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Vladimir Gerdjikov
Bulgarian Academy of Sciences, gerjikov@inrne.bas.bg

Georgi Grahovski
Dublin Institute of Technology, georgi.grahovski@dit.ie

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Boltzmanngasse 9
A-1090 Wien, Austria

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Vladimir S. Gerdjikov
Georgi G. Grahovski

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Multi-component NLS Models on Symmetric Spaces: Spectral Properties versus Representations Theory

Vladimir S. Gerdjikov[†1] **and Georgi G. Grahovski**[†,‡2]

[†] *Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko chaussee, 1784 Sofia, Bulgaria*
[‡] *School of Mathematical Sciences, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland*

Abstract

The algebraic structure and the spectral properties of a special class of multicomponent NLS equations, related to the symmetric spaces of BD.I-type are analyzed. The focus of the study is on the spectral theory of the associated Lax operator to these nonlinear evolutionary equations for different fundamental representations of the underlying simple Lie algebra $\mathfrak{g}$. Special attention is paid to the spinor representation of the orthogonal Lie algebras of **B** type.

AMS subject classification numbers 37K20; 35Q51; 74J30; 78A60

Key Words: Multicomponent MNLS Equations, Reduction Group, Riemann-Hilbert Problem, Spectral Decompositions, Representation Theory

1 Introduction

The nonlinear Schrödinger equation

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \qquad q = q(x,t) \tag{1}$$

has natural multi-component generalizations. The first multi-component NLS type model with applications to physics is so-called vector NLS equation (Manakov model) [38, 2]:

$$i\mathbf{v}_t + \mathbf{v}_{xx} + 2||\mathbf{v}||^2\mathbf{v} = 0, \quad \mathbf{v} = \begin{pmatrix} v_1(x,t) \\ \vdots \\ v_n(x,t) \end{pmatrix}. \tag{2}$$

Here $\mathbf{v}$ is an n-component vector and $||\cdot||$ is the (standard) Euclidean norm.

Later on, the applications of the differential geometric and Lie algebraic methods to soliton type equations had lead to the discovery of a close relationship between the multi-component (matrix) NLS equations and the homogeneous and symmetric spaces [12]. It was shown that the integrable MNLS type models have Lax representation with the generalized Zakharov–Shabat system as the Lax operator:

$$L\psi(x,t,\lambda) \equiv i\frac{d\psi}{dx} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0, \tag{3}$$

[1]E-mail: gerjikov@inrne.bas.bg
[2]E-mails: grah@inrne.bas.bg ggrahovski@dit.ie

where J is a constant element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of the simple Lie algebra $\mathfrak{g}$ and $Q(x,t) \equiv [J, \widetilde{Q}(x,t)] \in \mathfrak{g}/\mathfrak{h}$. In other words $Q(x,t)$ belongs to the co-adjoint orbit $\mathcal{M}_J$ of $\mathfrak{g}$ passing through J. The Hermitian symmetric spaces, compatible with the NLS dispersion law, are labeled by the Cartan classification: **A.III**, **C.I**, **D.III** and **BD.I** (all others are listed, e.g. in [32]).

The choice of J determines the dimension of $\mathcal{M}_J$ which can be viewed as the phase space of the relevant nonlinear evolution equations (NLEE). It is is equal to the number of roots of $\mathfrak{g}$ such that $\alpha(J) \neq 0$. Taking into account that if α is a root, then and $-\alpha$ is also a root of $\mathfrak{g}$ then $\dim \mathcal{M}_J$ is always even.

The interpretation of the ISM as a generalized Fourier transforms and the expansion over the so-called "squared" solutions (see [14] for regular and [20, 22] for non-regular J) are based on the spectral theory for Lax operators in the form (3). This allow one to study all the fundamental properties of the corresponding NLEE's: i) the description of the class of NLEE related to a given Lax operator $L(\lambda)$ and solvable by the ISM; ii) derivation of the infinite family of integrals of motion; and iii) their hierarchy of Hamiltonian structures.

Since the Lax operator (3) takes values in a simple Lie algebra $\mathfrak{g}$, the choice of the representation of $\mathfrak{g}$ is crucial for the spectral theory of $L(\lambda)$. Our aim in this paper is to explore the spectral theory of the Lax operator in the form (3) for different fundamental representations of the underlying simple Lie algebra $\mathfrak{g}$. Special attention will be payed to the adjoint representation, which gives the expansion over the so-called "squared" solutions. Furthermore, based on examples, related to orthogonal simple Lie algebras (of $\mathbf{B}_r$ and $\mathbf{D}_r$ type, in the Cartan classification), the spectral properties of the Lax operator in the spinor representations of such algebras are discussed. The results are demonstrated on examples related to low-rank orthogonal Lie algebras.

The paper is organised as follows: In Section 2, we give some preliminaries about MNLS type equations over symmetric spaces, their Lax representation, direct scattering problem, scattering data, dressing factors and soliton solutions, in addition to some algebraic notations. In Section 3 we describe the spectral properties of the Lax operator in the typical representation for **BD.I** symmetric spaces and the effect of the dressing on the scattering data. In Section 4 we describe the spectral decomposition of the Lax operator in the adjoint representation: the expansions over the "squared" solutions and the generating (recursion) operator. In the next Section 5 we study the spectral properties of the same Lax operator on the spinor representation: starting with the algebraic structure of the Gauss factors for the scattering matrix, associated to $L(\lambda)$, the dressing factors, etc. The matrix realisations of both adjoint and spinor representations for $\mathbf{B}_2 \simeq so(5,\mathbb{C})$ and $\mathbf{B}_7 \simeq so(7,\mathbb{C})$ are presented in Appendices A, B and C.

2 Preliminaries

We start with some basic facts about the MNLS type models, related to **BD.I** symmetric spaces. Here we will present all result in the typical representation of the corresponding Lie algebra $\mathfrak{g} \simeq \mathbf{B}_r, \mathbf{B}_r$. These models admit a Lax representation in the form:

$$L\psi(x,t,\lambda) \equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \tag{4}$$

$$M\psi(x,t,\lambda) \equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \tag{5}$$

$$V_1(x,t) = Q(x,t), \qquad V_0(x,t) = i\mathrm{ad}_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right]. \tag{6}$$

Here, in general, J is an element of the corresponding Cartan subalgebra $\mathfrak{h}$ and $Q(x,t)$ is an off-diagonal matrix, taking values in a simple Lie algebra $\mathfrak{g}$. In the case of symmetric spaces, the corresponding Cartan involution provides a grading in $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is the subalgebra of all elements of $\mathfrak{g}$ commuting with J.

For symmetric spaces of **BD.I** type, the potential has the form:

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \qquad J = \mathrm{diag}(1, 0, \ldots 0, -1). \tag{7}$$

For $n = 2r+1$ the n-component vectors $\vec{q}$ and $\vec{p}$ have the form $\vec{q} = (q_1, \ldots, q_r, q_0, q_{\bar{r}}, \ldots, q_{\bar{1}})^T$, $\vec{p} = (p_1, \ldots, p_r, p_0, p_{\bar{r}}, \ldots, p_{\bar{1}})^T$), while the matrix $s_0 = S_0^{(n)}$ enters in the definition of $so(n)$: $X \in so(n)$, $\qquad X + S_0^{(n)} X^T S_0^{(n)} = 0$, where

$$S_0^{(n)} = \sum_{s=1}^{n+1} (-1)^{s+1} E^{(n)}_{s,n+1-s}. \tag{8}$$

For $n = 2r$ the n-component vectors $\vec{q}$ and $\vec{p}$ have the form $\vec{q} = (q_1, \ldots, q_r, q_{\bar{r}}, \ldots, q_{\bar{1}})^T$, $\vec{p} = (p_1, \ldots, p_r, p_{\bar{r}}, \ldots, p_{\bar{1}})^T$) and

$$S_0^{(n)} = \sum_{s=1}^{r} (-1)^{s+1} (E^{(n)}_{s,n+1-s} + E^{(n)}_{n+1-s,s}) \tag{9}$$

With this definition of orthogonality the Cartan subalgebra generators are represented by diagonal matrices. By $E^{(n)}_{sp}$ above we mean $n \times n$ matrix whose matrix elements are $(E^{(n)}_{sp})_{ij} = \delta_{si}\delta_{pj}$.

Let us comment briefly on the algebraic structure of the Lax pair, which is related to the symmetric space $SO(n+2)/(SO(n) \times SO(2))$. The element of the Cartan subalgebra J, which is dual to $e_1 \in \mathbb{E}^r$ allows us to introduce a grading in it: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which satisfies:

$$[X_1, X_2] \in \mathfrak{g}_0, \qquad [X_1, Y_1] \in \mathfrak{g}_1, \qquad [Y_1, Y_2] \in \mathfrak{g}_0, \tag{10}$$

for any choice of the elements $X_1, X_2 \in \mathfrak{g}_0$ and $Y_1, Y_2 \in \mathfrak{g}_1$. The grading splits the set of positive roots of $so(n)$ into two subsets $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ where Δ_0^+ contains all the positive roots of $\mathfrak{g}$ which are orthogonal to e_1, i.e. $(\alpha, e_1) = 0$; the roots in $\beta \in \Delta_1^+$ satisfy $(\beta, e_1) = 1$. For more details see the appendix below and [32].

The Lax pair can be considered in any representation of $so(n)$, then the potential Q will take the form:

$$Q(x,t) = \sum_{\alpha \in \Delta_1^+} \left(q_\alpha(x,t) E_\alpha + p_\alpha(x,t) E_{-\alpha} \right). \tag{11}$$

Next we introduce n-component 'vectors' formed by the Weyl generators of $so(n+2)$ corresponding to the roots in Δ_1^+:

$$\vec{E}_1^{\pm} = (E_{\pm(e_1-e_2)}, \ldots, E_{\pm(e_1-e_r)}, E_{\pm e_1}, E_{\pm(e_1+e_r)}, \ldots, E_{\pm(e_1+e_2)}), \tag{12}$$

for $n = 2r+1$ and

$$\vec{E}_1^{\pm} = (E_{\pm(e_1-e_2)}, \ldots, E_{\pm(e_1-e_r)}, E_{\pm(e_1+e_r)}, \ldots, E_{\pm(e_1+e_2)}), \tag{13}$$

for $n = 2r$. Then the generic form of the potentials $Q(x,t)$ related to these type of symmetric spaces can be written as sum of two "scalar" products

$$Q(x,t) = (\vec{q}(x,t) \cdot \vec{E}_1^+) + (\vec{p}(x,t) \cdot \vec{E}_1^-). \tag{14}$$

In terms of these notations the generic MNLS type equations connected to **BD.I.** acquire the form

$$\begin{aligned} i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q},\vec{p})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{p} = 0, \\ i\vec{p}_t - \vec{p}_{xx} - 2(\vec{q},\vec{p})\vec{p} + (\vec{p}, s_0\vec{p})s_0\vec{q} = 0, \end{aligned} \tag{15}$$

The Hamiltonian for the MNLS equations (15) with the "canonical" reduction $\vec{p} = \epsilon\vec{q}^*$, $\epsilon = \pm 1$ imposed, is given by:

$$H_{\text{MNLS}} = \int_{-\infty}^{\infty} dx \left((\partial_x\vec{q}, \partial_x\vec{q^*}) - \epsilon(\vec{q}, \vec{q^*})^2 + \epsilon(\vec{q}, s_0\vec{q})(\vec{q^*}, s_0\vec{q^*}) \right), \tag{16}$$

2.1 Direct Scattering Problem for L

The starting point for solving the direct and the inverse scattering problem (ISP) for L are the so-called Jost solutions, which are defined by their asymptotics (see, e.g. [18] and the references therein):

$$\lim_{x\to-\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1} \tag{17}$$

and the scattering matrix $T(\lambda,t) \equiv \psi^{-1}\phi(x,t,\lambda)$. The special choice of J and the fact that the Jost solutions and the scattering matrix take values in the group $SO(n+2)$ we can use the following block-matrix structure of $T(\lambda,t)$

$$T(\lambda,t) = \begin{pmatrix} m_1^+ & -\vec{b}^{-T} & c_1^- \\ \vec{b}^+ & \mathbf{T}_{22} & -s_0\vec{B}^- \\ c_1^+ & \vec{B}^{+T}s_0 & m_1^- \end{pmatrix}, \qquad \hat{T}(\lambda,t) = \begin{pmatrix} m_1^- & \vec{b}^{-T} & c_1^- \\ -\vec{B}^+ & \hat{\mathbf{T}}_{22} & s_0\vec{b}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix}, \tag{18}$$

where $\vec{b}^\pm(\lambda,t)$ and $\vec{B}^\pm(\lambda,t)$ are n-component vectors, $\mathbf{T}_{22}(\lambda)$ and $\boldsymbol{m}^\pm(\lambda)$ are $n\times n$ block matrices, and $m_1^\pm(\lambda)$, $c_1^\pm(\lambda)$ are scalar functions. Such parametrization is compatible with the generalized Gauss decompositions of $T(\lambda)$.

With this notations we introduce the generalized Gauss factors of $T(\lambda)$ as follows:

$$T(\lambda,t) = T_J^- D_J^+ \hat{S}_J^+ = T_J^+ D_J^- \hat{S}_J^-, \tag{19}$$

$$T_J^- = e^{(\vec{\rho}^+, \vec{E}^-)} = \begin{pmatrix} 1 & 0 & 0 \\ \vec{\rho}^+ & \mathbb{1} & 0 \\ c_1^{'',+} & \vec{\rho}^{+,T}s_0 & 1 \end{pmatrix}, \quad T_J^+ = e^{(-\vec{\rho}^-, \vec{E}^+)} = \begin{pmatrix} 1 & -\vec{\rho}^{-,T} & c_1^{''-} \\ 0 & \mathbb{1} & -s_0\vec{\rho}^- \\ 0 & 0 & 1 \end{pmatrix},$$

$$S_J^+ = e^{(\vec{\tau}^+, \vec{E}^+)} = \begin{pmatrix} 1 & \vec{\tau}^{+,T} & c_1^{'-} \\ 0 & \mathbb{1} & s_0\vec{\tau}^+ \\ 0 & 0 & 1 \end{pmatrix}, \quad S_J^- = e^{(-\vec{\tau}^-, \vec{E}^-)} = \begin{pmatrix} 1 & 0 & 0 \\ -\vec{\tau}^- & \mathbb{1} & 0 \\ c_1^{'+} & -\vec{\tau}^{-,T}s_0 & 1 \end{pmatrix},$$

$$D_J^+ = \begin{pmatrix} m_1^+ & 0 & 0 \\ 0 & \mathbf{m}_2^+ & 0 \\ 0 & 0 & 1/m_1^+ \end{pmatrix}, \qquad D_J^- = \begin{pmatrix} 1/m_1^- & 0 & 0 \\ 0 & \mathbf{m}_2^- & 0 \\ 0 & 0 & m_1^- \end{pmatrix}, \tag{20}$$

$$c_1^{'',\pm} = \frac{1}{2}(\vec{\rho}^{\pm,T} s_0 \vec{\rho}^{\pm}), \qquad c_1^{',\pm} = \frac{1}{2}(\vec{\tau}^{\mp,T} s_0 \vec{\tau}^{\mp}) \tag{21}$$

where

$$\begin{aligned}
&\vec{\rho}^{-} = \frac{\vec{B}^-}{m_1^-}, \qquad \vec{\tau}^{-} = \frac{\vec{B}^+}{m_1^-}, \qquad \vec{\rho}^{+} = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\tau}^{+} = \frac{\vec{b}^-}{m_1^+}, \\
&c_1^- = \frac{m_1^-}{2}(\vec{\rho}^{-,T} s_0 \vec{\rho}^{-}) = \frac{m_1^+}{2}(\vec{\tau}^{+,T} s_0 \vec{\tau}^{+}), \qquad c_1^+ = \frac{m_1^+}{2}(\vec{\rho}^{+,T} s_0 \vec{\rho}^{+}) = \frac{m_1^-}{2}(\vec{\tau}^{-,T} s_0 \vec{\tau}^{-}), \\
&\vec{b}^- = \frac{\mu_2^{-,T}}{m_1^-}\vec{B}^-, \qquad \vec{b}^+ = \frac{\mu_2^-}{m_1^-}\vec{B}^+, \qquad \vec{B}^+ = \frac{s_0 \mu_2^{+,T} s_0}{m_1^+}\vec{b}^+, \qquad \vec{B}^- = \frac{s_0 \mu_2^+ s_0}{m_1^+}\vec{b}^-, \\
&\mu^+ = \boldsymbol{m}_2^+ - \frac{\vec{b}^+ \vec{b}^{-,T}}{2m_1^+}, \qquad \mu^- = \boldsymbol{m}_2^- - \frac{s_0 \vec{B}^+ \vec{B}^{-,T} s_0}{2m_1^-}
\end{aligned} \tag{22}$$

These notations satisfy a number of relations which ensure that both $T(\lambda)$ and its inverse $\hat{T}(\lambda)$ belong to the corresponding orthogonal group $SO(n{+}2)$ and that $T(\lambda)\hat{T}(\lambda) = \mathbb{1}$. Some of them take the form:

$$\begin{aligned}
m_1^+ m_1^- + (\vec{b}^-, \vec{B}^+) + c_1^+ c_1^- &= 1, & \vec{b}^+ \vec{B}^{-T} + \mathbf{T}_{22} s_0 \mathbf{T}_{22}^T s_0 + s_0 \vec{B}^- \vec{b}^{+T} s_0 &= \mathbb{1}, \\
2m_1^+ c_1^- - \vec{b}^{-,T} s_0 \vec{b}^- &= 0, & 2m_1^- c_1^+ - \vec{B}^{+,T} s_0 \vec{B}^+ &= 0, \\
m_1^- \vec{b}^+ - \mathbf{T}_{22} \vec{B}^+ - s_0 \vec{B}^- c_1^+ &= 0, & m_1^+ \vec{B}^- - \hat{\mathbf{T}}_{22}^T \vec{b}^- - s_0 \vec{b}^+ c_1^- &= 0.
\end{aligned} \tag{23}$$

Important tools for reducing the ISP to a Riemann-Hilbert problem (RHP) are the fundamental analytic solution (FAS) $\chi^{\pm}(x,t,\lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda,t)$, see [43, 14, 16]:

$$\chi^{\pm}(x,t,\lambda) = \phi(x,t,\lambda) S_J^{\pm}(t,\lambda) = \psi(x,t,\lambda) T_J^{\mp}(t,\lambda) D_J^{\pm}(\lambda). \tag{24}$$

More precisely, this construction ensures that $\xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda) e^{i\lambda J x}$ are analytic functions of λ for $\lambda \in \mathbb{C}_{\pm}$. Here $S_J^{\pm}$, $T_J^{\pm}$ upper- and lower- block-triangular matrices, while $D_J^{\pm}(\lambda)$ are block-diagonal matrices with the same block structure as $T(\lambda,t)$ above. Skipping the details, we give here the explicit expressions of the Gauss factors in terms of the matrix elements of $T(\lambda,t)$

$$S_J^{\pm}(t,\lambda) = \exp\left(\pm(\vec{\tau}^{\pm}(\lambda,t) \cdot \vec{E}_1^{\pm})\right), \quad T_J^{\pm}(t,\lambda) = \exp\left(\mp(\vec{\rho}^{\mp}(\lambda,t) \cdot \vec{E}_1^{\pm})\right),$$

where

$$\vec{\tau}^{+}(\lambda,t) = \frac{\vec{b}^-}{m_1^+}, \qquad \vec{\tau}^{-}(\lambda,t) = \frac{\vec{B}^+}{m_1^-}, \qquad \vec{\rho}^{+}(\lambda,t) = \frac{\vec{b}^+}{m_1^+}, \qquad \vec{\rho}^{-}(\lambda,t) = \frac{\vec{B}^-}{m_1^-}, \tag{25}$$

and

$$\begin{aligned}
\mathbf{T}_{22} &= \boldsymbol{m}_2^+ - \frac{\vec{b}^+ \vec{b}^{-T}}{2m_1^+}, & \mathbf{T}_{22} &= \boldsymbol{m}_2^- - \frac{s_0 \vec{b}^- \vec{b}^{+T} s_0}{2m_1^-}, \\
\hat{\mathbf{T}}_{22} &= \hat{\boldsymbol{m}}_2^+ - \frac{s_0 \vec{b}^- \vec{b}^{+T} s_0}{2m_1^+}, & \hat{\mathbf{T}}_{22} &= \hat{\boldsymbol{m}}_2^- - \frac{\vec{B}^+ \vec{B}^{-T}}{2m_1^-},
\end{aligned} \tag{26}$$

If $Q(x,t)$ evolves according to (15) then the scattering matrix and its elements satisfy the following linear evolution equations

$$i\frac{d\vec{b}^{\pm}}{dt} \pm \lambda^2 \vec{b}^{\pm}(t,\lambda) = 0, \qquad i\frac{d\vec{B}^{\pm}}{dt} \pm \lambda^2 \vec{B}^{\pm}(t,\lambda) = 0, \qquad i\frac{dm_1^{\pm}}{dt} = 0, \qquad i\frac{d\mathbf{m}_2^{\pm}}{dt} = 0, \tag{27}$$

so the block-diagonal matrices $D^{\pm}(\lambda)$ can be considered as generating functionals of the integrals of motion. The fact that all $(2r-1)^2$ matrix elements of $\boldsymbol{m}_2^{\pm}(\lambda)$ for $\lambda \in \mathbb{C}_{\pm}$ generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (15). We remind that $D_J^{\pm}(\lambda)$ allow analytic extension for $\lambda \in \mathbb{C}_{\pm}$ and that their zeroes and poles determine the discrete eigenvalues of L.

2.2 Riemann-Hilbert Problem and Minimal Set of Scattering Data for L

The FAS for real λ are linearly related

$$\chi^{+}(x,t,\lambda) = \chi^{-}(x,t,\lambda) G_{0,J}(\lambda,t), \qquad G_{0,J}(\lambda,t) = \hat{S}_J^{-}(\lambda,t) S_J^{+}(\lambda,t). \tag{28}$$

One can rewrite eq. (28) in an equivalent form for the FAS $\xi^{\pm}(x,t,\lambda) = \chi^{\pm}(x,t,\lambda)e^{i\lambda Jx}$ which satisfy the equation:

$$i\frac{d\xi^{\pm}}{dx} + Q(x)\xi^{\pm}(x,\lambda) - \lambda[J,\xi^{\pm}(x,\lambda)] = 0, \qquad \lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}. \tag{29}$$

Then these FAS satisfy

$$\xi^{+}(x,t,\lambda) = \xi^{-}(x,t,\lambda) G_J(x,\lambda,t), \qquad G_J(x,\lambda,t) = e^{-i\lambda Jx} G_{0,J}^{-}(\lambda,t) e^{i\lambda Jx}. \tag{30}$$

Obviously the sewing function $G_J(x,\lambda,t)$ is uniquely determined by the Gauss factors $S_J^{\pm}(\lambda,t)$. Equation (30) is a Riemann-Hilbert problem (RHP) in multiplicative form. Since the Lax operator has no discrete eigenvalues, $\det \xi^{\pm}(x,\lambda)$ have no zeroes for $\lambda \in \mathbb{C})_{\pm}$ and $\xi^{\pm}(x,\lambda)$ are regular solutions of the RHP. As it is well known the regular solution $\xi^{\pm}(x,\lambda)$ of the RHP is uniquely determined.

Given the solutions $\xi^{\pm}(x,t,\lambda)$ one recovers $Q(x,t)$ via the formula

$$Q(x,t) = \lim_{\lambda\to\infty} \lambda \left(J - \xi^{\pm} J \widehat{\xi^{\pm}}(x,t,\lambda) \right) = [J, \xi_1(x)], \tag{31}$$

which is obtained from eq. (29) taking the limit $\lambda \to \infty$. By $\xi_1(x)$ above we have denoted $\xi_1(x) = \lim_{\lambda\to\infty} \lambda(\xi(x,\lambda) - \mathbb{1})$.

If the potential $Q(x,t)$ is such that the Lax operator L has no discrete eigenvalues, then the minimal set of scattering data is given one of the sets $\mathfrak{T}_i$, $i = 1,2$

$$\begin{aligned} \mathfrak{T}_1 &\equiv \{\rho_\alpha^{+}(\lambda,t), \rho_\alpha^{-}(\lambda,t), \quad \alpha \in \Delta_1^{+}, \quad \lambda \in \mathbb{R}\}, \\ \mathfrak{T}_2 &\equiv \{\tau_\alpha^{+}(\lambda,t), \tau_\alpha^{-}(\lambda,t), \quad \alpha \in \Delta_1^{+}, \quad \lambda \in \mathbb{R}\}. \end{aligned} \tag{32}$$

Any of these sets determines uniquely the scattering matrix $T(\lambda,t)$ and the corresponding potential $Q(x,t)$. For more details, we refer to [18, 11, 43] and references therein.

Most of the known examples of MNLS on symmetric spaces are obtained after imposing the reduction:

$$
\begin{aligned}
&Q(x,t) = K_0^{-1} Q^\dagger(x,t) K_0, \qquad K_0 = \exp\left(\frac{\pi i}{2}\sum_{j=1}^{r}(3+\epsilon_j)H_j\right), \\
&p_k = \tilde{K}_0 q_k^*, \qquad \tilde{K}_0 = \mathrm{diag}\,(K_{0,22},\ldots,K_{0,n+1,n+1})
\end{aligned}
\tag{33}
$$

or in components $p_k = \epsilon_1\epsilon_k q_k^*$. As a consequence the corresponding MNLS takes the form:

$$
i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}, \tilde{K}_0\vec{q}^*)\vec{q} - (\vec{q}, s_0\vec{q})s_0\tilde{K}_0\vec{q}^* = 0. \tag{34}
$$

The scattering data are restricted by $\vec{\rho}^-(\lambda,t) = \tilde{K}_0\vec{\rho}^{+,*}(\lambda,t)$ and $\vec{\tau}^-(\lambda,t) = \tilde{K}_0\vec{\tau}^{+,*}(\lambda,t)$.

If all $\epsilon_j = 1$ then the reduction becomes the "canonical" one: $Q(x,t) = Q^\dagger(x,t)$ and $\vec{\rho}^-(\lambda,t) = \vec{\rho}^{+,*}(\lambda,t)$ and $\vec{\tau}^-(\lambda,t) = \vec{\tau}^{+,*}(\lambda,t)$.

2.3 Dressing Factors and Soliton Solutions

The main goal of the dressing method [23, 33, 24, 31, 15] is, starting from a known solutions $\chi_0^\pm(x,t,\lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x,t)$ to construct new singular solutions $\chi_1^\pm(x,t,\lambda)$ of L with a potential $Q_{(1)}(x,t)$ with two additinal singularities located at prescribed positions $\lambda_1^\pm$; the reduction $\vec{p} = \vec{q}^*$ ensures that $\lambda_1^- = (\lambda_1^+)^*$. It is related to the regular one by a dressing factor $u(x,t,\lambda)$

$$
\chi_1^\pm(x,t,\lambda) = u(x,\lambda)\chi_0^\pm(x,t,\lambda)u_-^{-1}(\lambda). \qquad u_-(\lambda) = \lim_{x\to-\infty} u(x,\lambda) \tag{35}
$$

Note that $u_-(\lambda)$ is a block-diagonal matrix. The dressing factor $u(x,\lambda)$ must satisfy the equation

$$
i\partial_x u + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x,\lambda)] = 0, \tag{36}
$$

and the normalization condition $\lim_{\lambda\to\infty} u(x,\lambda) = \mathbb{1}$. The construction of $u(x,\lambda) \in SO(n+2)$ is based on an appropriate anzatz specifying explicitly the form of its λ-dependence (see [44, 31] and the references therein). Here we will consider a special choice of dressing factors:

$$
u(x,\lambda) = \mathbb{1} + (c(\lambda)-1)P(x) + \left(\frac{1}{c(\lambda)} - 1\right)\overline{P}(x), \qquad \overline{P} = S_0^{-1}P^T S_0, \qquad c(\lambda) = \frac{\lambda-\lambda_1^+}{\lambda-\lambda_1^-} \tag{37}
$$

where $P(x)$ and $\overline{P}(x)$ are mutually orthogonal projectors with rank 1. More specifically we have

$$
P(x,t) = \frac{|n(x,t)\rangle\langle m(x,t)|}{\langle n(x,t)|m(x,t)\rangle}, \tag{38}
$$

where $\langle m(x,t)| = \langle m_0|(\chi_0^-(x,\lambda_1^-))^{-1}$ and $|n(x,t)\rangle = \chi_0^+(x,\lambda_1^+)|n_0\rangle$; $\langle m_0|$ and $|n_0\rangle$ are (constant) polarization vectors [23]. Taking the limit $\lambda\to\infty$ in eq. (36) we get that

$$
Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^- - \lambda_1^+)[J, P(x,t) - \overline{P}(x,t)].
$$

Below we assume that $Q_{(0)} = 0$ and impose $\mathbb{Z}_2$ reduction condition

$$
KQK^{-1} = Q, \qquad K = \mathrm{diag}\,(\epsilon_1,\epsilon_2,1,\epsilon_2,\epsilon_1). \tag{39}
$$

This in its turn leads to $\lambda_1^{\pm} = \mu \pm i\nu$ and $|m_a\rangle = K|n_a\rangle^*$. As a result we get

$$q_k^{(1s)}(x,t) = -2i\nu \left(P_{1k}(x,t) + (-1)^k P_{\bar{k},n+2}(x,t)\right), \tag{40}$$

where $\bar{k} = n+3-k$. For more details on the soliton solutions satisfying the standard reduction $Q(x,t) = Q^\dagger(x,t)$ see [28, 33, 35, 24, 31, 22].

The effect of the dressing on the scattering data (22) is as follows:

$$\begin{aligned} \vec{\rho}_1^{+} &= \frac{\vec{b}^{+}}{m_1^{+}} = \frac{1}{c(\lambda)}\vec{\rho}_0^{+}, \qquad & \vec{\rho}_1^{-} &= \frac{\vec{B}^{-}}{m_1^{-}} = c(\lambda)\vec{\rho}_0^{-}; \\ \vec{\tau}_1^{+} &= \frac{\vec{b}^{-}}{m_1^{+}} = \frac{1}{c(\lambda)}\vec{\tau}_0^{+}, \qquad & \vec{\tau}_1^{-} &= \frac{\vec{B}^{+}}{m_1^{-}} = c(\lambda)\vec{\tau}_0^{-}. \end{aligned} \tag{41}$$

Applying N times the dressing method, one gets a Lax operator with N pars of prescribed discrete eigenvalues $\lambda_j^{\pm}$, $j = 1, ..., N$. The minimal sets of scattering data for the "dressed" Lax operator contains in addition the discrete eigenvalues $\lambda_j^{\pm}$, $j = 1, ..., N$ and the corresponding reflection/transmission coefficient at these points:

$$\begin{aligned} \mathfrak{T}_1' &\equiv \{\rho_\alpha^{+}(\lambda,t), \rho_\alpha^{-}(\lambda,t), \rho_{\alpha,j}^{+}(t), \rho_{\alpha,j}^{+}(t), \lambda_j^{\pm} \quad \alpha \in \Delta_1^{+}, \quad \lambda \in \mathbb{R}, \quad j = 1, \ldots, N\}, \\ \mathfrak{T}_2' &\equiv \{\tau_\alpha^{+}(\lambda,t), \tau_\alpha^{-}(\lambda,t), \tau_{\alpha,j}^{+}(t), \tau_{\alpha,j}^{+}(t), \lambda_j^{\pm} \quad \alpha \in \Delta_1^{+}, \quad \lambda \in \mathbb{R}, \quad j = 1, \ldots, N\}. \end{aligned} \tag{42}$$

3 Resolvent and spectral decompositions in the typical representation of $\mathfrak{g} \simeq B_r$

Here we first formulate the interrelation between the 'naked' and dressed FAS. We will use these relations to determine the order of pole singularities of the resolvent which of course will influence the contribution of the discrete spectrum to the completeness relation.

We first start with the simplest and more general case of the generalized Zakharov-Shabat system in which all eigenvalues of J are different. Then we discuss the additional construction necessary to treat cases when J has vanishing and/or equal eigenvalues.

3.1 The effect of dressing on the scattering data

Let us first determine the effect of dressing on the Jost solutions with the simplest dressing factor $u_1(x,\lambda)$. In what follows below we denote the Jost solutions corresponding to the regular solutions of RHP by $\psi_0(x,\lambda)$ and the dressed one by $\psi_1(x,\lambda)$. In order to preserve the definition in eq. (17) we put:

$$\psi_1(x,\lambda) = u_1(x,\lambda)\psi_0(x,\lambda)\hat{u}_{1,+}(\lambda), \qquad \phi_1(x,\lambda) = u_1(x,\lambda)\phi_0(x,\lambda)\hat{u}_{1,-}(\lambda), \tag{43}$$

where $u_{1,\pm}(x,\lambda) = \lim_{x\to\pm\infty} u_1(x,\lambda)$. We will also use the fact that $u_{1,\pm}(\lambda)$ are x-independent elements belonging to the Cartan subgroup of $\mathfrak{g}$. For the typical representation of $so(2r+1)$ and for the case only two singularities $\lambda_1^{\pm}$ are added we have:

$$\begin{aligned} u(x,\lambda) &= \exp\left(\ln c_1(\lambda)(P_1 - \bar{P}_1)\right), \\ u_{1,+}(\lambda) &= \mathbb{1} + (c_1(\lambda)-1)E_{11} + \left(\frac{1}{c_1(\lambda)} - 1\right)E_{n+2,n+2}, \\ u_{1,-}(\lambda) &= \mathbb{1} + (c_1(\lambda)-1)E_{n+2,n+2} + \left(\frac{1}{c_1(\lambda)} - 1\right)E_{11}, \\ u_{1,\pm}(\lambda) &= \exp\left(\pm \ln c_1(\lambda) J\right), \end{aligned} \tag{44}$$

Then from eq. (24) we get:

$$\chi_1^{\pm}(x,\lambda) = u_1(x,\lambda)\chi_0^{\pm}(x,\lambda)\hat{u}_{1,-}(\lambda), \tag{45}$$

As a consequence we find that

$$\begin{aligned} T_1(\lambda) &= u_{1,+}(\lambda)T_0(\lambda)\hat{u}_{1,-}(\lambda), \qquad & D_1^{\pm}(\lambda) &= u_{1,+}(\lambda)D_0(\lambda)\hat{u}_{1,-}(\lambda), \\ S_1^{\pm}(\lambda) &= u_{1,-}(\lambda)S_0(\lambda)\hat{u}_{1,-}(\lambda), \qquad & T_1^{\pm}(\lambda) &= u_{1,+}(\lambda)T_0(\lambda)\hat{u}_{1,+}(\lambda), \end{aligned} \tag{46}$$

One can repeat the dressing procedure N times by using dressing factor:

$$u(x,\lambda) = u_N(x,\lambda)u_{N-1}(x,\lambda)\cdots u_1(x,\lambda), \tag{47}$$

Note that the projector P_k of the k-th dressing factor has the form of (38) but the x-dependence of the polarization vectors is determined by the $k-1$ dressed FAS:

$$\begin{aligned} \chi_k^{\pm}(x,\lambda) &= u_k(x,\lambda)u_{k-1}(x,\lambda)\cdots u_1(x,\lambda)\chi_0^{\pm}(x,\lambda)\hat{u}_{1,-}(\lambda)\cdots\hat{u}_{k-1,-}(\lambda)\hat{u}_{k,-}(\lambda), \\ u_k(x,\lambda) &= \mathbb{1} + (c_k(\lambda)-1)P_k(x) + \left(\frac{1}{c_k(\lambda)} - 1\right)\overline{P}_k(x), \qquad \overline{P}_k = S_0^{-1}P_k^T S_0, \end{aligned} \tag{48}$$

$$\begin{aligned} c_k(\lambda) &= \frac{\lambda-\lambda_k^+}{\lambda_k-\lambda_1^-}, \qquad & P_k(x,t) &= \frac{|n_k(x,t)\rangle\langle m_k(x,t)|}{\langle n_k(x,t)|m_k(x,t)\rangle}, \\ \langle m_k(x,t)| &= \langle m_{0,k}|(\chi_{k-1}^-(x,\lambda_k^-))^{-1}, \qquad & |n_k(x,t)\rangle &= \chi_{k-1}^+(x,\lambda_k^+)|n_{0,k}\rangle; \end{aligned} \tag{49}$$

and $\langle m_{0,k}|$ and $|n_{0,k}\rangle$ are (constant) polarization vectors

Using eq. (51) and assuming that all $\lambda_j \in \mathbb{C}_{\pm}$ are different we can treat the general case of RHP with $2N$ singular points. The corresponding relations between the 'naked' and dressed FAS are;

$$\begin{aligned} T(\lambda) &= u_-(\lambda)T_0(\lambda)\hat{u}_-(\lambda), \qquad & D^{\pm}(\lambda) &= u_+(\lambda)D_0(\lambda)\hat{u}_-(\lambda), \\ S^{\pm}(\lambda) &= u_-(\lambda)S_0(\lambda)\hat{u}_-(\lambda), \qquad & T^{\pm}(\lambda) &= u_+(\lambda)T_0(\lambda)\hat{u}_+(\lambda), \end{aligned} \tag{50}$$

where

$$\begin{aligned} u_+(\lambda) &= \mathbb{1} + (c(\lambda)-1)E_{11} + \left(\frac{1}{c(\lambda)} - 1\right)E_{n+2,n+2}, \\ u_-(\lambda) &= \mathbb{1} + (c(\lambda)-1)E_{n+2,n+2} + \left(\frac{1}{c(\lambda)} - 1\right)E_{11}, \\ u_{\pm}(\lambda) &= \exp\left(\pm\ln c(\lambda)J\right), \qquad c(\lambda) = \prod_{k=1}^{N} c_k(\lambda). \end{aligned} \tag{51}$$

In components eqs. (50) give:

$$\begin{aligned} m_1^+(\lambda) &= m_{1,0}^+(\lambda)c^2(\lambda), \qquad & m_1^-(\lambda) &= \frac{m_{1,0}^-(\lambda)}{c^2(\lambda)}, \\ \rho_1^+(\lambda) &= \frac{\rho_0^+(\lambda)}{c(\lambda)}, \qquad & \rho_1^-(\lambda) &= c(\lambda)\rho_0^-(\lambda), \\ \tau_1^+(\lambda) &= \frac{\tau_0^+(\lambda)}{c(\lambda)}, \qquad & \tau_1^-(\lambda) &= c(\lambda)\tau_0^-(\lambda), \end{aligned} \tag{52}$$

and $\boldsymbol{m}_2^{\pm}(\lambda) = \boldsymbol{m}_{2,0}^{\pm}(\lambda)$.

In what follows we will need the residues of $u(x,\lambda)\chi_0^{\pm}(x,\lambda)$ and its inverse $\hat{\chi}_0^{\pm}(x,\lambda)\hat{u}(x,\lambda)$ at $\lambda = \lambda_k^{\pm}$ respectively. From eqs. (48) and (49) we get:

$$\begin{aligned}
u(x,\lambda)\chi_0^{+}(x,\lambda) &\simeq \frac{(\lambda_k^- - \lambda_k^+)\chi^{+,(k)}(x)}{\lambda - \lambda_k^+} + \dot{\chi}^{+,(k)}(x) + \mathcal{O}(\lambda - \lambda_k^+),\\
u(x,\lambda)\chi_0^{-}(x,\lambda) &\simeq \frac{(\lambda_k^+ - \lambda_k^-)\chi^{-,(k)}(x)}{\lambda - \lambda_k^-} + \dot{\chi}^{-,(k)}(x) + \mathcal{O}(\lambda - \lambda_k^-),\\
\hat{\chi}_0^{+}(x,\lambda)\hat{u}(x,\lambda) &\simeq \frac{(\lambda_k^- - \lambda_k^+)\hat{\chi}^{+,(k)}(x)}{\lambda - \lambda_k^+} + \hat{\dot{\chi}}^{+,(k)}(x) + \mathcal{O}(\lambda - \lambda_k^+),\\
\hat{\chi}_0^{-}(x,\lambda)\hat{u}(x,\lambda) &\simeq \frac{(\lambda_k^+ - \lambda_k^-)\hat{\chi}^{-,(k)}(x)}{\lambda - \lambda_k^-} + \hat{\dot{\chi}}^{-,(k)}(x) + \mathcal{O}(\lambda - \lambda_k^-),
\end{aligned} \tag{53}$$

where

$$\begin{aligned}
\chi^{+,(k)}(x) &= u_N(x,\lambda_k^+)\cdots u_{k+1}(x,\lambda_k^+)\bar{P}_k\chi^{+}_{(k-1)}(x,\lambda_k^+),\\
\chi^{-,(k)}(x) &= u_N(x,\lambda_k^-)\cdots u_{k+1}(x,\lambda_k^-)P_k\chi^{-}_{(k-1)}(x,\lambda_k^-),\\
\hat{\chi}^{+,(k)}(x) &= \hat{\chi}^{+}_{(k-1)}(x,\lambda_k^+)\hat{u}_{k+1}(x,\lambda_k^+)\cdots\hat{u}_N(x,\lambda_k^+)P_k,\\
\hat{\chi}^{-,(k)}(x) &= \hat{\chi}^{-}_{(k-1)}(x,\lambda_k^-)\hat{u}_{k+1}(x,\lambda_k^-)\cdots\hat{u}_N(x,\lambda_k^-)\bar{P}_k,
\end{aligned} \tag{54}$$

These results will be used below to find the residues of the resolvent at $\lambda = \lambda_k^{\pm}$.

3.2 Spectral decompositions for the generalized Zakharov-Shabat system: $sl(n)$-case

The FAS are the basic tool in constructing the spectral theory of the corresponding Lax operator. For the generic Lax operators related to the $sl(n)$ algebras:

$$L_{\text{gen}} \equiv i\frac{\partial\chi_{\text{gen}}}{\partial x} + (Q_{\text{gen}}(x) - \lambda J_{\text{gen}})\chi_{\text{gen}}(x,\lambda) = 0, \qquad (Q_{\text{gen}})_{jj}(x) = 0, \tag{55}$$

this theory is well developed, see [43, 19, 14, 17]. In the generic case all eigenvalues of $J_{\text{gen}} = \operatorname{diag}(J_1, J_2, \ldots, J_{n+2})$ are different and nonvanishing:

$$J_1 > J_2 > \cdots > J_k > 0 > J_{k+1} > \cdots > J_{n+2}, \operatorname{tr} J_{\text{gen}} = 0. \tag{56}$$

The Jost solutions $\psi_{\text{gen}}(x,\lambda)$, $\phi_{\text{gen}}(x,\lambda)$, the scattering matrix $T_{\text{gen}}(\lambda)$ and the FAS $\chi_{\text{gen}}^{\pm}(x,\lambda)$ are introduced by [41, 43] (see also [14, 17, 18, 19, 22]):

$$\begin{aligned}
&\lim_{x\to-\infty}\phi_{\text{gen}}(x,\lambda)e^{iJ_{\text{gen}}x\lambda} = \mathbb{1}, && \lim_{x\to\infty}\psi_{\text{gen}}(x,\lambda)e^{iJ_{\text{gen}}x\lambda} = \mathbb{1},\\
&T_{\text{gen}}(\lambda) = \psi_{\text{gen}}^{-1}(x,\lambda)\phi_{\text{gen}}(x,\lambda) &&\\
&\chi_{\text{gen}}^{\pm}(x,\lambda) = \phi_{\text{gen}}(x,\lambda)S_{\text{gen}}^{\pm}(\lambda), && \chi_{\text{gen}}^{\pm}(x,\lambda) = \psi_{\text{gen}}(x,\lambda)T_{\text{gen}}^{\mp}(\lambda)D_{\text{gen}}^{\pm}(\lambda),
\end{aligned} \tag{57}$$

where $S_{\text{gen}}^{\pm}(\lambda)$, $T_{\text{gen}}^{\pm}(\lambda)$ and $D_{\text{gen}}^{\pm}(\lambda)$ are the factors in the Gauss decompositions of $T_{\text{gen}}(\lambda)$:

$$T_{\text{gen}}(\lambda) = T_{\text{gen}}^{-}(\lambda)D_{\text{gen}}^{+}(\lambda)\hat{S}_{\text{gen}}^{+}(\lambda) = T_{\text{gen}}^{+}(\lambda)D_{\text{gen}}^{-}(\lambda)\hat{S}_{\text{gen}}^{-}(\lambda). \tag{58}$$

More specifically $S^+_{\rm gen}(\lambda)$ and $T^+_{\rm gen}(\lambda)$ (resp. $S^-_{\rm gen}(\lambda)$ and $T^-_{\rm gen}(\lambda)$) are upper (resp. lower) triangular matrices whose diagonal elements are equal to 1. The diagonal matrices $D^+_{\rm gen}(\lambda)$ and $D^-_{\rm gen}(\lambda)$ allow analytic extension in the upper and lower half planes respectively.

The dressing factors $u_k(x,\lambda)$ are the simplest possible ones [43]:

$$u_{k,\rm gen}(x,\lambda) = \mathbb{1} + (c_k(\lambda)-1)P_k(x), \qquad u^{-1}_{k,\rm gen}(x,\lambda) = \mathbb{1} + \left(\frac{1}{c_k(\lambda)} - 1\right) P_k(x), \tag{59}$$

where the rank-1 projectors $P_k(x)$ are expressed through the regular solutions analogously to eqs. (47)–(49) with $\chi^\pm_0(x,\lambda^\pm_k)$ replaced by $\chi^\pm_{0,\rm gen}(x,\lambda^\pm_k)$

The relations between the dressed and 'naked' scattering data are the same like in eq. (46) only now the asymptotic values $u_{\rm gen;\pm}(\lambda)$ are different. Assuming that all projectors P_k have rank 1 we get:

$$u_{\rm gen;\pm}(\lambda) = \textstyle\prod_{k=1}^N u_{k,\rm gen;\pm}(\lambda), \qquad u_{k,\rm gen;\pm}(\lambda) = \mathbb{1} + (c_k(\lambda)-1)P_{k,\pm}, \tag{60}$$

$$P_{k,+} = E_{s_k,s_k}, \qquad P_{k,-} = E_{p_k,p_k}, \tag{61}$$

where s_k (resp. p_k) labels the position of the first (resp. the last) nonvanishing component of the polarization vector $|n_{0,k}\rangle$.

Using the FAS we introduce the resolvent $R_{\rm gen}(\lambda)$ of $L_{\rm gen}$ in the form:

$$R_{\rm gen}(\lambda)f(x) = \int_{-\infty}^{\infty} R_{\rm gen}(x,y,\lambda)f(y). \tag{62}$$

The kernel $R_{\rm gen}(x,y,\lambda)$ of the resolvent is given by:

$$R_{\rm gen}(x,y,\lambda) = \begin{cases} R^+_{\rm gen}(x,y,\lambda) \text{ for } \lambda\in\mathbb{C}^+, \\ R^-_{\rm gen}(x,y,\lambda) \text{ for } \lambda\in\mathbb{C}^-, \end{cases} \tag{63}$$

where

$$R^\pm_{\rm gen}(x,y,\lambda) = \pm i\chi^\pm_{\rm gen}(x,\lambda)\Theta^\pm(x-y)\hat\chi^\pm_{\rm gen}(y,\lambda), \tag{64}$$

$$\Theta^\pm(z) = \theta(\mp z)\Pi_0 - \theta(\pm z)(\mathbb{1}-\Pi_0), \qquad \Pi_0 = \sum_{s=1}^k E_{ss},$$

Theorem 3.1 *Let $Q(x)$ be a potential of L which falls off fast enough for $x\to\pm\infty$ and the corresponding RHP has a finite number of simple singularities at the points $\lambda^\pm_j\in\mathbb{C}_\pm$, i.e. $\chi^\pm_{\rm gen}(x,\lambda)$ have simple poles and zeroes at $\lambda^\pm_j$. Then*

1. *$R^\pm_{\rm gen}(x,y,\lambda)$ is an analytic function of λ for $\lambda\in\mathbb{C}_\pm$ having pole singularities at $\lambda^\pm_j\in\mathbb{C}_\pm$;*
2. *$R^\pm_{\rm gen}(x,y,\lambda)$ is a kernel of a bounded integral operator for $\operatorname{Im}\lambda\neq 0$;*
3. *$R_{\rm gen}(x,y,\lambda)$ is uniformly bounded function for $\lambda\in\mathbb{R}$ and provides a kernel of an unbounded integral operator;*
4. *$R^\pm_{\rm gen}(x,y,\lambda)$ satisfy the equation:*

$$L_{\rm gen}(\lambda)R^\pm_{\rm gen}(x,y,\lambda) = \mathbb{1}\delta(x-y). \tag{65}$$

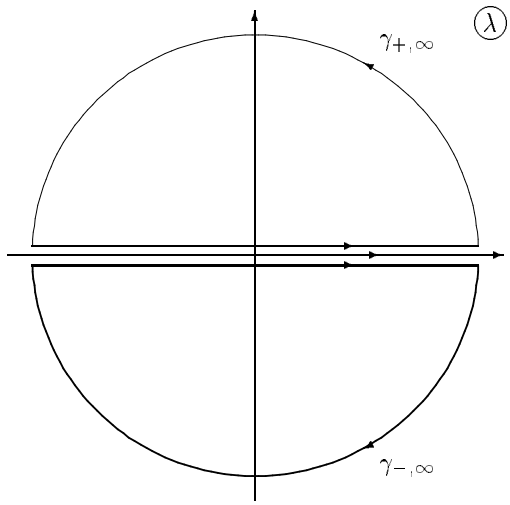

Figure 1: The contours $\gamma_\pm = \mathbb{R} \cup \gamma_{\pm\infty}$.

Skipping the details (see [17]) we formulate the completeness relation for the eigenfunctions of the Lax operator L_{gen}. It is derived by applying the contour integration method (see e.g. [29, 1]) to the integral:

$$\mathcal{J}_{\text{gen}}(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda R^+_{\text{gen}}(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda R^-_{\text{gen}}(x,y,\lambda), \tag{66}$$

where the contours $\gamma_\pm$ are shown on the Figure 1 and has the form:

The explicit form of the dressing factors $u_{\text{gen}}(x,\lambda)$ makes it obvious that the kernel of the resolvent has only simple poles at $\lambda = \lambda_k^\pm$. Therefore the final form of the completeness relation for the Jost solutions of L_{gen} takes the form [17]:

$$\begin{aligned}
&\delta(x-y)\sum_{s=1}^{n}\frac{1}{a_s}E_{ss} \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left\{ \sum_{s=1}^{k_0} |\chi^{[s]+}_{\text{gen}}(x,\lambda)\rangle\langle\hat{\chi}^{[s]+}_{\text{gen}}(y,\lambda)| - \sum_{s=k_0+1}^{n} |\chi^{[s]-}_{\text{gen}}(x,\lambda)\rangle\langle\hat{\chi}^{[s]-}_{\text{gen}}(y,\lambda)| \right\} \\
&\quad + \sum_{j=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} R^+(x,y) + \operatorname*{Res}_{\lambda=\lambda_j^-} R^-(x,y) \right),
\end{aligned} \tag{67}$$

It is easy to check that the residues in (67) can be expressed by the properly normalized eigenfunctions of L_{gen} corresponding to the eigenvalues $\lambda_j^\pm$ [17].

Thus we conclude that the continuous spectrum of L_{gen} has multiplicity n and fills up the whole real axis $\mathbb{R}$ of the complex λ-plane; the discrete eigenvalues of L_{gen} constructed using the dressing factors $u_{\text{gen}}(x,\lambda)$ are simple and the resolvent kernel $R_{\text{gen}}(x,y,\lambda)$ has poles of order one at $\lambda = \lambda_k^\pm$.

3.3 Resolvent and spectral decompositions for BD.I-type Lax operators

In our case J has n vanishing eigenvalues which makes the problem more difficult.

We can rewrite the Lax operator in the form:

$$
\begin{aligned}
i\frac{\partial \chi_1}{\partial x} + \vec{q}^T \vec{\chi}_0 &= \lambda \chi_1, \\
i\frac{\partial \vec{\chi}_0}{\partial x} + \vec{q}^* \chi_1 + s_0 \vec{q} \chi_{-1} &= 0, \\
i\frac{\partial \chi_{-1}}{\partial x} + \vec{q}^\dagger s_0 \vec{\chi}_0 &= \lambda \chi_{-1},
\end{aligned}
\tag{68}
$$

where we have split the eigenfunction $\chi(x,\lambda)$ of L into three according to the natural block-matrix structure compatible with J: $\chi(x,\lambda) = \left(\chi_1, \vec{\chi}_0^T, \chi_{-1}\right)^T$. Note that the equation for $\vec{\chi}_0$ can not be treated as eigenvalue equations; they can be formally integrated with:

$$
\vec{\chi}_0(x,\lambda) = \vec{\chi}_{0,\text{as}} + i\int^x dy\ \left(\vec{q}^* \chi_1 + s_0 \vec{q} \chi_{-1}\right), \tag{69}
$$

which eventually casts the Lax operator into the following integro-differential system with non-degenerate λ dependence.

$$
\begin{aligned}
i\frac{\partial \chi_1}{\partial x} + i\vec{q}^T(x) \int^x dy\ \left(\vec{q}^* \chi_1 + s_0 \vec{q} \chi_{-1}\right)(y,\lambda) &= \lambda \chi_1, \\
i\frac{\partial \chi_{-1}}{\partial x} + i\vec{q}^\dagger(x) s_0 \int^x dy\ \left(\vec{q}^* \chi_1 + s_0 \vec{q} \chi_{-1}\right)(y,\lambda) &= -\lambda \chi_{-1},
\end{aligned}
\tag{70}
$$

Similarly we can treat the operator which is adjoint to L whose FAS $\hat{\chi}(x,\lambda)$ are the inverse to $\chi(x,\lambda)$, i.e. $\hat{\chi}(x,\lambda) = \chi^{-1}(x,\lambda)$. Splitting each of the rows of $\hat{\chi}(x,\lambda)$ into components as follows $\hat{\chi}(x,\lambda) = (\hat{\chi}_1, \hat{\vec{\chi}}_0, \hat{\chi}_{-1})$ we get:

$$
\begin{aligned}
i\frac{\partial \hat{\chi}_1}{\partial x} - (\hat{\vec{\chi}}_0, \vec{q}^*) - \lambda \hat{\chi}_1 &= 0, \\
i\frac{\partial \hat{\vec{\chi}}_0}{\partial x} - \hat{\chi}_1 \vec{q}^T - \hat{\chi}_{-1} \vec{q}^\dagger s_0 &= 0, \\
i\frac{\partial \hat{\chi}_{-1}}{\partial x} - (\hat{\vec{\chi}}_0, s_0 \vec{q}) - \lambda \hat{\chi}_{-1} &= 0,
\end{aligned}
\tag{71}
$$

Again the equation for $\hat{\vec{\chi}}_0$ can be formally integrated with:

$$
\hat{\vec{\chi}}_0(x,\lambda) = \hat{\vec{\chi}}_{0,\text{as}} + i\int^x dy\ \left(\hat{\chi}_1(y,\lambda)\vec{q}^T(y) + \hat{\chi}_{-1}(y,\lambda)\vec{q}^\dagger(y) s_0\right), \tag{72}
$$

Now we get the following integro-differential system with non-degenerate λ dependence.

$$
\begin{aligned}
i\frac{\partial \hat{\chi}_1}{\partial x} - i\int^x dy\ \left(\hat{\chi}_1(y,\lambda)(\vec{q}^T(y), \vec{q}^*(x)) + \hat{\chi}_{-1}(y,\lambda)(\vec{q}^\dagger(y) s_0 \vec{q}^*(x))\right) + \lambda \hat{\chi}_1 &= 0, \\
i\frac{\partial \hat{\chi}_{-1}}{\partial x} - i\int^x dy\ \left(\hat{\chi}_1(y,\lambda)(\vec{q}^T(y) s_0 \vec{q}(x)) + \hat{\chi}_{-1}(y,\lambda)(\vec{q}^\dagger(y), \vec{q}(x))\right) - \lambda \hat{\chi}_{-1} &= 0,
\end{aligned}
\tag{73}
$$

Now we are ready to generalize the standard approach to the case of BD.I-type Lax operators.

The kernel $R(x,y,\lambda)$ of the resolvent is given by:

$$
R(x,y,\lambda) = \begin{cases} R^+(x,y,\lambda) \text{ for } \lambda \in \mathbb{C}^+, \\ R^-(x,y,\lambda) \text{ for } \lambda \in \mathbb{C}^-, \end{cases} \tag{74}
$$

where

$$R^{\pm}(x,y,\lambda) = \pm i\chi^{\pm}(x,\lambda)\Theta^{\pm}(x-y)\hat{\chi}^{\pm}(y,\lambda), \qquad (75)$$
$$\Theta^{\pm}(z) = \theta(\mp z)E_{11} - \theta(\pm z)(\mathbb{1} - E_{11}),$$

The completeness relation for the eigenfunctions of the Lax operator L is derived by applying the contour integration method (see e.g. [29, 1]) to the integral:

$$\mathcal{J}'(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda \Pi_1 R^+(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda \Pi_1 R^-(x,y,\lambda), \qquad (76)$$

where the contours $\gamma_\pm$ are shown on the Figure 1 and $\Pi_1 = E_{11} + E_{n+2,n+2}$. Using eqs. (53) and (54) we are able to check that the kernel of the resolvent has poles of second order at $\lambda = \lambda_k^{\pm}$. Therefore the completeness relation takes the form:

$$\Pi_1\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \Pi_1 \left\{ |\chi_{\rm gen}^{[1]+}(x,\lambda)\rangle\langle\hat{\chi}_{\rm gen}^{[1]+}(y,\lambda)| - |\chi_{\rm gen}^{[n+2]-}(x,\lambda)\rangle\langle\hat{\chi}_{\rm gen}^{[n+2]-}(y,\lambda)| \right\}$$
$$+ 2i\sum_{j=1}^{N}\left\{ \operatorname*{Res}_{\lambda=\lambda_k^+} R^+(x,y,\lambda) + \operatorname*{Res}_{\lambda=\lambda_k^-} R^-(x,y,\lambda) \right\}. \qquad (77)$$

where

$$\operatorname*{Res}_{\lambda=\lambda_k^{\pm}} R^{\pm}(x,y,\lambda) = \pm(\lambda_k^- - \lambda_k^+)\Pi_1 \left(\chi^{+,(k)}(x)\hat{\chi}^{+,(k)}(y) + \dot{\chi}^{+,(k)}(x)\hat{\chi}^{+,(k)}(y) \right) \qquad (78)$$

In other words the continuous spectrum of L has multiplicity 2 and fills up the whole real axis $\mathbb{R}$ of the complex λ-plane; the discrete eigenvalues of L constructed using the dressing factors $u(x,\lambda)$ (51) lead to second order poles of resolvent kernel $R_{\rm gen}(x,y,\lambda)$ at $\lambda = \lambda_k^{\pm}$.

4 Resolvent and spectral decompositions in the adjoint representation of $\mathfrak{g} \simeq B_r$

The simplest realization of L in the adjoint representation is to make use of the adjoint action of $Q(x) - \lambda J$ on $\mathfrak{g}$:

$$L_{\rm ad}e_{\rm ad} \equiv i\frac{\partial e_{\rm ad}}{\partial x} + [Q(x) - \lambda J_{\rm ad}, e_{\rm ad}(x,\lambda)] = 0. \qquad (79)$$

Note that the eigenfunctions of $L_{\rm ad}$ take values in the Lie algebra $\mathfrak{g}$. They are known also as the 'squared solutions' of L and appear in a natural way in the analysis of the transform from the potential $Q(x,t)$ to the scattering data of L, [1]; see also [14, 17] and the numerous references therein.

The idea for the interpretation of the ISM as a generalized Fourier transform was launched in [1]. It is based on the Wronskian relations which allow one to maps the potential $Q(x,t)$ onto the minimal sets of scattering data $\mathcal{T}_i$. These ideas have been generalized also to the symmetric spaces, see [14, 25, 22] and the references therein.

The 'squared solutions' that play the role of generalized exponentials are determined by the FAS and the Cartan-Weyl basis of the corresponding algebra as follows. First we introduce:

$$e^{\pm}_{\alpha,\mathrm{ad}}(x,\lambda) = \chi^{\pm} E_\alpha \hat{\chi}^{\pm}(x,\lambda), \qquad e^{\pm}_{j,\mathrm{ad}}(x,\lambda) = \chi^{\pm} H_j \hat{\chi}^{\pm}(x,\lambda), \tag{80}$$

where $\chi^{\pm}(x,\lambda)$ are the FAS of L and E_α, H_j form the Cartan-Weyl basis of $\mathfrak{g}$. Next we note that in the adjoint representation $J_{\mathrm{ad}}\cdot \equiv \mathrm{ad}_J\cdot \equiv [J,\cdot]$ has kernel. Just like in the previous Section we have to project out that kernel, i.e. we need to introduce the projector:

$$\pi_J X \equiv \mathrm{ad}_J^{-1}\mathrm{ad}_J X, \tag{81}$$

for any $X \in \mathfrak{g}$. In particular, choosing $\mathfrak{g} \simeq so(2r+1)$ and J as in eq. (4) we find that the potential Q provides a generic element of the image of π_J, i.e. $\pi_J Q \equiv Q$.

Next, the analysis of the Wronskian relations allows one to introduce two sets of squared solutions:

$$\mathbf{\Psi}^{\pm}_\alpha = \pi_J(\chi^{\pm}(x,\lambda)E_\alpha\hat{\chi}^{\pm}(x,\lambda)), \qquad \mathbf{\Phi}^{\pm}_\alpha = \pi_J(\chi^{\pm}(x,\lambda)E_{-\alpha}\hat{\chi}^{\pm}(x,\lambda)), \qquad \alpha \in \Delta_1^+. \tag{82}$$

We remind that the set Δ_1 contains all roots of $so(r+1)$ for which $\alpha(J) \neq 0$.

Let us introduce the sets of 'squared solutions':

$$\{\mathbf{\Psi}\} = \{\mathbf{\Psi}\}_{\mathrm{c}} \cup \{\mathbf{\Psi}\}_{\mathrm{d}}, \qquad \{\mathbf{\Phi}\} = \{\mathbf{\Phi}\}_{\mathrm{c}} \cup \{\mathbf{\Phi}\}_{\mathrm{d}}, \tag{83}$$

$$\begin{aligned} \{\mathbf{\Psi}\}_{\mathrm{c}} &\equiv \left\{\mathbf{\Psi}^{+}_\alpha(x,\lambda), \quad \mathbf{\Psi}^{-}_{-\alpha}(x,\lambda), \quad i<r, \quad \lambda\in\mathbb{R}\right\}, \\ \{\mathbf{\Psi}\}_{\mathrm{d}} &\equiv \left\{\mathbf{\Psi}^{+}_{\alpha;j}(x), \quad \dot{\mathbf{\Psi}}^{+}_{\alpha;j}(x), \quad \mathbf{\Psi}^{-}_{-\alpha;j}(x), \quad \dot{\mathbf{\Psi}}^{-}_{-\alpha;j}(x)\right\}_{j=1}^{N}, \end{aligned} \tag{84}$$

$$\begin{aligned} \{\mathbf{\Phi}\}_{\mathrm{c}} &\equiv \left\{\mathbf{\Phi}^{+}_{-\alpha}(x,\lambda), \quad \mathbf{\Phi}^{-}_{\alpha}(x,\lambda), \quad i<r, \quad \lambda\in\mathbb{R}\right\}, \\ \{\mathbf{\Phi}\}_{\mathrm{d}} &\equiv \left\{\mathbf{\Phi}^{+}_{-\alpha;j}(x), \quad \dot{\mathbf{\Phi}}^{+}_{-\alpha;j}(x), \quad \mathbf{\Phi}^{-}_{\alpha;j}(x), \quad \dot{\mathbf{\Phi}}^{-}_{\alpha;j}(x)\right\}_{j=1}^{N}, \end{aligned} \tag{85}$$

where the subscripts 'c' and 'd' refer to the continuous and discrete spectrum of L.

Each of the above two sets are complete sets of functions in the space of allowed potentials. This fact van be proved by applying the contour integration method to the integral

$$\mathcal{J}_G(x,y) = \frac{1}{2\pi i}\oint_{\gamma_+} d\lambda G^{+}(x,y,\lambda) - \frac{1}{2\pi i}\oint_{\gamma_-} d\lambda G^{-}(x,y,\lambda), \tag{86}$$

where the Green function is defined by:

$$\begin{aligned} G^{\pm}(x,y,\lambda) &= G^{\pm}_1(x,y,\lambda)\theta(y-x) - G^{\pm}_2(x,y,\lambda)\theta(x-y), \\ G^{\pm}_1(x,y,\lambda) &= \sum_{\alpha\in\Delta_1^+} \mathbf{\Psi}^{\pm}_{\pm\alpha}(x,\lambda)\otimes\mathbf{\Phi}^{\pm}_{\mp\alpha}(y,\lambda), \\ G^{\pm}_2(x,y,\lambda) &= \sum_{\alpha\in\Delta_0\cup\Delta_1^-} \mathbf{\Phi}^{\pm}_{\pm\alpha}(x,\lambda)\otimes\mathbf{\Psi}^{\pm}_{\mp\alpha}(y,\lambda) + \sum_{j=1}^{r}\boldsymbol{h}^{\pm}_j(x,\lambda)\otimes\boldsymbol{h}^{\pm}_j(y,\lambda), \\ \boldsymbol{h}^{\pm}_j(x,\lambda) &= \chi^{\pm}(x,\lambda)H_j\hat{\chi}^{\pm}(x,\lambda), \end{aligned} \tag{87}$$

Skipping the details we give the result [25, 27]:

$$\delta(x-y)\Pi_{0J} = \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda)) \\ - 2i\sum_{j=1}^{N}(G_{1,j}^+(x,y) + G_{1,j}^-(x,y)), \tag{88}$$

where

$$\Pi_{0J} = \sum_{\alpha\in\Delta_1^+}(E_\alpha \otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha), \tag{89}$$

$$G_{1,}j^{\pm}(x,y) = \sum_{\alpha\in\Delta_1^+}(\dot{\mathbf{\Psi}}^{\pm}_{\pm\alpha;j}(x)\otimes \mathbf{\Phi}^{\pm}_{\mp\alpha;j}(y) + \mathbf{\Psi}^{\pm}_{\pm\alpha;j}(x)\otimes \dot{\mathbf{\Phi}}^{\pm}_{\mp\alpha;j}(y)). \tag{90}$$

4.1 Expansion over the 'squared solutions'

The completeness relation of the 'squared solutions' allows one to expand any function over the 'squared solutions'

Using the Wronskian relations one can derive the expansions over the 'squared solutions' of two important functions. Skipping the calculational details we formulate the results [25]. The expansion of $Q(x)$ over the systems $\{\mathbf{\Phi}^{\pm}\}$ and $\{\mathbf{\Psi}^{\pm}\}$ takes the form:

$$Q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\tau_\alpha^+(\lambda)\mathbf{\Phi}_\alpha^+(x,\lambda) - \tau_\alpha^-(\lambda)\mathbf{\Phi}_{-\alpha}^-(x,\lambda)\right) \\ + 2\sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\tau_{\alpha;j}^+\mathbf{\Phi}_{\alpha;j}^+(x) + \tau_{\alpha;j}^-\mathbf{\Phi}_{-\alpha;j}^-(x)\right), \tag{91}$$

$$Q(x) = -\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\rho_\alpha^+(\lambda)\mathbf{\Psi}_{-\alpha}^+(x,\lambda) - \rho_\alpha^-(\lambda)\mathbf{\Psi}_{\alpha}^-(x,\lambda)\right) \\ - 2\sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\rho_{\alpha;j}^+\mathbf{\Psi}_{-\alpha;j}^+(x) + \rho_{\alpha;j}^-\mathbf{\Psi}_{\alpha;j}^-(x)\right), \tag{92}$$

The next expansion is of $\mathrm{ad}_J^{-1}\delta Q(x)$ over the systems $\{\mathbf{\Phi}^{\pm}\}$ and $\{\mathbf{\Psi}^{\pm}\}$:

$$\mathrm{ad}_J^{-1}\delta Q(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\delta\tau_\alpha^+(\lambda)\mathbf{\Phi}_\alpha^+(x,\lambda) + \delta\tau_\alpha^-(\lambda)\mathbf{\Phi}_{-\alpha}^-(x,\lambda)\right) \\ + \sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\delta W_{\alpha;j}^+(x) - \delta' W_{-\alpha;j}^-(x)\right), \tag{93}$$

$$\mathrm{ad}_J^{-1}\delta Q(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda \sum_{\alpha\in\Delta_1^+}\left(\delta\rho_\alpha^+(\lambda)\mathbf{\Psi}_{-\alpha}^+(x,\lambda) + \delta\rho_\alpha^-(\lambda)\mathbf{\Psi}_{\alpha}^-(x,\lambda)\right) \\ + \sum_{k=1}^{N}\sum_{\alpha\in\Delta_1^+}\left(\delta\tilde{W}_{-\alpha;j}^+(x) - \delta\tilde{W}_{\alpha;j}^-(x)\right), \tag{94}$$

where

$$\delta W^{\pm}_{\pm\alpha;j}(x) = \delta\lambda^{\pm}_j \tau^{\pm}_{\alpha;j} \dot{\mathbf{\Phi}}^{\pm}_{\pm\alpha;j}(x) + \delta\tau^{\pm}_{\alpha;j} \mathbf{\Phi}^{\pm}_{\pm\alpha;j}(x), \tag{95}$$

$$\delta \tilde{W}^{\pm}_{\mp\alpha;j}(x) = \delta\lambda^{\pm}_j \rho^{\pm}_{\alpha;j} \dot{\mathbf{\Psi}}^{\pm}_{\mp\alpha;j}(x) + \delta\rho^{\pm}_{\alpha;j} \mathbf{\Psi}^{\pm}_{\mp\alpha;j}(x) \tag{96}$$

and $\mathbf{\Phi}^{\pm}_{\pm\alpha;j}(x) = \mathbf{\Phi}^{\pm}_{\pm\alpha}(x, \lambda^{\pm}_j)$, $\dot{\mathbf{\Phi}}^{\pm}_{\pm\alpha;j}(x) = \partial_\lambda \mathbf{\Phi}^{\pm}_{\pm\alpha}(x, \lambda)|_{\lambda=\lambda^{\pm}_j}$.

The expansions (91), (92) is another way to establish the one-to-one correspondence between $Q(x)$ and each of the minimal sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$ (32). Likewise the expansions (93), (94) establish the one-to-one correspondence between the variation of the potential $\delta Q(x)$ and the variations of the scattering data $\delta\mathcal{T}_1$ and $\delta\mathcal{T}_2$.

The expansions (93), (94) have a special particular case when one considers the class of variations of $Q(x,t)$ due to the evolution in t. Then

$$\delta Q(x,t) \equiv Q(x, t+\delta t) - Q(x,t) = \frac{\partial Q}{\partial t}\delta t + (O)((\delta t)^2). \tag{97}$$

Assuming that δt is small and keeping only the first order terms in δt we get the expansions for $\mathrm{ad}_J^{-1} Q_t$. They are obtained from (93), (94) by replacing $\delta\rho^{\pm}_\alpha(\lambda)$ and $\delta\tau^{\pm}_\alpha(\lambda)$ by $\partial_t\rho^{\pm}_\alpha(\lambda)$ and $\partial_t\rho^{\pm}_\alpha(\lambda)$.

4.2 The generating operators

To complete the analogy between the standard Fourier transform and the expansions over the 'squared solutions' we need the analogs of the operator $D_0 = -\mathrm{i}\mathrm{d}/\mathrm{d}x$. The operator D_0 is the one for which $\mathrm{e}^{\mathrm{i}\lambda x}$ is an eigenfunction: $D_0 \mathrm{e}^{\mathrm{i}\lambda x} = \lambda \mathrm{e}^{\mathrm{i}\lambda x}$. Therefore it is natural to introduce the generating operators $\Lambda_\pm$ through:

$$\begin{aligned} &(\Lambda_+ - \lambda)\mathbf{\Psi}^+_{-\alpha}(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda)\mathbf{\Psi}^-_{\alpha}(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda^{\pm}_j)\mathbf{\Psi}^+_{\mp\alpha;j}(x) = 0, \\ &(\Lambda_- - \lambda)\mathbf{\Phi}^+_{\alpha}(x,\lambda) = 0, \qquad (\Lambda_- - \lambda)\mathbf{\Phi}^-_{-\alpha}(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda^{\pm}_j)\mathbf{\Phi}^+_{\pm\alpha;j}(x) = 0. \end{aligned} \tag{98}$$

where the generating operators $\Lambda_\pm$ are given by:

$$\Lambda_\pm X(x) \equiv \mathrm{ad}_J^{-1}\left(\mathrm{i}\frac{\mathrm{d}X}{\mathrm{d}x} + \mathrm{i}\left[Q(x), \int_{\pm\infty}^{x} \mathrm{d}y\,[Q(y), X(y)]\right]\right). \tag{99}$$

The rest of the squared solutions are not eigenfunctions of neither Λ_+ nor Λ_-:

$$\begin{aligned} (\Lambda_+ - \lambda^+_j)\dot{\mathbf{\Psi}}^+_{-\alpha;j}(x) &= \mathbf{\Psi}^+_{-\alpha;j}(x), \qquad (\Lambda_+ - \lambda^-_j)\dot{\mathbf{\Psi}}^-_{\alpha;j}(x) = \mathbf{\Psi}^-_{\alpha;j}(x), \\ (\Lambda_- - \lambda^+_j)\dot{\mathbf{\Phi}}^+_{ir;j}(x) &= \mathbf{\Phi}^+_{\alpha;j}(x), \qquad (\Lambda_- - \lambda^-_j)\dot{\mathbf{\Phi}}^-_{\alpha;j}(x) = \mathbf{\Phi}^-_{\alpha;j}(x), \end{aligned} \tag{100}$$

i.e., $\dot{\mathbf{\Psi}}^+_{\alpha;j}(x)$ and $\dot{\mathbf{\Phi}}^+_{\alpha;j}(x)$ are adjoint eigenfunctions of Λ_+ and Λ_-. This means that $\lambda^{\pm}_j$, $j = 1, \ldots, N$ are also the discrete eigenvalues of $\Lambda_\pm$ but the corresponding eigenspaces of $\Lambda_\pm$ have double the dimensions of the ones of L; now they are spanned by both $\mathbf{\Psi}^{\pm}_{\mp\alpha;j}(x)$ and $\dot{\mathbf{\Psi}}^{\pm}_{\mp\alpha;j}(x)$. Thus the sets $\{\Psi\}$ and $\{\Phi\}$ are the complete sets of eigen- and adjoint functions of Λ_+ and Λ_-.

Therefore the completeness relation (88) can be viewed as the spectral decompositions of the recursion operators $\Lambda_\pm$. It is also obvious that the continuous spectrum of these operators fills up the real axis $\mathbb{R}$ and has multiplicity $2n$; theis discrete spectra consists of the eigenvalues $\lambda^{\pm}_k$ and each of them has multiplicity 2.

5 Resolvent and spectral decompositions in the spinor representation of $\mathfrak{g} \simeq B_r$

Using the general theory one can calculate the explicit form of the Cartan-Weyl basis in the spinor representations. In appendices B and C below we give the results for $r = 2$ and $r = 3$. Therefore in the spinor representation the Lax operators take the form:

$$L_{\rm sp}\psi_{\rm sp} = i\frac{\partial\psi_{\rm sp}}{\partial x} + (Q_{\rm sp} - \lambda J_{\rm sp})\psi_{\rm sp}(x,\lambda) = 0, \tag{101}$$

where $Q_{\rm sp}(x,t)$ and $J_{\rm sp}$ are $2^r \times 2^r$ matrices of the form:

$$Q_{\rm sp} = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{q}^\dagger & 0 \end{pmatrix}, \qquad J_{\rm sp} = \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \tag{102}$$

where the explicit form of $\boldsymbol{q}(x)$ for $r = 2$ and $r = 3$ is given by eqs. (137) and (142) below.

It is well known [5] that the spinor representations of $so(2r+1)$ are realized by symplectic (resp. orthogonal) matrices if $r(r+1)/2$ is odd (resp. even). Thus in what follows we will view the spinor representations of $so(2r+1)$ as typical representations of $sp(2^r)$ (resp. $so(2^r)$) algebra. Combined with the corresponding value of $J_{\rm sp}$ (137) one can conclude that that the Lax operator $L_{\rm sp}$ for odd values of $r(r+1)/2$ can be related to the C.III-type symmetric spaces. The potential $Q_{\rm sp}(x)$ however is not a generic one; it may be obtained from the generic potential as a special reduction, which picks up $so(2r+1)$ as the subalgebra of $sp(2^r)$. Below we will construct an automorphism whose kernel will pick up $so(2r+1)$ as a subalgebra of $sp(2^r)$.

Similarly the Lax operator $L_{\rm sp}$ above for even values of $r(r+1)/2$ can be related to the $so(2^r)$ algebra. The element $J_{\rm sp}$ (137) is characteristic for the **D.III**-type symmetric spaces. The potential $Q_{\rm sp}(x)$ may be obtained from the generic potential as a special reduction, which picks up $so(2r+1)$ as the subalgebra of $so(2^r)$.

The spectral problem (101) is technically more simple to treat. It has the form of block-matrix AKNS which means that the corresponding Jost solutions and FAS are determined as follows:

$$\begin{aligned}
&\psi(x,\lambda) \underset{x\to\infty}{\simeq} e^{-i\lambda Jx}, && \phi(x,\lambda) \underset{x\to-\infty}{\simeq} e^{-i\lambda Jx}, \\
&T(\lambda) = \begin{pmatrix} \boldsymbol{a}^+ & -\boldsymbol{b}^- \\ \boldsymbol{b}^+ & \boldsymbol{a}^- \end{pmatrix}, && \\
&\psi(x,\lambda) = (\psi^-(x,\lambda), \psi^+(x,\lambda)), && \phi(x,\lambda) = (\phi^+(x,\lambda), \phi^-(x,\lambda)), \\
&\chi^+(x,\lambda) = (\phi^+(x,\lambda), \psi^+(x,\lambda)), && \chi^-(x,\lambda) = (\psi^-(x,\lambda), \phi^-(x,\lambda)),
\end{aligned} \tag{103}$$

5.1 The Gauss factors in the spinor representation

The spectral theory of the Lax operators related to the symmetric spaces of C.III and D.III types were developed in [20, 36]. What is different here is the special choice of the rank and the additional reduction $\pi_{\rm B_r}$ which picks up the spinor representation of $so(2r+1)$. In our considerations below we will assume that this reduction is applied. So though all our $2^r \times 2^r$ matrices are split into blocks of dimension $2^{r-1} \times 2^{r-1}$, the

corresponding group (resp. algebraic) elements belong to the (spinor representation of) group $SO(2r+1)$ (resp. algebra $so(2r+1)$). Thus we define the FAS of $L_{\rm sp}$ by:

$$\begin{aligned}\chi^+_{\rm sp}(x,\lambda) &\equiv \left(|\phi^+\rangle, |\psi^+\hat{c}^+\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^+_{\rm sp}(\lambda) = \psi_{\rm sp}(x,\lambda)\boldsymbol{T}^-_{\rm sp}(\lambda)D^+_{\rm sp}(\lambda),\\ \chi^-_{\rm sp}(x,\lambda) &\equiv \left(|\psi^-\hat{c}^-\rangle, |\phi^-\rangle\right)(x,\lambda) = \phi(x,\lambda)\boldsymbol{S}^-_{\rm sp}(\lambda) = \psi_{\rm sp}(x,\lambda)\boldsymbol{T}^+_{\rm sp}(\lambda)D^-_{\rm sp}(\lambda),\end{aligned} \tag{104}$$

where the block-triangular functions $\boldsymbol{S}^\pm_{\rm sp}(\lambda)$ and $\boldsymbol{T}^\pm_{\rm sp}(\lambda)$ are given by:

$$\begin{aligned}\boldsymbol{S}^+_{\rm sp}(\lambda) &= \begin{pmatrix} \mathbb{1} & \boldsymbol{d}^-\hat{\boldsymbol{c}}^+(\lambda)\\ 0 & \mathbb{1}\end{pmatrix}, \qquad \boldsymbol{T}^-_{\rm sp}(\lambda) = \begin{pmatrix} \mathbb{1} & 0\\ \boldsymbol{b}^+\hat{\boldsymbol{a}}^+(\lambda) & \mathbb{1}\end{pmatrix},\\ \boldsymbol{S}^-_{\rm sp}(\lambda) &= \begin{pmatrix} \mathbb{1} & 0\\ -\boldsymbol{d}^+\hat{\boldsymbol{c}}^-(\lambda) & \mathbb{1}\end{pmatrix}, \qquad \boldsymbol{T}^+_{\rm sp}(\lambda) = \begin{pmatrix} \mathbb{1} & -\boldsymbol{b}^-\hat{\boldsymbol{a}}^-(\lambda)\\ 0 & \mathbb{1}\end{pmatrix},\end{aligned} \tag{105}$$

The matrices $D^\pm_{\rm sp}(\lambda)$ are block-diagonal and equal:

$$D^+_{\rm sp}(\lambda) = \begin{pmatrix} \boldsymbol{a}^+(\lambda) & 0\\ 0 & \hat{\boldsymbol{c}}^+(\lambda)\end{pmatrix}, \qquad D^-_{\rm sp}(\lambda) = \begin{pmatrix} \hat{\boldsymbol{c}}^-(\lambda) & 0\\ 0 & \boldsymbol{a}^-(\lambda)\end{pmatrix}. \tag{106}$$

The supper scripts $\pm$ here refer to their analyticity properties for $\lambda \in \mathbb{C}_\pm$.

All factors $\boldsymbol{S}^\pm_{\rm sp}$, $\boldsymbol{T}^\pm_{\rm sp}$ and $D^\pm_{\rm sp}$ take values in the spinor representation of the group $SO(2r+1)$ and are determined by the minimal sets of scattering data (42). Besides, since:

$$\begin{aligned}T_{\rm sp}(\lambda) &= \boldsymbol{T}^-_{\rm sp}(\lambda)D^+_{\rm sp}(\lambda)\hat{\boldsymbol{S}}^+_{\rm sp}(\lambda) = \boldsymbol{T}^+_{\rm sp}(\lambda)D^-_{\rm sp}(\lambda)\hat{\boldsymbol{S}}^-_{\rm sp}(\lambda),\\ \hat{T}_{\rm sp}(\lambda) &= \boldsymbol{S}^+_{\rm sp}(\lambda)\hat{D}^+_{\rm sp}(\lambda)\hat{\boldsymbol{T}}^-_{\rm sp}(\lambda) = \boldsymbol{S}^-_{\rm sp}(\lambda)\hat{D}^-_{\rm sp}(\lambda)\hat{\boldsymbol{T}}^+_{\rm sp}(\lambda).\end{aligned} \tag{107}$$

we can view the factors $\boldsymbol{S}^\pm_{\rm sp}$, $\boldsymbol{T}^\pm_{\rm sp}$ and $D^\pm_{\rm sp}$ as generalized Gauss decompositions (see [32]) of $T_{\rm sp}(\lambda)$ and its inverse.

From eqs. (104), (105) one can derive:

$$\chi^+_{\rm sp}(x,\lambda) = \chi^-_{\rm sp}(x,\lambda)G_{0,\rm sp}(\lambda), \qquad \chi^-_{\rm sp}(x,\lambda) = \chi^+_{\rm sp}(x,\lambda)\hat{G}_{0,\rm sp}(\lambda), \tag{108}$$

$$G_{0,\rm sp}(\lambda) = \begin{pmatrix} \mathbb{1} & \tau^+\\ \tau^- & \mathbb{1}+\tau^-\tau^+\end{pmatrix}, \qquad \hat{G}_{0,\rm sp}(\lambda) = \begin{pmatrix} \mathbb{1}+\tau^+\tau^- & -\tau^+\\ -\tau^- & \mathbb{1}\end{pmatrix} \tag{109}$$

valid for $\lambda \in \mathbb{R}$. Below we introduce:

$$X^\pm_{\rm sp}(x,\lambda) = \chi^\pm_{\rm sp}(x,\lambda)\mathrm{e}^{i\lambda Jx}. \tag{110}$$

Strictly speaking it is $X^\pm_{\rm sp}(x,\lambda)$ that allow analytic extension for $\lambda \in \mathbb{C}_\pm$. They have also another nice property, namely their asymptotic behavior for $\lambda \to \pm\infty$ is given by:

$$\lim_{\lambda\to\infty} X^\pm_{\rm sp}(x,\lambda) = \mathbb{1}, \tag{111}$$

Along with $X^\pm_{\rm sp}(x,\lambda)$ we can use another set of FAS $\tilde{X}^\pm_{\rm sp}(x,\lambda) = X^\pm_{\rm sp}(x,\lambda)\hat{D}^\pm_{\rm sp}$, which also satisfy eq. (111) due to the fact that:

$$\lim_{\lambda\to\infty} D^\pm_{\rm sp}(\lambda) = \mathbb{1}. \tag{112}$$

The equations (108) and (109) can be written down as:

$$X^+_{\rm sp}(x,\lambda) = X^-_{\rm sp}(x,\lambda)G_{\rm sp}(x,\lambda), \qquad \lambda \in \mathbb{R} \tag{113}$$

with

$$\tilde{G}_{\rm sp}(x,\lambda) = {\rm e}^{-i\lambda Jx}\tilde{G}_{0,{\rm sp}}(\lambda)e^{i\lambda Jx} \qquad \tilde{G}_{0,{\rm sp}}(\lambda) = \begin{pmatrix} \mathbb{1} + \rho^-\rho^+ & \rho^- \\ \rho^+ & \mathbb{1} \end{pmatrix}. \tag{114}$$

Equations (113) combined with (111) are known as a Riemann-Hilbert problem (RHP) with canonical normalization [13]. It has unique regular solution; the matrix-valued solutions $X^+_{0,{\rm sp}}(x,\lambda)$ and $X^-_{0,{\rm sp}}(x,\lambda)$ of (113), (111) is called regular if $\det X^\pm_{0,{\rm sp}}(x,\lambda)$ does not vanish for any $\lambda \in \mathbb{C}_\pm$.

One can derive the following integral decomposition for $X^\pm_{\rm sp}(x,\lambda)$:

$$X^+_{\rm sp}(x,\lambda) = \mathbb{1} + \frac{1}{2\pi {\rm i}}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}X^-_{\rm sp}(x,\mu)K_1(x,\mu) + \sum_{j=1}^{N}\frac{X^-_{j,{\rm sp}}(x)K_{1,j}(x)}{\lambda^-_j - \lambda} \tag{115}$$

$$X^-_{\rm sp}(x,\lambda) = \mathbb{1} + \frac{1}{2\pi {\rm i}}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}X^-_{\rm sp}(x,\mu)K_2(x,\mu) - \sum_{j=1}^{N}\frac{X^+_{j,{\rm sp}}(x)K_{2,j}(x)}{\lambda^+_j - \lambda} \tag{116}$$

where $X^\pm_{j,{\rm sp}}(x) = X^\pm_{\rm sp}(x,\lambda^\pm_j)$ and

$$K_{1,j}(x) = {\rm e}^{-{\rm i}\lambda^-_j Jx}\begin{pmatrix} 0 & \rho^+_j \\ \tau^-_j & 0 \end{pmatrix}{\rm e}^{{\rm i}\lambda^-_j Jx} \quad K_{2,j}(x) = {\rm e}^{-{\rm i}\lambda^+_j Jx}\begin{pmatrix} 0 & \tau^+_j \\ \rho^-_j & 0 \end{pmatrix}{\rm e}^{{\rm i}\lambda^+_j Jx}. \tag{117}$$

Equations (115), (116) can be viewed as a set of singular integral equations which are equivalent to the RHP. For the MNLS these were first derived in [38].

Finally, the potential $Q(x,t)$ can be recovered from the solutions $X^\pm_{\rm sp}(x,\lambda)$ of RHP (113) with a canonical normalisation (111). Skipping the details, we provide here only the final result:

$$Q_{\rm sp}(x,t) = \lim_{\lambda\to\infty}\lambda(J - X^\pm_{\rm sp}(x,\lambda)J\hat{X}^\pm_{\rm sp}(x,\lambda)]) = [J, X_1(x)]. \tag{118}$$

where $X_1(x) = \lim_{\lambda\to\infty}(X_{\rm sp}(x,\lambda) - \mathbb{1})$.

5.2 Definition and properties of $R^\pm(x,y,\lambda)$

The resolvent $R_{\rm sp}(\lambda)$ of $L_{\rm sp}$ is again expressed through the FAS in the form:

$$R_{\rm sp}(\lambda)f(x) = \int_{-\infty}^{\infty} R_{\rm sp}(x,y,\lambda)f(y). \tag{119}$$

where $R_{\rm sp}(x,y,\lambda)$ are given by:

$$R_{\rm sp}(x,y,\lambda) = \begin{cases} R^+_{\rm sp}(x,y,\lambda) \text{ for } \lambda \in \mathbb{C}^+, \\ R^-_{\rm sp}(x,y,\lambda) \text{ for } \lambda \in \mathbb{C}^-, \end{cases} \tag{120}$$

and

$$R^\pm_{\rm sp}(x,y,\lambda) = \pm i\chi^\pm_{\rm sp}(x,\lambda)\Theta^\pm(x-y)\hat{\chi}^\pm_{\rm sp}(y,\lambda), \qquad \Theta^\pm(z) = \begin{pmatrix} \theta(\mp z)\mathbb{1} & 0 \\ 0 & -\theta(\pm z)\mathbb{1} \end{pmatrix} \tag{121}$$

5.3 The dressing factors in the spinor representation

The asymptotics of the dressing factor for $x \to \pm\infty$ are:

$$u_\pm = \exp\left(\ln c_1(\lambda) J_{\rm sp}\right), \tag{122}$$

where

$$u_+ = \begin{pmatrix} \sqrt{c_1}\mathbb{1}_{2^r} & 0 \\ 0 & 1/\sqrt{c_1}\mathbb{1}_{2^{r-1}} \end{pmatrix}, \qquad u_- = \begin{pmatrix} 1/\sqrt{c_1}\mathbb{1}_{2^{r-1}} & 0 \\ 0 & \sqrt{c_1}\mathbb{1}_{2^{r-1}} \end{pmatrix}. \tag{123}$$

Thus the asymptotics of the projectors P and $\bar{P}$ in the spinor representation are projectors of rank 2^{r-1}. Since the dynamics of the MNLS does not change the rank of the projectors we conclude that the dressing factor in the spinor representation must be of the form:

$$u(x,\lambda) = \exp\left(\frac{1}{2}\ln c_1(\lambda)(P_1 - \bar{P}_1)\right) = \sqrt{c_1(\lambda)}P_1(x,t) + \frac{1}{\sqrt{c_1(\lambda)}}\bar{P}_1(x,t) \tag{124}$$

dressing factors of such form with non-rational dependence on λ were considered for the first time in by Ivanov [33] for the MNLS related to symplectic algebras. Here we see that such construction can be used also for the orthogonal algebras.

However when it comes to analyze the relations between the 'naked' and the dressed scattering matrices and their Gauss factors we get integer powers of $c_1(\lambda)$. Indeed, for the simplest case when the dressing procedure is applied just once we have:

$$\begin{aligned} T_{\rm sp}(\lambda) &= \hat{u}_+ T_{0,\rm sp}(\lambda) u_-(\lambda), & D^\pm_{\rm sp}(\lambda) &= \hat{u}_+ D^\pm_{0,\rm sp}(\lambda) u_-(\lambda), \\ T^\pm_{\rm sp}(\lambda) &= \hat{u}_+ T^\pm_{0,\rm sp}(\lambda) u_+(\lambda), & S^\pm_{\rm sp}(\lambda) &= \hat{u}_- S^\pm_{0,\rm sp}(\lambda) u_-(\lambda), \end{aligned} \tag{125}$$

and as a consequence

$$\begin{aligned} \boldsymbol{a}^+_{\rm sp}(\lambda) &= \frac{1}{c_1^+(\lambda)}\boldsymbol{a}^+_{0,\rm sp}(\lambda), & \boldsymbol{a}^-_{\rm sp}(\lambda) &= c_1^+(\lambda)\boldsymbol{a}^-_{0,\rm sp}(\lambda), \\ \rho^+_{\rm sp}(\lambda) &= \frac{1}{c_1^+(\lambda)}\rho^+_{0,\rm sp}(\lambda), & \rho^-_{\rm sp}(\lambda) &= c_1^+(\lambda)\rho^-_{0,\rm sp}(\lambda), \\ \tau^+_{\rm sp}(\lambda) &= \frac{1}{c_1^+(\lambda)}\tau^+_{0,\rm sp}(\lambda), & \tau^-_{\rm sp}(\lambda) &= c_1^+(\lambda)\tau^-_{0,\rm sp}(\lambda). \end{aligned} \tag{126}$$

5.4 The spectral decompositions of $L_{\rm sp}$

Again apply the contour integration method to the kernel $R^\pm_{\rm sp}$ and derive the following completeness relation:

$$\begin{aligned} \delta(x-y)\mathbb{1}_{2^r} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left\{ |\phi^+(x,\lambda)\rangle \hat{\boldsymbol{a}}^+(\lambda)\langle\psi^+(y,\lambda)| - |\phi^-(x,\lambda)\rangle\hat{\boldsymbol{a}}^-(\lambda)\langle\psi^-(y,\lambda)| \right\} \\ + \sum_{j=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_j^+} R^+(x,y) + \operatorname*{Res}_{\lambda=\lambda_j^-} R^-(x,y) \right), \end{aligned} \tag{127}$$

The residues in (127) can be expressed by the properly normalized eigenfunctions of $L_{\rm sp}$ corresponding to the eigenvalues $\lambda_j^\pm$.

Thus we conclude that the continuous spectrum of $L_{\rm sp}$ fills up $\mathbb{R}$ and has multiplicity 2^r. The discrete eigenvalues $\lambda_k^\pm$ are simple poles of the resolvent.

6 Conclusions

We have analyzed the spectral properties of the Lax operators related to three different representations of $\mathfrak{g} \simeq B_r$: the typical, the adjoint and the spinor representation. In all these cases the spectral properties such as: i) the multiplicity of the continuous spectra and of the discrete eigenvalues; ii) the explicit form of the dressing factors; iii) the completeness relations of the eigenfunctions are substantially different. However the minimal sets of scattering data $\mathfrak{T}_i$ are provided by the same sets of functions, i.e. the sets $\mathfrak{T}_i$ are invariant with respect to the choice of the representation of $\mathfrak{g}$.

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A The adjoint representations of $so(5)$ and $so(7)$

The adjoint representations of all orthogonal algebras $so(2r+1)$ and $so(2r)$ are characterized by the fundamental weight $\omega_2 = e_1 + e_2$. By definition the corresponding weight system is

$$\Gamma_{\omega_2} \equiv \Delta \cup \{0\}_r, \tag{128}$$

where Δ is the root system of the algebra and $\{0\}$ is a vanishing weight with multiplicity r. We will order the weights in Γ_{ω_2} according to their scalar products with e_1.

Let us consider in more detail the two special cases of $so(2r+1)$ with $r = 2$ and $r = 3$. For $r = 2$ the adjoint representation is 10-dimensional. Ordering the roots as mentioned above we get:

$$\Delta_1^+ \simeq \{e_1+e_2, e_1, e_1-e_2\}, \qquad \Delta_0 \simeq \{e_2, -e_2\}, \qquad \Delta_1^- \simeq \{-e_1+e_2, e_1, -e_1-e_2\}, \tag{129}$$

As a result the element J in the adjoint representation takes the form:

$$J_{\rm ad} = \begin{pmatrix} \mathbb{1}_3 & 0 & 0 \\ 0 & 0_6 & 0 \\ 0 & 0 & -\mathbb{1}_3 \end{pmatrix}, \qquad Q_{\rm ad} = \begin{pmatrix} 0 & Q_{\rm ad;12} & 0 \\ Q_{\rm ad;21} & 0_6 & Q_{\rm ad;23} \\ 0 & Q_{\rm ad;32} & 0 \end{pmatrix}, \tag{130}$$

Analogously for $r = 3$ the adjoint representation is 21-dimensional. Ordering the roots as mentioned above we get:

$$\begin{aligned} &\Delta_1^+ \simeq \{e_1+e_2, e_1+e_3, e_1, e_1-e_3, e_1-e_2\}, \qquad \Delta_0 \simeq \{e_2 \pm e_3, -(e_2 \pm e_3)\}, \\ &\Delta_1^- \simeq \{-e_1+e_2, -e_1+e_3, e_1, -e_1-e_3, -e_1-e_2\}, \end{aligned} \tag{131}$$

As a result the element J in the adjoint representation takes the form:

$$J_{\rm ad} = \begin{pmatrix} \mathbb{1}_5 & 0 & 0 \\ 0 & 0_{11} & 0 \\ 0 & 0 & -\mathbb{1}_5 \end{pmatrix}, \qquad Q_{\rm ad} = \begin{pmatrix} 0 & Q_{\rm ad;12} & 0 \\ Q_{\rm ad;21} & 0_{11} & Q_{\rm ad;23} \\ 0 & Q_{\rm ad;32} & 0 \end{pmatrix}, \tag{132}$$

It is not difficult to write down the explicit form of $Q_{\rm ad}$ but it will not be necessary. We have used above a more compact realization of $Q_{\rm ad}$ as ad_Q.

B The spinor representation of $so(5)$

The highest weight and the weight system of $so(5)$ are given by [32]. It is well known that $so(5) \simeq sp(4)$ so the spinor representation of $so(5)$ is realized through symplectic $sp(4)$ matrices.

$$\begin{aligned} \omega_2 \equiv \gamma_1 &= \frac{1}{2}(e_1 + e_2), & \gamma_2 &= \frac{1}{2}(e_1 - e_2), \\ \gamma_3 &= -\gamma_2, & \gamma_4 &= -\gamma_1. \end{aligned} \tag{133}$$

The Cartan-Weyl basis of $so(5)$ is given by

$$\begin{aligned} E_{e_1-e_2} &= \Gamma_{2,\bar{2}} = \mathcal{E}_{2\epsilon_2}, & E_{e_1+e_2} &= \Gamma_{1,\bar{1}} = \mathcal{E}_{2\epsilon_1}, \\ E_{e_1} &= \Gamma_{1,\bar{2}} - \Gamma_{2,\bar{1}} = \mathcal{E}_{\epsilon_1+\epsilon_2}, & E_{e_2} &= \Gamma_{1,2} - \Gamma_{2,\bar{1}} = \mathcal{E}_{\epsilon_1-\epsilon_2}, \\ H_{e_1} &= \frac{1}{2}(\Gamma_{1,1} + \Gamma_{2,2}), & H_{e_2} &= \frac{1}{2}(\Gamma_{1,1} - \Gamma_{2,2}), \end{aligned} \tag{134}$$

where $\bar{k} = 5 - k$ and

$$\Gamma_{k,p} = |\gamma_k\rangle\langle\gamma_p| \qquad 1 \le k \le p \le 4. \tag{135}$$

By $\mathcal{E}_{\epsilon_i \pm \epsilon_j}$ above we have denoted the Weyl generators of $sp(4)$. The Lax operator L in the spinor representation of $so(5)$ takes the form:

$$L_{\rm sp}\psi_{\rm sp} = i\frac{\partial \psi_{\rm sp}}{\partial x} + (Q_{\rm sp} - \lambda J_{\rm sp})\psi_{\rm sp}(x,\lambda) = 0, \tag{136}$$

where $Q_{\rm sp}(x,t)$ and $J_{\rm sp}$ are 4×4 symplectic matrices of the form:

$$Q_{\rm sp} = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{q}^\dagger & 0 \end{pmatrix}, \qquad J_{\rm sp} = \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \boldsymbol{q}(x,t) = \begin{pmatrix} q_0 & q_{\bar{1}} \\ q_1 & -q_0 \end{pmatrix}, \tag{137}$$

C The spinor representation of $so(7)$.

The highest weight and the weight system of $so(7)$ are given by [32]

$$\begin{aligned} \omega_3 \equiv \gamma_1 &= \frac{1}{2}(e_1 + e_2 + e_3), & \gamma_2 &= \frac{1}{2}(e_1 + e_2 - e_3), \\ \gamma_3 &= \frac{1}{2}(e_1 - e_2 + e_3), & \gamma_4 &= \frac{1}{2}(e_1 - e_2 - e_3), \\ \gamma_5 &= -\gamma_4, \quad \gamma_6 = -\gamma_3, & \gamma_7 &= -\gamma_2, \quad \gamma_8 = -\gamma_1. \end{aligned} \tag{138}$$

The Cartan-Weyl basis of $so(7)$ is given by

$$\begin{aligned}
E_{e_1-e_2} &= \Gamma_{3,\bar{4}} = \mathcal{E}_{\epsilon_3+\epsilon_4}, & E_{e_2-e_3} &= \Gamma_{2,3} = \mathcal{E}_{\epsilon_2-\epsilon_3},\\
E_{e_1-e_3} &= \Gamma_{2,\bar{4}} = \mathcal{E}_{\epsilon_2+\epsilon_4}, & E_{e_1+e_2} &= \Gamma_{1,\bar{2}} = \mathcal{E}_{\epsilon_2+\epsilon_4},\\
E_{e_1+e_3} &= \Gamma_{1,\bar{3}} = \mathcal{E}_{\epsilon_1+\epsilon_3}, & E_{e_2+e_3} &= \Gamma_{1,4} = \mathcal{E}_{\epsilon_1-\epsilon_4},\\
E_{e_1} &= \Gamma_{1,\bar{4}} + \Gamma_{2,\bar{3}} = \mathcal{E}_{\epsilon_1+\epsilon_4} + \mathcal{E}_{\epsilon_2+\epsilon_3}, & E_{e_2} &= \Gamma_{1,3} + \Gamma_{2,4} = \mathcal{E}_{\epsilon_1-\epsilon_4} + \mathcal{E}_{\epsilon_2-\epsilon_4},\\
E_{e_3} &= \Gamma_{1,2} - \Gamma_{3,4} = \mathcal{E}_{\epsilon_1-\epsilon_2} - \mathcal{E}_{\epsilon_3-\epsilon_4}, & H_{e_1} &= \frac{1}{2}(\Gamma_{1,1} + \Gamma_{2,2} + \Gamma_{3,3} + \Gamma_{4,4}),\\
H_{e_1} &= \frac{1}{2}(\Gamma_{1,1} + \Gamma_{2,2} - \Gamma_{3,3} - \Gamma_{4,4}), & H_{e_1} &= \frac{1}{2}(\Gamma_{1,1} - \Gamma_{2,2} + \Gamma_{3,3} - \Gamma_{4,4}),
\end{aligned} \tag{139}$$

where $\bar{k} = 9 - k$ and

$$\begin{aligned}
\Gamma_{k,p} &= |\gamma_k\rangle\langle\gamma_p| - (-1)^{k+p}|\gamma_{\bar{p}}\rangle\langle\gamma_{\bar{k}}|, \qquad 1 \le k \le p \le 4,\\
\Gamma_{k,\bar{p}} &= |\gamma_k\rangle\langle\gamma_p| + (-1)^{k+p}|\gamma_{\bar{p}}\rangle\langle\gamma_{\bar{k}}|, \qquad 1 \le k \le p \le 4,
\end{aligned} \tag{140}$$

Note that the typical representation of $so(8)$ is also 8-dimensional. So by $\mathcal{E}_{\epsilon_i \pm \epsilon_j}$ above we have denoted the Weyl generators of $so(8)$. So we can also consider the spinor representation of $so(7)$ as an imbedding of $so(7)$ into $so(8)$. Such imbedding can be realized by taking the average of the Cartan-Weyl basis of $so(8)$ with respect to the $\mathbb{Z}_2$ external automorphism of $so(8)$ (the mirror reflection) which changes $\epsilon_4 \leftrightarrow -\epsilon_4$. In short the Lax operator L in the spinor representation of $so(7)$ take the form:

$$L_{\rm sp}\psi_{\rm sp} = i\frac{\partial \psi_{\rm sp}}{\partial x} + (Q_{\rm sp} - \lambda J_{\rm sp})\psi_{\rm sp}(x,\lambda) = 0, \tag{141}$$

where $Q_{\rm sp}(x,t)$ and $J_{\rm sp}$ are 8×8 matrices of the form:

$$Q_{\rm sp} = \begin{pmatrix} 0 & \boldsymbol{q} \\ \boldsymbol{q}^\dagger & 0 \end{pmatrix}, \qquad J_{\rm sp} = \frac{1}{2}\begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & -\mathbb{1}_4 \end{pmatrix}, \qquad \boldsymbol{q}(x,t) = \begin{pmatrix} q_0 & q_{\bar{2}} & q_{\bar{1}} & 0 \\ q_2 & q_0 & 0 & -q_{\bar{1}} \\ q_1 & 0 & -q_0 & q_{\bar{2}} \\ 0 & -q_1 & q_2 & -q_0 \end{pmatrix}. \tag{142}$$

Our final remark here concerns the spinor representation with highest weight $\omega = 3\omega_3$. Then

$$\Gamma \simeq \left\{ \frac{3}{2}(\pm e_1 \pm e_2 \pm e_3), \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3) \right\} \tag{143}$$

The dimension of this representation is 8. The explicit form of the element J in this representation is $J = \mathrm{diag}\,(\frac{3}{2}\mathbb{1}, \frac{1}{2}\mathbb{1}, -\frac{3}{2}\mathbb{1}, -\frac{1}{2}\mathbb{1})$, i.e. all eigenvalues of J are non-vanishing. The potential Q will have block-matrix structure compatible with the one of J.

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