

## TRANSITIVE AND FULLY TRANSITIVE GROUPS

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ABSTRACT. The notions of transitivity and full transitivity for abelian  $p$ -groups were introduced by Kaplansky in the 1950s. Important classes of transitive and fully transitive  $p$ -groups were discovered by Hill, among others. Since a 1976 paper by Corner, it has been known that the two properties are independent of one another. We examine how the formation of direct sums of  $p$ -groups affects transitivity and full transitivity. In so doing, we uncover a far-reaching class of  $p$ -groups for which transitivity and full transitivity are equivalent. This result sheds light on the relationship between the two properties for all  $p$ -groups.

### 1. INTRODUCTION

Throughout this note, we will denote the  $p$ -height sequence of an element  $x$  in a  $p$ -local abelian group  $G$  by  $U_G(x)$  or simply  $U(x)$ . Recall that  $G$  is *transitive* if  $x$  can be mapped to  $y$  by an automorphism of  $G$  whenever  $x, y \in G$  satisfy  $U(x) = U(y)$ ; and *fully transitive* if this can be accomplished by an endomorphism of  $G$  whenever  $U(x) \leq U(y)$  pointwise. Extensive classes of abelian  $p$ -groups with both transitivity properties—including separable and totally projective  $p$ -groups—are set forth in [Co], [Gr], [Hi] and [Ka]. Examples of  $p$ -groups with neither of the properties are given in [Me] and [Hi].

Can an abelian  $p$ -group be fully transitive but not transitive, or vice versa? In the earliest account [Ka], Kaplansky proved that transitive  $p$ -groups are indeed fully transitive provided  $p \neq 2$ . More than twenty years later, however, Corner [Co] answered the question in the negative by constructing fully transitive  $p$ -groups which fail to be transitive, and a transitive 2-group which is not fully transitive.

Despite the independence of transitivity and full transitivity for abelian  $p$ -groups, it has become increasingly clear (see e.g. [CaGo]) that there is, indeed, some basic connection between the two. The most striking of the results in this present note is the fundamental, but apparently unknown

**Corollary 3.** *A  $p$ -group  $G$  is fully transitive if and only if its square  $G \oplus G$  is transitive.*

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In Theorem 1, we will set forth an extensive class of  $p$ -groups for which transitivity and full transitivity are equivalent. Corollary 3 is symptomatic of the fact that this class contains the square of every abelian  $p$ -group.

Throughout this note all groups are reduced  $p$ -local abelian groups, and we refer to them simply as groups. Notation follows the standard works of Fuchs [Fu] and Kaplansky [Ka] with the exception that maps are written on the right; all undefined terms may be found in these references.

## 2. TRANSITIVITY AND FULL TRANSITIVITY

It was shown by Megibben [Me, Theorem 2.4] that the direct sum of two fully transitive  $p$ -groups need not be fully transitive. In order to obtain closure under direct sums, we first extend the notion of full transitivity in a fairly obvious way.

**Definition 1.** If  $G_1$  and  $G_2$  are groups, then  $\{G_1, G_2\}$  is a *fully transitive pair* if for every  $x \in G_i$ ,  $y \in G_j$  ( $i, j \in \{1, 2\}$ ) which satisfy  $U_{G_i}(x) \leq U_{G_j}(y)$ , there exists  $\alpha \in \text{Hom}(G_i, G_j)$  with  $x\alpha = y$ .

For example, it is an easy exercise to verify that  $\{G_1, G_2\}$  is a fully transitive pair whenever  $G_1$  and  $G_2$  are direct summands of a fully transitive group. The next result shows that all fully transitive pairs of  $p$ -groups arise in this way.

**Proposition 1.** *Let  $\{G_i\}_{i \in I}$  be a collection of  $p$ -groups such that for each  $i, j \in I$ ,  $\{G_i, G_j\}$  is a fully transitive pair. Then the (external) direct sum  $\bigoplus_{i \in I} G_i$  is fully transitive.*

*Proof.* It suffices to consider the case where  $I = \{1, \dots, n\}$  is finite. Denote  $G = G_1 \oplus \dots \oplus G_n$  and suppose  $x, y \in G$  satisfy  $U_G(x) \leq U_G(y)$ . We will obtain an endomorphism of  $G$  mapping  $x$  to  $y$  by inducting on the order of  $y$ . First suppose  $py = 0$ . Write  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ . By relabelling, we may assume that the  $p$ -heights satisfy  $ht_G(x) = ht_{G_1}(x_1)$ . Observe, since  $py = 0$ , that  $U_{G_1}(x_1) \leq U_G(y) \leq U_{G_i}(y_i)$  for all  $i$ . By assumption, there exist  $\alpha_i \in \text{Hom}(G_1, G_i)$  with  $x_1\alpha_i = y_i$  for  $1 \leq i \leq n$ . Clearly, the  $n \times n$  matrix with first row  $(\alpha_1, \dots, \alpha_n)$  and other rows zero represents an endomorphism of  $G$  mapping  $x$  to  $y$ .

Now assume  $o(y) > p$ . Note  $U_G(px) \leq U_G(py)$ . Since  $o(py) < o(y)$ , induction yields  $\theta \in \text{End}(G)$  with  $(px)\theta = py$ . Set  $x' = x\theta$ . Then  $y - x' \in G[p]$  and  $U_G(x) \leq U_G(y - x')$ ; hence by the first paragraph there exists  $\alpha \in \text{End}(G)$  with  $x\alpha = y - x'$ . Now  $\theta + \alpha$  maps  $x$  to  $x' + y - x' = y$ , as desired.  $\square$

We will require the following consequence of Proposition 1, indicating how a single fully transitive  $p$ -group can be used to produce many more.

**Corollary 1.** *Let  $G$  be a fully transitive  $p$ -group and  $\lambda$  any cardinal. Then all direct summands of the power  $\bigoplus_\lambda G$  are fully transitive.*

*Proof.* Because  $G$  is fully transitive,  $\{G, G\}$  is a fully transitive pair. Proposition 1 implies  $\bigoplus_\lambda G$  is fully transitive, and the claim follows since direct summands of fully transitive groups are fully transitive.  $\square$

In [Co, Proposition 2.2], Corner proves that if  $G = G_1 \oplus G_2$  is a fully transitive  $p$ -group such that the Ulm subgroups  $p^\omega G_1, p^\omega G_2$  are nontrivial and  $p^\omega G$  is homocyclic, then  $G$  is transitive. His example of a non-transitive, fully transitive  $p$ -group  $G$  is such that  $p^{\omega+1}G = 0$ , and he notes the ‘‘curious consequence’’ of Proposition 2.2 that  $G \oplus G$  must be transitive (note that it is fully transitive by Corollary 1

above). It was Corner’s result that motivated the theorem we shall soon prove. We need two preparatory lemmas.

**Lemma 1.** *Assume  $G = G_1 \oplus G_2$  is a fully transitive group and  $x_i, y_i \in G_i$  ( $i = 1, 2$ ). If  $U_{G_1}(x_1) \leq U_{G_2}(y_2 - x_2)$  and  $U_{G_2}(y_2) \leq U_{G_1}(y_1 - x_1)$ , then there is an automorphism of  $G$  mapping  $(x_1, x_2)$  to  $(y_1, y_2)$ .*

*Proof.* Because  $\{G_1, G_2\}$  is a fully transitive pair, there exist  $\alpha \in \text{Hom}(G_1, G_2)$  and  $\beta \in \text{Hom}(G_2, G_1)$  with  $x_1\alpha = y_2 - x_2$  and  $y_2\beta = y_1 - x_1$ . The matrix  $\phi = \begin{pmatrix} 1 + \alpha\beta & \alpha \\ \beta & 1 \end{pmatrix}$  represents an automorphism of  $G_1 \oplus G_2$ , and an easy check verifies that  $(x_1, x_2)\phi = (y_1, y_2)$ . □

If  $G$  is a group and  $\sigma$  an ordinal number, we use  $f_G(\sigma)$  to denote the classical Ulm invariant of  $G$  at  $\sigma$  (see [Fu] or [Ka]).

**Definition 2.** If  $G$  is a reduced group, the *Ulm support*  $\text{supp}(G)$  of  $G$  is the set of all ordinal numbers  $\sigma$  less than the  $p$ -length of  $G$  for which  $f_G(\sigma)$  is nonzero.

If  $G_1$  and  $G_2$  are  $p$ -groups with  $\text{supp}(G_1) \subseteq \text{supp}(G_2)$ , it follows that every  $U$ -sequence relative to  $G_1$  is also a  $U$ -sequence relative to  $G_2$ . In particular, we note that for every  $x \in G_1$  there is an element  $y \in G_2$  such that  $U_{G_1}(x) = U_{G_2}(y)$  (see [Ka, Lemma 24]). We employ this fact in the proof of the following crucial lemma.

**Lemma 2.** *Assume  $G = G_1 \oplus G_2$  is a fully transitive  $p$ -group and  $\text{supp}(p^\omega G_1) \subseteq \text{supp}(p^\omega G_2)$ . If  $x \in p^\omega G$ , there is an automorphism of  $G$  mapping  $x$  to an element  $(c, d) \in G_1 \oplus G_2$  with  $U_G(x) = U_{G_2}(d)$ .*

*Proof.* Write  $x = (a, b)$  and assume for the moment that we have shown that there exists an automorphism  $\phi$  of  $G$  with  $x\phi = (a_1, b_1)$  and  $ht_{G_1}(p^i a_1) \neq ht_{G_2}(p^i b_1)$  whenever  $p^i a_1 \neq 0$ . But then, as noted above, our assumption on the Ulm supports means we can choose  $b_2 \in p^\omega G_2$  such that  $U_{G_1}(a_1) = U_{G_2}(b_2)$ . By full transitivity,  $b_2 = a_1\alpha$  for some homomorphism  $\alpha : G_1 \rightarrow G_2$ . The composite automorphism  $\phi \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$  of  $G_1 \oplus G_2$  maps  $x$  to  $(a_1, b_1 + b_2)$ . Since  $ht(p^i b_1) \neq ht(p^i b_2)$  when  $p^i b_1 \neq 0$ , we compute

$$U_{G_2}(b_1 + b_2) = U_{G_2}(b_1) \wedge U_{G_2}(b_2) = U_{G_2}(b_1) \wedge U_{G_1}(a_1) = U_G(x).$$

Choosing  $d = (b_1 + b_2)$  gives the desired result. It remains only to establish the existence of the elements  $a_1, b_1$  and the automorphism  $\phi$  as above.

We prove this by induction on the maximum  $m$  of the set

$$\mathcal{S}_G(a, b) = \{i < \omega : ht_{G_1}(p^i a) = ht_{G_2}(p^i b) \neq \infty\}.$$

If  $\mathcal{S} = \emptyset$ , simply take  $\phi = 1_G$ . If  $m = 0$ , we proceed as follows. Clearly  $ht(pa) > ht(pb)$  or  $ht(pa) < ht(pb)$  by definition of  $\mathcal{S}_G(a, b)$ , say the former. Then  $ht_{G_1}(pa) > ht_{G_1}(a) + 1$ ; hence  $pa = pa_1$  for some  $a_1 \in G_1$  with  $ht(a_1) > ht(a)$ . Put  $b_1 = b$ . Clearly,  $ht(p^i a_1) = ht(p^i b_1)$  only if  $p^i a_1 = 0$ . Since  $U_{G_1}(a_1 - a) = (ht(a), \infty, \dots) \geq U_{G_2}(b_1)$  and  $\{G_1, G_2\}$  is a fully transitive pair, we have  $a_1 - a = b_1\alpha$  for some  $\alpha \in \text{Hom}(G_2, G_1)$ . The automorphism  $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$  of  $G_1 \oplus G_2$  maps  $(a, b)$  to  $(a_1, b_1)$  as desired. If  $ht(pa) < ht(pb)$ , we proceed as above to obtain a suitable automorphism of the form  $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ , finishing the case  $m = 0$ .

Now assume that  $\mathcal{S}_G(a, b)$  is nonempty and has maximum  $m > 0$ . Note that  $\mathcal{S}_G(pa, pb)$  has maximum  $< m$ . By induction, there exists  $\psi \in \text{Aut}(G)$  such that  $(pa, pb)\psi = (a_2, b_2)$  and  $ht(p^i a_2) \neq ht(p^i b_2)$  whenever  $p^i a_2 \neq 0$ . Set  $x' = (a', b') = x\psi$ . Because  $px' = (a_2, b_2)$ , it follows that  $\mathcal{S}_G(a', b')$  is empty or has maximum 0. By the above paragraph, there exists  $\phi \in \text{Aut}(G)$  such that  $(a', b')\phi = x\psi\phi = (a_1, b_1)$  and  $ht(p^i a_1) \neq ht(p^i b_1)$  whenever  $p^i a_1 \neq 0$ . This establishes our claim.  $\square$

Before turning to the main theorem, we observe a consequence of Lemmas 1 and 2 which indicates that a fully transitive  $p$ -group  $G$  with transitive direct summand  $H$  is itself transitive provided  $p^\omega G$  and  $p^\omega H$  have the same Ulm supports.

**Proposition 2.** *Assume  $G = G_1 \oplus G_2$  is a fully transitive  $p$ -group and  $\text{supp}(p^\omega G_1) \subseteq \text{supp}(p^\omega G_2)$ . If  $G_2$  is transitive, then  $G$  is transitive.*

*Proof.* By [Co, Lemma 2.1], we need only verify that  $\text{Aut}(G)$  acts transitively on  $p^\omega G$ . Suppose  $x, y \in p^\omega G$  have the same Ulm sequences in  $G$ . By Lemma 2 there exist  $\phi_1, \phi_2 \in \text{Aut}(G)$  such that if  $x\phi_1 = (x_1, x_2)$  and  $y\phi_2 = (y_1, y_2)$ , then  $U_{G_2}(x_2) = U_G(x) = U_G(y) = U_{G_2}(y_2)$ . Note that  $U_{G_2}(x_2) \leq U_{G_1}(y_1 - x_1)$ , since  $U_G(x_1 - y_1, x_2 - y_2) \geq U_G(x\phi_1)$ . Since  $G_2$  is transitive and  $G$  is fully transitive, there are  $\beta \in \text{Aut}(G_2)$  and  $\alpha \in \text{Hom}(G_2, G_1)$  with  $x_2\beta = y_2$  and  $x_2\alpha = y_1 - x_1$ . Put  $\psi = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$ , an automorphism of  $G$ . An easy check verifies that  $x\phi_1\psi\phi_2^{-1} = y$ , as required.  $\square$

The following result is complementary to Corollary 1.

**Corollary 2.** *Assume the  $p$ -group  $G$  is transitive and fully transitive. If  $\{H_i\}$  is a collection of direct summands of any power of  $G$ , then the external direct sum  $G \oplus (\bigoplus H_i)$  is transitive and fully transitive.*

*Proof.* Proposition 1 implies  $G \oplus (\bigoplus H_i)$  is fully transitive. Since  $\text{supp}(p^\omega G)$  contains  $\text{supp}(p^\omega H_i)$ , the direct sum is also transitive by Proposition 2.  $\square$

We now give a result indicating when transitivity and full transitivity are equivalent.

**Theorem 1.** *Assume  $G$  is a  $p$ -group which has a decomposition  $G = G_1 \oplus G_2$  such that  $p^\omega G_1$  and  $p^\omega G_2$  have the same Ulm supports. Then  $G$  is fully transitive if and only if  $G$  is transitive.*

*Proof.* Suppose  $G$  is fully transitive and that  $x, y \in p^\omega G$  satisfy  $U_G(x) = U_G(y)$ . By Lemma 2, there are  $\phi_1, \phi_2 \in \text{Aut}(G)$  such that  $x\phi_1 = (x_1, x_2)$ ,  $y\phi_2 = (y_1, y_2) \in G_1 \oplus G_2$  satisfy  $U_G(x) = U_{G_1}(x_1)$  and  $U_G(y) = U_{G_2}(y_2)$ . Because  $U_G(x) = U_G(y)$  we have  $U_{G_1}(x_1) \leq U_{G_2}(x_2), U_{G_2}(y_2)$ ; hence  $U_{G_1}(x_1) \leq U_{G_2}(y_2 - x_2)$ . Similarly,  $U_{G_2}(y_2) \leq U_{G_1}(y_1 - x_1)$ . The conditions of Lemma 1 are fulfilled, hence there exists  $\psi \in \text{Aut}(G)$  with  $(x_1, x_2)\psi = (y_1, y_2)$ . Now  $x\phi_1\psi\phi_2^{-1} = y$ , and we see that  $\text{Aut}(G)$  acts transitively on  $p^\omega G$ . By [Co, Lemma 2.1],  $G$  is transitive.

Conversely, assume  $G$  is transitive. Let  $B$  denote the square of the standard basic  $p$ -group. Then  $H = G \oplus B$  is transitive since  $B$  is separable ([CaGo, Proposition 2.6]). The structure of the groups  $B$  and  $p^\omega H = p^\omega G_1 \oplus p^\omega G_2$  implies that  $H$  has no Ulm invariants equal to one. Therefore  $H$  is fully transitive by [Ka, Theorem 26(b)], whence  $G$  is fully transitive.  $\square$

Theorem 1 has many corollaries. Corollary 3 is merely a noteworthy special case of Corollary 4.

**Corollary 3.** *A  $p$ -group  $G$  is fully transitive if and only if  $G \oplus G$  is transitive.*

**Corollary 4.** *The following conditions are equivalent for a  $p$ -group  $G$ .*

- (i) *For all cardinals  $\lambda$ ,  $\bigoplus_\lambda G$  is fully transitive.*
- (ii) *For some  $\lambda > 0$ ,  $\bigoplus_\lambda G$  is fully transitive.*
- (iii) *For all  $\lambda > 1$ ,  $\bigoplus_\lambda G$  is transitive.*
- (iv) *For some  $\lambda > 1$ ,  $\bigoplus_\lambda G$  is transitive.*

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (iv) are trivial. Assume (ii) holds, and  $\lambda > 1$  is a fixed cardinal. Note that  $G$  is a summand of a fully transitive group, hence is fully transitive. By Corollary 1,  $\bigoplus_\lambda G$  is fully transitive. Because  $\lambda > 1$  we can obviously decompose  $\bigoplus_\lambda G = G_1 \oplus G_2$  in such a way that

$$\text{supp}(p^\omega G_1) = \text{supp}(p^\omega G_2) = \text{supp}(p^\omega G).$$

Hence  $\bigoplus_\lambda G$  is transitive by Theorem 1. Therefore (iii) holds.

Finally, assume (iv) holds. Writing  $\bigoplus_\lambda G = G_1 \oplus G_2$  as above, it follows from Theorem 1 that  $\bigoplus_\lambda G$  is fully transitive since it is transitive. Therefore  $G$  is fully transitive, and Corollary 1 yields condition (i).  $\square$

Since transitive  $p$ -groups are fully transitive if  $p \neq 2$ , and squares of fully transitive  $p$ -groups are necessarily transitive, we obtain

**Corollary 5.** *For  $p \neq 2$ , the class of fully transitive  $p$ -groups is precisely the class of direct summands of transitive  $p$ -groups.*

Corner [Co] has given an example of a 2-group  $G$  which is transitive but not fully transitive. It follows from Corollary 4 that for all  $\lambda > 1$ , the power  $\bigoplus_\lambda G$  is neither transitive nor fully transitive. In particular, the square of a transitive 2-group need not be transitive. For  $p \neq 2$ , Corollary 2 implies that all powers of a transitive  $p$ -group are both transitive and fully transitive, simply because the group itself is also fully transitive in this case.

The final corollary extends Theorem 1 and Proposition 1 by exploiting Hill's powerful criteria for transitivity and full transitivity.

**Corollary 6.** *Let  $\{G_i\}_{i \in I}$  be a collection of  $p$ -groups. Assume there exists an ordinal  $\sigma$  such that  $G_i/p^\sigma G_i$  is totally projective and  $\{p^\sigma G_i, p^\sigma G_j\}$  is a fully transitive pair for each  $i, j \in I$ . Then  $\bigoplus_{i \in I} G_i$  is fully transitive. If there exists a partition  $I = J \cup K$  such that the groups  $\bigoplus_{i \in J} p^{\sigma+\omega} G_i$  and  $\bigoplus_{i \in K} p^{\sigma+\omega} G_i$  have equal Ulm supports, then  $\bigoplus_{i \in I} G_i$  is also transitive.*

*Proof.* Let  $G = \bigoplus_{i \in I} G_i$ . Proposition 1 shows that  $p^\sigma G = \bigoplus_{i \in I} p^\sigma G_i$  is fully transitive. Since  $\frac{G}{p^\sigma G} \cong \bigoplus_{i \in I} \frac{G_i}{p^\sigma G_i}$  is totally projective,  $G$  is fully transitive by [Hi, Theorem 4]. If the second condition in the corollary is also met, then  $p^\sigma G$  is transitive by Theorem 1, and it follows again from [Hi] that  $G$  is transitive.  $\square$

Observe that the total projectivity of the quotients  $G_i/p^\sigma G_i$  in Corollary 6 is automatic if  $\sigma$  is a finite ordinal. Megibben's result [Me, Theorem 2.4] and Hill's result [Hi, Theorem 6] demonstrate that Corollary 6 can fail if one of these quotients is not totally projective.

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