of the trigonometric circle due to the fact that the numerator is positive and the denominator is negative. Figure 2 shows the variation of signs for the numerator (Im(ω)) and the denominator (Re(ω)) around the trigonometric circle.

Figure 2: Trigonometric circle



As explained above, the computation of the phase depends on the sign of the numerator and the denominator of the transfer function. The following rule, labelled Rule 1, computes the correct phase, depending on the sign of the real and imaginary parts.

- *If Im(ω) > 0 and Re(ω) > 0 or if Im(ω) < 0 and Re(ω) > 0*

$$\phi=\tan^{-1}\left[\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right]$$

- *If Im(ω) <0 and Re(ω)< 0 or if Im(ω) > 0 and Re(ω) < 0*

$$\phi=-\left[\pi+\tan^{-1}\left[\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right]\right]$$

A comparison between phase values using equation (5) and expression (6), modified by the implementation of Rule 1, shows that the results obtained are identical. The determination of the limiting phase also shows that the calculated phase tends to the same limits as the correct phase.

$$\lim_{\omega\to+\infty}\phi_c=\lim_{\omega\to+\infty}-3\tan^{-1}[\omega]=-3*90=-270°$$

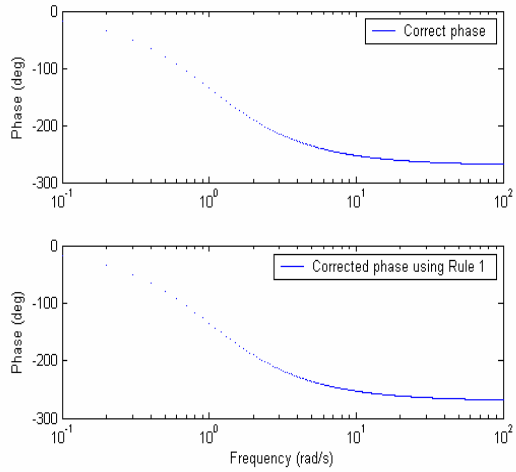
$$\lim_{\omega\to+\infty}\phi=\lim_{\omega\to+\infty}-\tan^{-1}\left[\frac{-\omega^3+3\omega}{-3\omega^2+1}\right]=\lim_{\omega\to+\infty}-\tan^{-1}\left[\frac{-\omega^3}{-3\omega^2}\right]$$

Using Rule 1 for the computation of the phase detailed previously:

$$\lim_{\omega\to+\infty}\phi=\lim_{\omega\to+\infty}-\left[180+\tan^{-1}[\omega]\right]=-270°$$

Using MATLAB, a program has been developed to integrate Rule 1 into the computation of the phase. Figure 3 shows the correct phase (equation (5)) and the corrected phase (equation (6) with Rule 1).

Figure 3: Correct phase, corrected phase using Rule 1



As can be noticed, equation (6) using Rule 1 gives the same phase as the correct phase (equation (5)). So, it is possible to obtain the correct phase by developing a specific program with computer packages such as MATLAB or MATHEMATICA.

III THIRD ORDER LAG WITH TIME DELAY PROCESS

Processes with time delay are common in control systems but may also be found in teleoperation,

networking and communication. For example, physical processes such as thermal processes, chemical processes, systems having transportation or diffusion or long transmission lines in pneumatic systems contain time delay. The presence of a time delay means that if a stimulus is applied to the system, the response to that stimulus is not seen directly but rather after a certain dead time has elapsed.

To investigate the consequences of the introduction of a dead time in the computation of the phase, the transfer function (equation (3)) is multiplied by the delay term:

$$G_{OL} = \frac{e^{-j\omega}}{(j\omega+1)^3} = \frac{e^{-j\omega}}{-j\omega^3 - 3\omega^2 + 3j\omega + 1} \quad (7)$$

The correct value of the phase for a third order lag with a delay is given by:

$$\phi_c = -\omega - 3\tan^{-1}[\omega] \quad (8)$$

A delay is considered to have a phase of -ωτ where τ is the value of the delay (here τ equals 1).

In practice, a computer package such as MATLAB or MATHEMATICA approximates the delay. So, the correct phase expression (equation (8)) cannot be obtained exactly.

The following sections of the paper compare the correct and computed phases for an approximated delay and then the consequences of the use of these approximations when applied to a third order lag with time delay process. This is done for two types of approximations: the Euler form and the Padé approximation.

a) Euler form

The delay may be exactly represented in Euler form:

$$e^{-j\omega\tau} = \cos(\tau\omega) - j\sin(\omega\tau) \quad (9)$$

The correct phase expression, based on the Euler form is:

$$\phi_c = \tan^{-1}\left[\frac{-\sin(\omega\tau)}{\cos(\omega\tau)}\right] = -\tan^{-1}[\tan(\omega\tau)] = -\omega\tau \quad (10)$$

The actual phase, as calculated by computer packages, is:

$$\phi = \tan^{-1}\left[\frac{-\sin(\omega\tau)}{\cos(\omega\tau)}\right] \quad (11)$$

At some frequencies, the function $\tan^{-1}$ is undefined:

$$\cos(\omega\tau) = 0 \Rightarrow \omega\tau = \frac{\pi}{2} \Rightarrow \omega = \frac{\pi}{2\tau}$$

Discontinuities in the phase occur at $\omega = \frac{n\pi}{2\tau}$ (12), where n is an odd integer. To illustrate the discontinuity phenomenon, the phase versus frequency characteristic of a delay of 1 second is plotted, Figure 4. The plot shows that the phase response of a "pure" delay represented by the Euler form is oscillatory between 90° and -90° and contains discontinuities. Table 1 also shows relevant data.

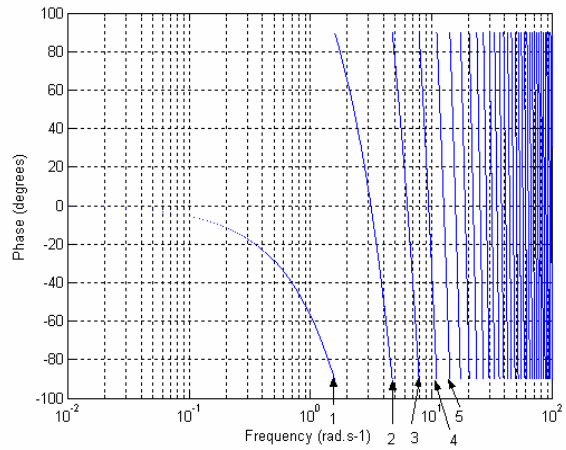


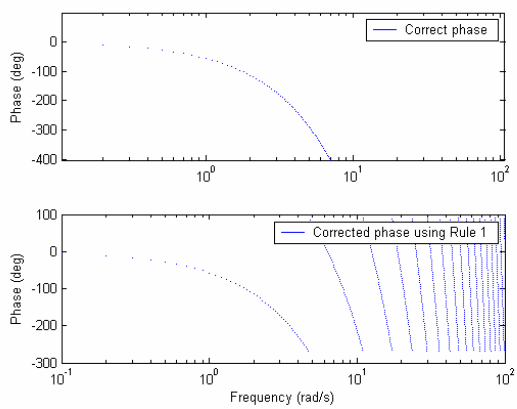
Figure 4: Plot of the Euler form

Points	Discontinuities at: (rad.s^{-1})
1	$1.66 \approx \frac{\pi}{2}$
2	$4.75 \approx \frac{3\pi}{2}$
3	$7.8 \approx \frac{5\pi}{2}$
4	$11.25 \approx \frac{7\pi}{2}$
5	$15 \approx \frac{9\pi}{2}$

Table 1: Frequencies at which discontinuities occur

When Figure 4 is studied closely, it can be noticed that the phase shifts occur at the discontinuities. This shift is 180°. So, Rule 1 has to be used to obtain a correct phase. Rule 1 permits only the completion of one rotation around the trigonometric circle. When the rotation is completed, the phase has to be shifted by -180° again at the second discontinuity to obtain the correct phase. Figure 5 shows the correct phase and the corrected phase using Rule 1 with the Euler form.

Figure 5: Correct phase, corrected phase, using Rule 1 with Euler form



As it can be noticed, the phase shift is 360°. It can also be noticed that Rule 1 corrects only the phase for the first discontinuity. So, Rule 1 has to be

modified. By using a modified version of Rule 1, which shifts the phase by -360° for the second rotation around the trigonometric circle, the correct phase may be recovered. However, these modifications are only valid for the second rotation; another phase shift has to be done for the third rotation and so on. This represents a programming difficulty, as the discontinuities have to be determined. A possibility is to consider the following assumption:

Assumption 1:
As processes are generally low pass in nature, it may be assumed that phases calculated at higher frequencies will be more negative than phases calculated at lower frequencies. Phase shifts observed represent discontinuities in the phase expression. These discontinuities are caused by trigonometric properties of the function tan^{-1}.

The same conclusions may be drawn using Capstick and Fidler's [4] time delay approximation. In the frequency domain, they approximate the delay as:

$$e^{-j\omega\tau} = \frac{1}{\left(1+\frac{j\omega\tau}{n}\right)^n} \quad (13)$$

The theoretical phase is given below:

$$\phi_{theo} = -n\tan^{-1}\left[\frac{\omega\tau}{n}\right] \quad (14)$$

If the phases are calculated manually, some problems with the phase expression arise. It may be shown that with a second order approximation for the phase expression:

$$\phi = -\tan^{-1}\left[\frac{4\omega\tau}{4-\omega^2\tau^2}\right] \quad (15)$$

There are discontinuities at $\pm\frac{2}{\tau}$ rad.s^{-1}.

It can be assumed that the phase response will be shifted at the discontinuity, which can be easily corrected by shifting the phase by -360° every time the imaginary and real parts of the transfer function become positive again.

The consequences of these observations may be analysed with the example of the third order lag with delay transfer function expressed by equation (7) and the most commonly used Euler form (9). The transfer function in the frequency domain becomes:

$$g = \frac{(\cos\omega - j\sin\omega)}{(j\omega+1)^3} = \frac{\cos\omega - j\sin\omega}{-j\omega^3 - 3\omega^2 + 3j\omega + 1} \quad (16)$$

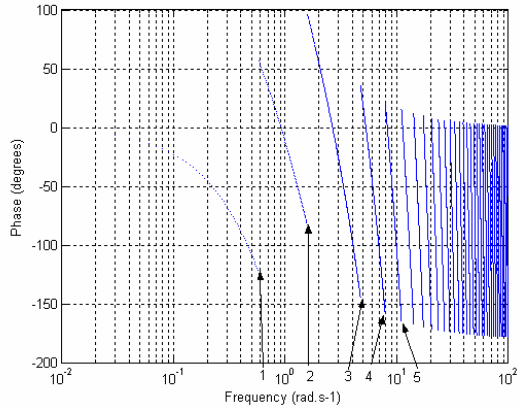
The phase may be calculated as follows:

$$\phi = \tan^{-1}\left[\frac{-\sin\omega}{\cos\omega}\right] - \tan^{-1}\left[\frac{-\omega^3+3\omega}{-3\omega^2+1}\right] \quad (17)$$

There are two types of discontinuities for equation (17). The first one corresponds to the value of ω when $\cos\omega = 0$. The frequencies at which discontinuities occur are given by equation (12). The second type of discontinuities corresponds to the (unique) ω value when $-3\omega^2+1=0$. Figure 6 represents the phase versus frequency characteristic of Equation (17).

Table 2 summarised the discontinuity values corresponding to the lower frequency values of Figure 6 (indicated by arrows in the figure).

Figure 6: Phase versus frequency characteristic of Equation (17)



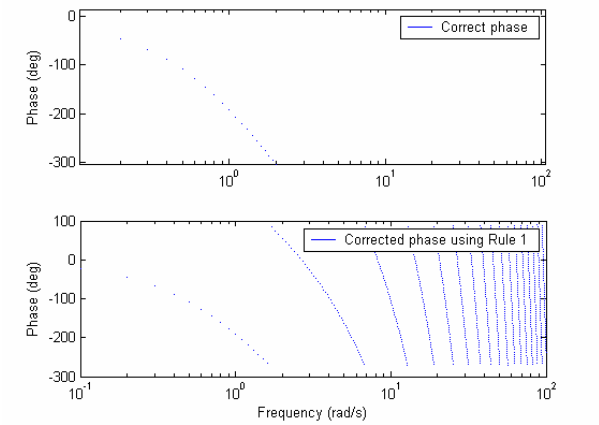
Points	Discontinuities at: (rad.s^{-1})
1	$0.577 \approx \sqrt{\frac{1}{3}}$
2	$1.66 \approx \frac{\pi}{2}$
3	$4.75 \approx \frac{3\pi}{2}$
4	$7.8 \approx \frac{5\pi}{2}$
5	$11 \approx \frac{7\pi}{2}$

Table 2: Values at the discontinuities

From Table 2, it can be noticed that the first discontinuity corresponds to the first discontinuity associated with the system with three lags (Figure 1) and that the rest of the discontinuities are associated with the delay term. It can be added that the phase shift is 180°. Figure 7 presents the corrected phase using Rule 1.

In conclusion, the correct phase for a third order lag with delay may be obtained using the phase expression evaluated by a computer package, applying Rule 1 and Assumption 1.

Figure 7: Correct phase and corrected phase using Rule 1 for third order lag with delay.



b) Padé approximations

Due to the irrational property of the Euler form in the time domain, another approximation, which rationalises the transfer function of the time delay has been developed to analyse systems with dead time. Any rational function has a numerator and a denominator, which can be a composite of real and imaginary parts. A rational function is the ratio of two polynomial functions:

$$y = \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_2 x^2 + b_1 x + b_0} \quad (18),$$

where n and m are non-negative integers that defines the degree of the numerator and denominator, respectively. Fitting rational function models to the delay may be referred as the Padé approximation process [3].

The following lists some of the advantages of rational function models [5]:

1. Rational function models have moderately simple form.
2. Rational functions have excellent extrapolatory powers.
3. Rational function models have excellent asymptotic properties. Rational function can be either finite or infinite for finite values or finite or infinite for infinite x values.
4. Rational function models can often be used to model complicated structures; with a fairly low degree in both the numerator and denominator. This means that fewer coefficients will be required compared to the polynomial model.

But rational function models also present the disadvantages listed below [5]:

1. The properties of the rational function family are not as well known to engineers and scientists as are those of the polynomial family. The literature on the rational function family is also more limited.
2. Unconstrained rational function fitting can at times, result in undesired nuisance asymptotes (vertically) due to roots in the denominator polynomial, called poles in control systems analysis. The range of x values affected by the function "blowing up" may be quite narrow, but such asymptotes, when they occur, are a nuisance for local interpolation in the neighborhood of the asymptote point. These asymptotes are easy to detect by a simple plot of the fitted function over the range of the data.

The Padé approximation has been developed to allow $e^{-\alpha s}$ to be represented by the ratio of low-order polynomials in the s-domain. The most general form is [3]:

$$e^{-s\alpha} = \frac{a_p(\alpha s)^p + a_{p-1}(\alpha s)^{p-1} + ... + a_0}{b_q(\alpha s)^q + b_{q-1}(\alpha s)^{q-1} + ... + b_0} \quad (19),$$

where the order of the polynomial, p and q, as well as the coefficients a_i, b_j are chosen so that the power series expansion of the polynomial ratio (19) matches the power series expansion of $e^{-s\alpha}$ as closely as possible. The Taylor series expansion for $e^{-s\alpha}$ is:

$$e^{-s\alpha} = 1 - \alpha s + \frac{(\alpha s)^2}{2!} - \frac{(\alpha s)^3}{3!} + ... \quad (20)$$

The coefficient set a_i, b_j which causes equations (19) and (20) to be equivalent for the maximum number of terms (for specific order p, q) is termed a "padé approximant".

As the parameters for the rational function are chosen to obtain a power series expansion as close as possible to the power series expansion of $e^{-s\tau}$, several Padé approximations exist.

The first approximation [6] is expressed by:

$$e^{-s\tau} \approx \frac{\left(1 - \frac{s\tau}{2}\right)^n}{\left(1 + \frac{s\tau}{2}\right)^n} \quad (21)$$

A second order approximation, using equation (21), in the frequency domain is given by:

$$e^{-j\omega\tau} \approx \frac{4 - 4j\omega\tau - \omega^2\tau^2}{4 + 4j\omega\tau - \omega^2\tau^2}$$

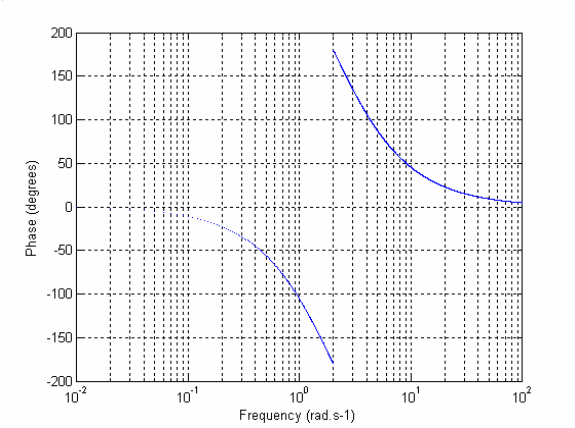
$$\phi = \tan^{-1}\left[\frac{-4\omega\tau}{4 - \omega^2\tau^2}\right] - \tan^{-1}\left[\frac{4\omega\tau}{4 - \omega^2\tau^2}\right] \quad (22)$$

There is a discontinuity at 2 rad.s^{-1} for a delay of 1 second. The phase versus frequency characteristic of the Padé approximation, expressed by equation (22), is plotted (Figure 8).

From Figure 8, it can be noticed that the phenomenon associated with the function tan^{-1}, described in the early part of the paper, arises again.

This time, the phase shift is 360°. Rule 1 corrects this problem. It has also been noticed that the number of discontinuities increase as the order of the approximation increases. This is a problem for correcting the phase using Rule 1. The solution is to use Assumption 1.

Figure 8: Phase versus Frequency: Equation (22)



A second Padé approximation is $e^{-s\tau} = \dfrac{\sum_{k=0}^{r} \frac{(2r-k)!}{k!(r-k)!}(-s\tau)^k}{\sum_{k=0}^{r} \frac{(2r-k)!}{k!(r-k)!}(s\tau)^k}$ which numerous authors ([3],[7-10]) use; MATLAB also uses this approximation in the function *pade(n, τ)*.

Using a second order approximation, the following equation is obtained in the frequency domain:

$$e^{-j\omega\tau} = \frac{12 - 6j\omega\tau - \omega^2\tau^2}{12 + 6j\omega\tau - \omega^2\tau^2}$$

$$\phi = \tan^{-1}\left[\frac{-6\omega\tau}{12-\omega^2\tau^2}\right] - \tan^{-1}\left[\frac{6\omega\tau}{12-\omega^2\tau^2}\right] \quad (23)$$

There are discontinuities at $\pm\frac{2\sqrt{3}}{\tau}$ rad.s^{-1}. Again, these discontinuities may be solved by shifting the phase by $-180°$, using Rule 1.

The same conclusion may also be drawn for the approximation used by [11] of the form $e^{-s\tau} = \dfrac{2 + \sum_{i=1}^{n} \frac{(-s\tau)^i}{i!}}{2 + \sum_{i=1}^{n} \frac{(s\tau)^i}{i!}}$.

Detailed analysis of existing approximations for time delays can be found in the technical report [12].

IV CONCLUSION

It has been shown that the phase formulae, as used in a computer package such as MATLAB, vary from the correct phase expressions. This is due to the trigonometric properties of the function tan^{-1}.The correct phase expression can be obtained by shifting the phase at the discontinuities by an angle of -180°.

Approximations are necessary to calculate transfer functions which have a time delay term, to produce an overall rational transfer functions. Rule 1 and Assumption 1 have to be used to obtain the correct phase values.

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