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Brendan Goldsmith
Dublin Institute of Technology, brendan.goldsmith@dit.ie

A. L. S. Corner

R. Gobel
Universitat Duisberg Essen

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On Torsion-free Crawley Groups *

In Memoriam: John T. Lewis, friend, colleague and outstanding mathematician.

A.L.S. Corner

10, Cornwall Street, St. Thomas,

Exeter EX4 1BU, England

e-mail: alscorner@waitrose.com

R. Göbel

FB6, Mathematik, Universität Duisburg Essen,

45117 Essen, Germany

e-mail: R.Goebel@uni-essen.de

and

B. Goldsmith

School of Mathematical Sciences, Dublin Institute of Technology,

Kevin Street, Dublin 8,Ireland

e-mail: brendan.goldsmith@dit.ie

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Abstract

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The notion of a Crawley p-group is well known in Abelian group theory. In this present work a corresponding concept is introduced for torsion-free groups. The principal result, which uses the set-theoretic notions of Diamond and Martin's Axiom, establishes an independence result for $\aleph_1$-free Crawley groups.

Introduction.

The notion of a Crawley group in the theory of separable Abelian p-groups is well known; recall that a p-group is said to be separable if $p^\omega G = 0$ and G is a Crawley group if it has the property that all p-groups A with $p^\omega A \cong \mathbb{Z}(p)$, the cyclic group of order p, and $A/p^\omega A \cong G$ are isomorphic. In the 1960's Crawley raised the question of whether every such group is necessarily a direct sum of cyclic groups. Megibben, in an elegant and surprising paper [9] in 1983, showed that the answer to Crawley's question is independent of the usual Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC); specifically he showed that in Gödel's Constructible Universe (V=L) (but in fact using only Jensen's diamond principle $\diamondsuit$) Crawley's question has an affirmative answer while this is not so if one assumes Martin's Axiom and the negation of the Continuum Hypothesis (MA + $\neg$ CH). A surprising feature of Megibben's work was that while the Crawley problem was about extensions, it was not, unlike the Whitehead Problem, equivalent to the vanishing of some group of extensions $\mathrm{Ext}(A, B)$. Further details and more recent developments on Crawley p-groups may be found in the papers of Mekler and Shelah [10], [11].

If we attempt to generalize the notion of a Crawley p-group to the category of torsion-free groups, it is clear that there are several possibilities including *inter alia*, the difficult question of determining for a given torsion-free group G, when all extensions of $\mathbb{Z}$ by G are isomorphic. Our choice of generalization is made easier by an observation of Megibben in [9] on earlier work of Richman [12]: a separable p-group G is a Crawley group if and only if the automorphism group of G acts transitively on the dense subsocles of codimension one of G. Accordingly we make the following definition.

Definition: A torsion-free Abelian group G is said to be a *Crawley* group if, given any pair of pure, dense subgroups of corank 1 in G, there is an automorphism of G mapping one onto the other.

It is clear that an entirely analogous definition of a torsion-free Crawley module can be made for modules over an integral domain. Indeed this type of generalization and a conjecture similar to our main results (Theorems 1 and 2) has been made by Luigi Salce. Salce's conjecture cannot be answered fully by our results, since we are unable to use Martin's Axiom in its usual form when working over uncountable rings.

The class of all Crawley groups is shown to be extensive with many possibilities arising from the fact that there are $2^{\aleph_0}$ non-isomorphic rank 1 groups, all of which are Crawley groups. To make the investigation of Crawley groups of infinite rank more tractable, we 'remove' the problems associated with the varying types of rank 1 groups by restricting to groups which are *almost free* in the sense of Eklof and Mekler [3].

Our principal results for almost free Crawley groups (Theorems 1 and 2) show that a situation analogous to that for separable p-groups holds: we show, *inter alia*, that assuming the diamond principle $\diamondsuit$, every $\aleph_1$-free Crawley group of size $\aleph_1$ is free but assuming (MA + $\neg$ CH) there exists a non-free, but strongly $\aleph_1$-free group of size $\aleph_1$ which is a Crawley group. It is, perhaps, worth remarking that our proofs are much more transparent than the corresponding ones for torsion groups.

Our notation is largely in accord with the standard works of Fuchs [6], [7]. Details of concepts such as $\aleph_1$-freeness etc. may be found in the work of Eklof and Mekler [3].

Preliminary Results.

Our primary focus in this paper is groups of infinite rank but there is, of course, nothing in the definition of a Crawley group which requires it to be of infinite rank;

indeed all reduced torsion-free groups of rank 1 are trivially Crawley groups since they have no pure dense subgroups of corank 1 while it is immediate from the fact that $\mathbb{Q}$ is pure simple, that it too is a Crawley group. Our first result gives us an elementary way to 'build up' larger Crawley groups.

Proposition 1. If G is a Crawley group then the direct sum $K = G \oplus F$, where F is free of finite rank, is again a Crawley group and so there exist completely decomposable Crawley groups of any finite rank.

Proof. Suppose that M is maximal pure dense in K. We claim that $G \not\leq M$, for if not $M/G = M + G/G \leq K/G$ and $K/G/M/G \cong K/M \cong \mathbb{Q}$. But $K/G \cong F$ is finitely generated whereas $\mathbb{Q}$ is not — contradiction. Thus $G \not\leq M$ and then we have that $G/G \cap M \cong G + M/M = K/M \cong \mathbb{Q}$ where the equality $G + M = K$ follows since the quotient $K/G + M$ is simultaneously divisible and finitely generated, and hence is zero. Thus $G \cap M$ is maximal pure dense in G. Moreover $M/G \cap M \cong M + G/G \cong K/G \cong F$ and so M splits as $M = (G \cap M) \oplus F_M$, where $F_M \cong F$. Notice that $K = G + F_M$ since the quotient $K/G + F_M$ is again both divisible and finitely generated. Thus $K = G + F_M = G \oplus F_M$. Now if N is a second maximal pure dense subgroup we also get that $N = (G \cap N) \oplus F_N$ and $K = G \oplus F_N$. Since G is a Crawley group there is an automorphism θ of G with $(G \cap M)\theta = G \cap N$; adding this (directly) to an isomorphism $F_M \cong F_N$ gives the required automorphism of K sending M onto N.

The final assertion is immediate since any rank one group is a Crawley group.

Example 1. For each positive integer n, there are indecomposable homogeneous groups C and N, of type $\mathbb{Z}$ and of rank n, such that C is a Crawley group but N is not.

Proof: Let C be the so-called Pontryagin group of rank n - see [7, Example 5 p.125] - is homogeneous of type $\mathbb{Z}$ and is indecomposable since it has endomorphism ring isomorphic to $\mathbb{Z}$. But the group is trivially a Crawley group since it has no epimorphic images equal to $\mathbb{Q}$ - see Exercise 6, p.128 in [7].

The construction of N is based on the well-known realization theorem in [2]. Let

P be the pure subring of $\hat{\mathbb{Z}}$ constructed in Lemma 1.5 of that paper and choose n algebraically independent elements $\pi_i (i = 1, 2, ...n)$ of P. Now take N to be the pure subgroup of $P^{(n)}$ generated by $\mathbb{Z}^{(n)}$ and the element $(\pi_1, \pi_2, ...\pi_n)$. It is easy to see that N is homogeneous of type $\mathbb{Z}$ and that $N/\mathbb{Z}^{(n)} \cong \mathbb{Q}$. A standard argument using algebraic independence shows that the endomorphism ring of N is $\mathbb{Z}$. However any maximal pure subgroup isomorphic to $\mathbb{Z}^{(n)}$ will also be dense; this follows from Corner's observation described in Exercise 13 of [7]. Since N has only ± 1 as automorphisms, it is clearly not a Crawley group.

Note: A group which is vacuously a Crawley group i.e. it does not have $\mathbb{Q}$ as an epimorphic image, is necessarily of finite rank: this follows immediately from the fact that $\mathbb{Q}$ is injective and the fact that a free group of infinite rank has $\mathbb{Q}$ as an epimorphic image.

Example 2. The completely decomposable group $\mathbb{Z}_{(p)} \oplus \mathbb{Q}^{(p)}$ is a Crawley group.

Proof: A routine calculation shows that the only mappings from $\mathbb{Z}_{(p)} \oplus \mathbb{Q}^{(p)}$ *onto* $\mathbb{Q}$ have the form $(x, y) \mapsto \alpha x + \beta y$ where $0 \neq \alpha \in \mathbb{Z}_{(p)}$ and $0 \neq \beta \in \mathbb{Q}^{(p)}$. Hence the only maximal pure dense subgroups correspond to kernels of such maps. But now it follows by direct calculation that any such kernel has the form $\{(x, y) \mid \alpha x = -\beta y\} = (\alpha^{-1}, \beta^{-1})\mathbb{Z}$, where α, β are units in the appropriate groups. To show that the group is a Crawley group, it suffices to show that we can find an automorphism mapping $(1, 1)\mathbb{Z}$ to $(\alpha^{-1}, \beta^{-1})\mathbb{Z}$. This however is immediate since we can express the automorphism group as matrices in the usual way .

In fact the argument in the proof of Example 2 carries over *mutatis mutandis* to give the following, once we recall that groups which do not have $\mathbb{Q}$ as an epimorphic image are trivially Crawley groups:

Proposition 2. If A, B are rank 1 groups with $\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A)$, then $A \oplus B$ is a Crawley group.

It is now rather easy to describe completely the situation for completely decomposable groups of rank 2.

Proposition 3. If $G = A \oplus B$ is a completely decomposable group of rank 2, then G is a Crawley group.

Proof: We give a sketch of the proof by looking briefly at the various possibilities: Case(i) $B = \mathbb{Q}$. In this case we can show, as in Example 2, that the only maximal pure dense subgroups occur as kernels of maps of the form $\alpha x + \beta y$ where α, β are in $\mathbb{Q}$. However since the automorphism group contains $\mathrm{Hom}(A, \mathbb{Q})$ it is now easy to construct the desired automorphism.

Case(ii) $t(A) \leq t(B)$. In this case it is immediate that $A \oplus B$ has $\mathbb{Q}$ as an epimorphic image if, and only if $B = \mathbb{Q}$. Hence either G is trivially a Crawley group or the result follows from case(i).

Case(iii) $t(A), t(B)$ are incomparable. In this case the result follows immediately from Proposition 2.

Even for completely decomposable groups the situation becomes quite complex once we exceed rank 2.

Example 3. There is a completely decomposable group of rank 3 which is not a Crawley group.

Proof: For $i = 1, 2, 3$, let R_i be the rank 1 group whose type consists entirely of symbols ∞, except in the i^{th}-place where it is 0. Set $G = R_1 \oplus R_2 \oplus R_3$. Then the direct sum of R_3 and the kernel of the map: $R_1 \oplus R_2 \to \mathbb{Q}$ given by $(x, y) \mapsto x + y$ is a maximal pure dense subgroup of G which is not even isomorphic to the maximal pure dense subgroup of G obtained as the direct sum of R_1 and the kernel of the corresponding map from $R_2 \oplus R_3 \to \mathbb{Q}$.

Infinite Rank Crawley Groups.

If we now turn our attention to groups of infinite rank, we can exhibit many groups which are Crawley groups. Our starting point is, inevitably, based on a well-known result of J.Erdös [4](or see [5, §51]).

Example 4. If G is free of arbitrary rank, then G is a Crawley group.

Proof: If G has finite rank then the result follows from Proposition 1. If rk$G$ is infinite and M,N are pure,dense subgroups of corank 1 in G, then $G/M \cong G/N \cong$

$\mathbb{Q}$. Hence rk$M$ = rk$N$ = rk$G$ and so by a theorem of J.Erdös [4], there is an automorphism θ of G with $M\theta = N$.

This example is, in fact, a particular case of the following proposition which uses a generalization of Erdös's Theorem to completely decomposable homogeneous groups. We are grateful to the referee for suggesting the more natural approach below which replaces our original *ad hoc* one.

Proposition 4. If G is a homogeneous completely decomposable group, then G is a Crawley group.

Proof: Since a pure subgroup of a finite rank homogeneous completely decomposable group is necessarily a direct summand [7, Lemma 86.8], all finite rank homogeneous completely decomposable groups are trivially Crawley groups. Furthermore if G is divisible then the result follows from elementary arguments for vector spaces. So we may assume G is reduced and of infinite rank. Clearly it will suffice to show that Erdös's Theorem holds for completely decomposable homogeneous groups.

It is well known and easy to show that Erdös's Theorem generalizes to homogeneous completely decomposable modules of reduced (or in the terminology of [1], non-nil) type since the role of the ring $\mathbb{Z}$ is taken by the so-called nucleus of G which is then a principal ideal domain. So suppose that $G = \bigoplus_{i \in I} A_i$, where each $A_i = A$, is completely decomposable of type A and that $R = \mathrm{End}_{\mathbb{Z}}(A)$ is the reduced type of A. Now if H_1, H_2 are pure subgroups of G satisfying the conditions of Erdös's Theorem, then we have the exact sequences

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A, H_i) \to \mathrm{Hom}_{\mathbb{Z}}(A, G) \to \mathrm{Hom}_{\mathbb{Z}}(A, G/H_i) \to 0$$

since by an easy extension of Baer's Lemma, [7, Lemma 86.4], maps may be lifted from $\mathrm{Hom}_{\mathbb{Z}}(A, G/H_i)$ to $\mathrm{Hom}_{\mathbb{Z}}(A, G)$; note that $\mathrm{Hom}_{\mathbb{Z}}(A, G)$ is a free R-module and that $\mathrm{Hom}_{\mathbb{Z}}(A, G/H_i) \cong G/H_i$. Everything is now an R-module and so, by the module version of Erdös's Theorem, we get an equivalence of $\mathrm{Hom}_{\mathbb{Z}}(A, H_1)$ and $\mathrm{Hom}_{\mathbb{Z}}(A, H_2)$. However in this situation – see e.g.[1, §5] –we have a natural equivalence $\theta_G : \mathrm{Hom}_{\mathbb{Z}}(A, G) \otimes_R A \to G$ and applying this to the commutative diagrams associated to the above, we get the desired automorphism of G mapping H_1 onto H_2.

It is, however, not too difficult to exhibit Crawley groups which are completely decomposable but not homogeneous: if $C = \bigoplus_{\aleph_0} R$, where R is any rank one group with type $t(R) > t(\mathbb{Z})$, then C is a Crawley group and hence, by Proposition 1, so also is the completely decomposable, non-homogeneous group $C \oplus \mathbb{Z}$.

In Proposition 1 we showed that the class of Crawley groups is closed under the (direct) addition of free groups of finite rank. On the surface this seems a rather weak result but our next example shows that it is the best result of this type that we can achieve. In fact it is possible to give a simpler example by constructing a group of the form $C \oplus F$, where C is a Crawley group and F is free of infinite rank but $C \oplus F$ has non-isomorphic maximal pure, dense subgroups e.g. take C as in Example 2 above. However, even if the maximal pure, dense subgroups are all isomorphic, the sum of a Crawley group and F need not be Crawley.

Example 5. If $G = F \oplus V$, where F is free of countable rank and V is a countable dimension $\mathbb{Q}$-vector space, then G is not a Crawley group.

Proof: The automorphism group of G may be regarded as 2×2 matrices in which the first column has the form $(\theta, 0)^T$ where θ is an automorphism of F. If M is a maximal pure dense subgroup of G of the form $F \oplus W$, where W is a subspace of codimension 1 of V, it follows that any image of M under an automorphism of G must have a first component equal to F. However if we choose $H \leq F$ with $F/H \cong \mathbb{Q}$, then $N = H \oplus V$ is a maximal pure dense subgroup of G which is not the image of M under any automorphism of G.

Notice that it follows from Proposition 1, that the group G of Example 5 is the union of an ascending chain of summands, each of which is a Crawley group, but G itself is not a Crawley group.

In fact it is possible to extend the arguments used in Example 5 to get a great deal more information about direct sums and summands of Crawley groups.

We begin with:

Proposition 5. A characteristic summand of a Crawley group is a Crawley group.

Proof: Let $G = H \oplus K$ be a Crawley group where K is a characteristic subgroup. Let $\varphi_1, \varphi_2 : K \to \mathbb{Q}$ be surjections. Clearly it suffices to show that there is an automorphism α of K mapping $\mathrm{Ker}\varphi_1$ onto $\mathrm{Ker}\varphi_2$. The homomorphisms φ_i of K extend to surjections $\psi_i : G \to \mathbb{Q}$ vanishing on H. Since G is a Crawley group, there is an automorphism θ of G mapping $\mathrm{Ker}\psi_1$ onto $\mathrm{Ker}\psi_2$. However K is invariant under both θ and θ^{-1} and so the restriction $\alpha = \theta \restriction K$ is an automorphism of K, and since $\mathrm{Ker}\varphi_i \leq K$, we have $K \cap \mathrm{Ker}\psi_i = K \cap (\mathrm{Ker}\varphi_i \oplus H) = \mathrm{Ker}\varphi_i \oplus (K \cap H) = \mathrm{Ker}\varphi_i$. But then $(\mathrm{Ker}\varphi_1)\alpha = (\mathrm{Ker}\varphi_1)\theta = K\theta \cap (\mathrm{Ker}\psi_1)\theta = K \cap \mathrm{Ker}\psi_2 = \mathrm{Ker}\varphi_2$ as required.

Proposition 6. If $G = F \oplus N$ where F is free of infinite rank and N has trivial dual, then G is a Crawley group if, and only if $N = 0$.

Proof: The sufficiency is immediate so suppose G is a Crawley group. If N has infinite rank then, as noted earlier, there is a surjection from N onto $\mathbb{Q}$. Moreover the automorphism group of G can be represented as 2×2 upper triangular matrices and then an identical argument to that used in Example 5, shows that G is not a Crawley group — contradiction. So it suffices to consider the case where N has finite rank.

Suppose then, for a contradiction, that $N \neq 0$. Let $\varphi : F \to \mathbb{Q}$ be any surjection with kernel F'. Then $F' \oplus N$ is a maximal pure dense subgroup of G. Now choose any maximal pure subgroup M of N (note that M need not be dense); then N/M is of rank 1, $\mathrm{rk}M < \mathrm{rk}N$ and M is the kernel of a non-zero homomorphism ν from N into $\mathbb{Q}$. Now define $\theta : G \to \mathbb{Q}$ to be the surjection which agrees with φ on F and ν on N. Write $H = \mathrm{Ker}\theta$ so that H is a maximal pure dense subgroup of G. Now $H \cap N = \mathrm{Ker}\nu = M$ and so it follows readily that $H = F_1 \oplus M$ where F_1 is free. Since G is a Crawley group, H must be isomorphic (indeed equivalent) to $F' \oplus N$. This implies N is isomorphic to a subgroup of $F_1 \oplus M$ and so as before, $N = F_2 \oplus (N \cap M) = F_2 \oplus M$. Since M has rank strictly less than N, F_2 is necessarily non-zero which contradicts the hypothesis that N has trivial dual. Thus we conclude that $N = 0$.

A corresponding result for the situation where the free group F is of finite rank is easily established.

Proposition 7. If $G = F \oplus N$ where F is free of finite rank and N has trivial dual, then G is a Crawley group if, and only if N is a Crawley group.

Proof: If N is a Crawley group the result follows immediately from Proposition 1. The converse follows directly from Proposition 5 since in this case N is even fully invariant.

Propositions 6 and 7 give us some insight into countable torsion-free Crawley groups since we have, by Stein's Theorem, that every such countable group G has the form $G = F \oplus N$ where F is free and N has trivial dual; we refer to F and N as the free and trivial-dual parts of G respectively. Thus we have

Corollary 8. If G is a countable torsion-free group then

(i) if the free part of G is of infinite rank, G is a Crawley group if, and only if the trivial-dual part of G is zero.

(ii) if the free part of G is of finite rank, G is a Crawley group if, and only if the trivial-dual part of G is a Crawley group.

Note that if a countable torsion-free group has a free summand of infinite rank, then it is a Crawley group if, and only if it is free. We shall see shortly that the situation is much more complicated for uncountable groups.

Almost Free Crawley Groups.

We have seen in the previous section that Crawley groups exist in abundance but note that we have not given any example of an $\aleph_1$-free Crawley group other than a free group. (We shall see shortly in Theorem 1 why this is so.) By restricting to $\aleph_1$-free groups we obtain a significant simplification in the structure of Crawley groups.

Example 6. There exist separable $\aleph_1$-free groups which are not Crawley groups.

Proof: Let $G = \prod_{\aleph_0} \mathbb{Z}$ be the Baer-Specker group which is well known to be $\aleph_1$-free. However the automorphism group of G cannot act transitively on the dense maximal pure subgroups: there are $2^{2^{\aleph_0}}$ such subgroups but the full endomorphism ring, and hence, *a fortiori*, the automorphism group, has power at most the continuum since the integers are a slender group.

Our next result is technical and follows from a well-known fact about countable extensions of free groups.

Proposition 9. Suppose that G is an $\aleph_1$-free group of arbitrary cardinality and M is a free subgroup of G with $G/M \cong \mathbb{Q}$, then G is free also.

Proof: The result follows immediately from the more general fact that if G is an arbitrary group having a free subgroup M with countable quotient G/M, then $G = M_1 \oplus N$ where M_1 is a summand of M and N is countable: for if A is the subgroup generated by the pre-images of G/M, then $G = A + M$. Moreover since $A \cap M$ is countable, there is a decomposition $M = M_0 \oplus M_1$ with $A \cap M \leq M_0$. Set $N = A + M_0$, a countable group and note that $G = N + M_1$ while $N \cap M_1 = 0$. Thus $G = N \oplus M_1$ as required.

Suppose now that G is κ-free of cardinality κ but not free, where κ is a regular cardinal; choose a continuous filtration $G = \bigcup_{\alpha<\kappa} G_\alpha$ where each G_α is free; we may assume without loss that $\mathrm{rk}(G_{\alpha+1}/G_\alpha) = 1$ for each α. If M is a fixed dense maximal pure subgroup of G then we get an induced filtration $M = \bigcup_{\alpha<\kappa} M_\alpha$, where $M_\alpha = G_\alpha \cap M$.

Let $E = \{\alpha < \kappa \mid M_{\alpha+1}/M_\alpha \ \text{ is not free}\}$; we claim E is stationary in κ. For if not, there is a cub C in κ with $C \cap E = \emptyset$. Since C is closed and unbounded we obtain a continuous filtration $M = \bigcup_{\alpha \in C} M_\alpha$. But $\alpha \in C$ implies $\alpha \notin E$ and so $M_{\alpha+1}/M_\alpha$ is free for each $\alpha \in C$. Since M_α is also free this implies that $\bigcup_{\alpha \in C} M_\alpha$ is a continuous chain of free groups with successive quotients being free groups, and so by [3, IV, Prop.1.7] M is free and hence by Proposition 9 above, G would also be free — contradiction. Thus E is stationary. Hence we may partition E into two disjoint stationary subsets $E = E_1 \cup E_2$; notice that E_1 is now a stationary set with unbounded complement in κ.

Theorem 1. ($\diamondsuit_\kappa$) If G is a κ-free Crawley group of cardinality κ, then G is free.

Proof: If $\kappa = \aleph_0$ then G is free by the Pontryagin Criterion, while if κ is singular the result follows from the Singular Compactness Theorem, see [3]. Now let κ be regular and assume for a contradiction that G is not free and fix a dense maximal pure subgroup $M \leq G$. Choose filtrations as above with associated stationary sets E, E_1 of κ. Fix $z \in G \setminus M$ so that $G = \langle M, z\rangle_*$.

Since we are assuming $\diamondsuit_\kappa$, we have that $\diamondsuit_{E_1}$ holds and so we can find Jensen functions $f_\alpha : G_\alpha \to G_\alpha$ such that for every function $g : G \to G$, the set $\{\alpha \mid g \restriction G_\alpha = f_\alpha\}$ is stationary.

We now construct a new maximal pure dense subgroup N of G such that $M\theta \neq N$ for every automorphism θ of G; the resulting contradiction will show that G is free as required.

We obtain N as the union of an ascending sequence of free subgroups $(N_\alpha)_{\alpha<\kappa}$ such that:

(0) $z \notin N_\alpha$

(1) $N_0 = \langle M_0, n!z_n - z(n \in \omega)\rangle_*$

where $z_n \in G$ is chosen so that $n!z_n \equiv z \bmod M$ — note that such elements exist since G/M is divisible

(2) $N_\mu = \bigcup_{\alpha<\mu} N_\alpha$ if μ is a limit ordinal

(3) assuming N_α has been constructed, lift back a maximal linearly independent subset of the quotient $(M_{\alpha+1} + N_\alpha)/N_\alpha$ to a family of elements $\{x_i\}$ in $M_{\alpha+1}$ and set $N_{\alpha+1} = \langle N_\alpha, \{x_i - z\}\rangle_*$

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(3(a)) $\alpha \in E_1, M_\alpha f_\alpha = N_\alpha$ and $f_\alpha = \varphi \restriction G_\alpha$ for some $\varphi \in \mathrm{Aut(G)}$ with $z \notin M\varphi$, in which case we choose $x_\alpha \in M$ with $x_\alpha \in M_{\alpha+1} \setminus M_\alpha$, write $y_\alpha = x_\alpha\varphi$ and set $N_{\alpha+1} = \langle N_\alpha, y_\alpha - z\rangle_*$.

The only condition needed for consistency of this construction is that $z \notin N_\alpha$ for any α. Trivially $z \notin N_0$ while $z \in N_{\alpha+1}$ in condition (3(a)) would violate the hypothesis $z \notin M\varphi$. The remaining case (3) is disposed of by an elementary argument on linear independence. Set $N = \bigcup_{\alpha<\kappa} N_\alpha$.

Note that as E_2 is unbounded either version of (3) guarantees that $\langle N, z\rangle_* \geq M$; since $z \notin N$ it follows that N has corank 1 in G, and our choice of N_0 ensures that N is dense in G.

Now suppose that there exists an automorphism θ of G such that $M\theta = N$. Then the set $C = \{\alpha \mid M_\alpha\theta = N_\alpha\}$ is a cub and, since the set $X = \{\alpha \in E_1 \mid \theta \restriction G_\alpha = f_\alpha\}$ is stationary in κ, there exists β such that $\theta \restriction G_\beta = f_\beta$ and $M_\beta\theta = N_\beta$. Note also that $z \notin M\theta$ can be assumed: the intersection of the cub C and the stationary set X is, in fact, stationary so that $z \in M\theta$ would imply $z \in M_\gamma\theta$ for some γ and hence $z \in M_\beta\theta$ for all $\beta \geq \gamma$. But now any $\beta \in C \cap X$ with $\beta \geq \gamma$ would imply $z \in M_\beta\theta = N_\beta$ — contradiction.

Hence the construction of $N_{\beta+1}$ must have taken place as in condition (3(a)); so $y_\beta = x_\beta\varphi$ where φ in $\mathrm{Aut}(G)$ agrees with θ on M_β since both are equal to f_β on M_β. Thus $(\varphi-\theta) \restriction M_\beta$ is zero and so $(\varphi-\theta)$ induces a map: $M_{\beta+1}/M_\beta \to N_{\beta+1}$, which in turn must be zero since $\beta \in E_1$ implies $M_{\beta+1}/M_\beta$ is a rank 1 group not isomorphic to $\mathbb{Z}$ and $N_{\beta+1}$ is free. So we have $y_\beta = x_\beta\theta \in M\theta = N$. However $y_\beta - z \in N$ and this forces $z \in N$ — contradiction. Thus G is free as required.

We are now in a position to establish the independence result discussed in the introduction.

We begin with a simple but crucial observation concerning $\aleph_1$-coseparable groups; recall that a group G is said to be $\aleph_1$-coseparable if it is $\aleph_1$-free and if H is a subgroup of G with G/H countable, then there is a direct summand K of G, with $K \leq H$ and G/K countable. It is well known that this is equivalent to $\mathrm{Ext}(G, \mathbb{Z}^{(\omega)}) = 0$. Note that this next result holds in ZFC.

Proposition 10. If G is an $\aleph_1$-coseparable group then G is a Crawley group.

Proof: Suppose M, N are maximal pure dense in G. Then $G/M \cap N$ is isomorphic to a subgroup of $G/M \oplus G/N \cong \mathbb{Q} \oplus \mathbb{Q}$ and so $\mid G/M \cap N \mid = \aleph_0$. Hence there is a summand K of G with $K \leq M \cap N$ and G/K is countable. Let $G = K \oplus F$ where F, being countable, is free. But now $M = K \oplus (F \cap M)$, $N = K \oplus (F \cap N)$ and the quotients are both isomorphic to $\mathbb{Q}$. Thus $\mathrm{rk}(F \cap M) = \mathrm{rk}(F \cap N) = \aleph_0$ and so, by the result of Erdös [4] already cited, there is an isomorphism $\theta : F \to F$ such that $(F \cap M)\theta = F \cap N$. But then the map $1_K \oplus \theta$ is an automorphism of G mapping M onto N. Thus G is a Crawley group as required.

Theorem 2. (MA + ¬ CH) For any uncountable cardinal $\kappa < 2^{\aleph_0}$, there exists an $\aleph_1$-free Crawley group G of cardinality κ which is not free.

Proof: Let G be any strongly $\aleph_1$-free group of power κ which is not free; such groups exist, since by [3, VII, 1.3], there is a strongly $\aleph_1$-free group of cardinality $\aleph_1$, and hence the direct sum of this group and a free group of rank κ has the desired properties. But G is then a Shelah group — see e.g. [3, XII, 2.4]. However, since we are assuming (MA + ¬ CH), it follows from [3, XII, 2.5] that $\mathrm{Ext}(G, \mathbb{Z}^{(\omega)}) = 0$ and so G is in fact $\aleph_1$-coseparable. This, of course, means by Proposition 10 above, that G is a Crawley group.

Remark. An examination of the proof of Theorem 1 shows that it will hold for modules over an arbitrary domain R provided that

(i) the quotient ring of R is countably generated

(ii) the notion of $\aleph_1$-freeness for such modules is in accord with the approach in [3, IV, 1.1]

(iii) $\mathrm{Hom}(I, R) = 0$ for all rank 1 modules I not isomorphic to R.

In particular the result holds for modules over uncountable rings such as $\widehat{\mathbb{Z}_p}$, the ring of p-adic integers or complete discrete valuation domains. However, we cannot use Theorem 2 to resolve the situation for modules over a complete discrete valuation domain since such domains are necessarily uncountable; see however the forthcoming paper of Göbel and Shelah [8]. Theorem 2 will, of course, hold for modules over any domain which is not left perfect and has cardinality $< 2^{\aleph_0}$.

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