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Ruiyao Gao
Dublin Institute of Technology

Aidan O'Dwyer
Dublin Institute of Technology, aidan.odwyer@dit.ie

Seamus McLoone
NUI Maynooth

Eugene Coyle
Dublin Institute of Technology

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On Stability of Affine Blending Systems

Ruiyao Gao, Aidan O'Dwyer, Seamus McLoone*, Eugene Coyle

School of Control Systems and Electrical Engineering
Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland
*Department of Electronic Engineering,
National University of Ireland, Maynooth, Co.Kildare, Ireland

Abstract—This paper presents a novel approach to stability analysis of affine blending systems. The analysis is based on Quadratic Lyapunov functions. The approach considers the nonlinear offset term in affine blending systems as non-vanishing perturbations added to the corresponding nominal linear blending systems. The affine blending systems will be bounded if the corresponding linear blending system is exponentially stable. The bound is determined by an ultimate limit, which is proportional to the maximum of the offset terms of each affine system.

I INTRODUCTION

The last decade has shown an increase in the use of local model representations of nonlinear dynamic systems for controller design, such as gain scheduled control, fuzzy systems, local model/controller networks. The attraction of the application of local model representations is that a nonlinear design task is simplified to linear design problems by first decomposing the task into a number of linear sub-problems solvable by established methods, then recombining, in some appropriate manner, the resultant collection of linear designs to obtain the required nonlinear design. In general, the local model structure has two categories, i.e. linear (homogeneous) local models (LLM) and affine (inhomogeneous) local models (ALM) (which have an extra offset term). The resulting linear blending systems inherit many valuable properties, but they could result in poor global representations of the nonlinear plant ([1]). In contrast, blending affine systems improve the modelling accuracy of LLM significantly with a benefit from the extra offset term introduced in the ALM.

In terms of control, the inherent nonlinearity in the blending systems is known as a major disadvantage of the approach. It has become evident that many basic issues remain to be further addressed ([2], [3]). Stability analysis and systematic design are certainly among the most important issues in this area.

However, the literature review for linear blending systems and affine blending systems shows unevenly distributed interest, although the ALM blending system has been widely applied in the modelling of nonlinear systems. Most research work has been devoted to analysis of linear systems ([4]-[7]), although there are several interesting recent contributions on affine fuzzy systems or piecewise affine systems ([8]-[10]). It is obvious that the LLM is linear and has equilibrium centred at the origin x=0. Comparably, the ALM is inhomogeneous and has a constant offset term, whose equilibrium is close to but not at the origin. Thus, it is more difficult to deal with the stability analysis and controller design for affine blending systems.

This paper proposes a novel method to analyse the influence of offset terms on stability issues in affine blending systems by using quadratic Lyapunov functions. It deals with the offset term as a 'non-vanishing' disturbance of a system, which stabilizes at the origin.

The paper is organised as follows. Section 2 discusses the stability issue for blending systems and introduces the sufficient conditions for ensuring the stability of linear blending systems using quadratic Lyapunov functions. In section 3, stability issues for linear blending systems are investigated, with section 4 discussing the stability issue for affine blending systems. Concluding remarks are provided in section 5.

II PROBLEM FORMULATION

Consider the nonlinear system

$$\dot{x} = f(x,u) \tag{1}$$

Utilizing a blended local model structure we approximate the nonlinear system (1) as follows:

$$\dot{x} = \sum_{i=1}^{N_m} \rho_i(x,u) f_i(x,u) \tag{2}$$

where state vector $x \in \Re^N$, input $u \in \Re^P$, the model

$f_i(:,:)$ is one of N_m vector functions of the states, the input and the output, and is valid in a region defined by the scalar validity function ρ_i, which is, in turn, a function of the above variables. Typically, the local models f_i are chosen to be of the affine form $f_i(x,u) = A_i x + B_i u + d_i$, resulting in constituent dynamic systems $\sum_i$ given by,

$$\dot{x} = A(x,u)x + B(x,u)u + d(x,u) \qquad (3)$$

where $A(x,u) = \sum_{i=1}^{N_m} \rho_i(x,u) A_i$, $B(x,u) = \sum_{i=1}^{N_m} \rho_i(x,u) B_i$ and $d(x,u) = \sum_{i=1}^{N_m} \rho_i(x,u) d_i$.

Assuming that all the local subsystems are stable, a question naturally arises as to whether the overall global system is stable? The answer is no, in general. Although the stability, performance and robustness properties of each linear local model controller are well understood and can be analyzed using standard tools, such as the Bode plot and Nyquist plot for each fixed operating point, these local properties do not naturally and necessarily lead to guaranteed global properties ([11]). Global properties cannot be guaranteed if proper modification (for example, controller gains) is not made when implementing the gain-scheduled controller. One example in the discrete time domain is given below to illustrate the issue, which also works in the continuous time domain.

Assuming an open-loop system has two subsystems as follows:

$$\dot{x}(k) = \sum_{i=1}^{2} \rho_i A_i x(k) \qquad (4)$$

where, $x(k) = [x_1(k), x_2(k)]^T$, $A_1 = \begin{bmatrix} 1 & -0.5 \\ 1 & 0 \end{bmatrix}$, and $A_2 = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}$.

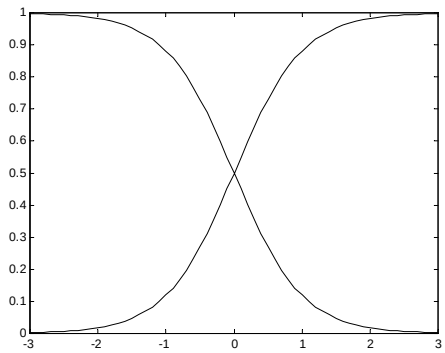
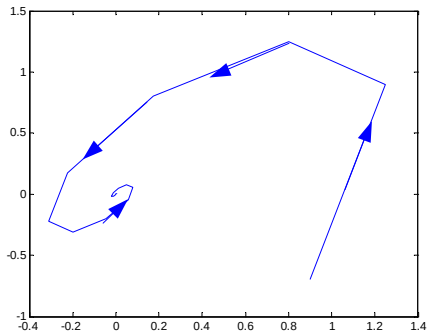
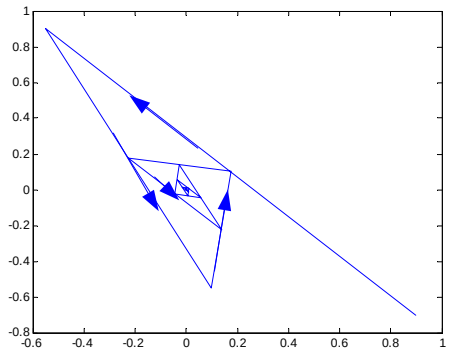


Figure 1. Normalized Validity function

Figure 1 shows the validity function for the interpolation of these two local models.



(a) Trajectory of A_1



(b) Trajectory of A_2

Figure 2. Trajectory of the subsystems

The eigenvalues of A_1 are $0.5 \pm j0.5$ and those of A_2 are $-0.5 \pm j0.5$. Since both A_1 and A_2 are Hurwitz, the linear subsystems are stable. Figure 2 shows the trajectory of A_1 in (a) and A_2 in (b), both of which finally converge at equilibrium point (0,0). However, when combining these two subsystems in a LM network, for some initial conditions, for example $x = [0.90, -0.7]^T$, the global system can be unstable as shown in figure 3.

As illustrated by the example, the blending procedure could cause an instability problem for the overall system, although each subsystem is locally stable. Thus stability issues should be taken into consideration when selecting validity functions and local models, and in the controller design of the blending system. How to systematically select validity functions, local models and approaches for controller

design to meet the required overall system stability is not clear so far.

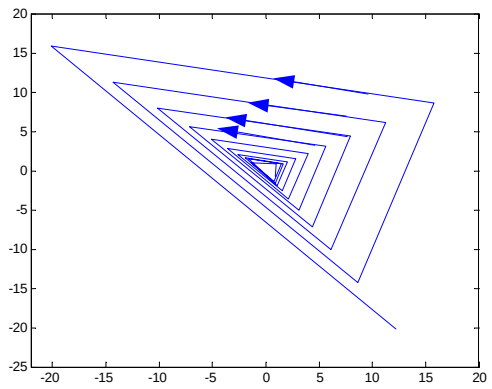


Figure 3. Trajectory of the overall system

Most of the time, a trial-and-error procedure has been used ([4], [5]). Hunt and Johansen ([12]) proposed one sufficient but not necessary condition to guarantee the overall stability for gain scheduling systems, which is based on the analysis of the effect of modeling errors. However, the condition is mainly of a qualitative nature, as no bounds on performance, robustness or design parameters are provided. The objective of this paper is to determine the bound for the stability condition of the blending affine systems.

III LINEAR BLENDING SYSTEMS

For a blending system with linear local models, whose offset terms fade to zero, a common sufficient condition for the stability is given by the Lyapunov function. Recall the state space representation of the nonlinear systems as given by equation (2), the open loop system corresponding to (2) is

$$\dot{x} = \sum_{i=1}^{Nm} \rho_i A_i x \qquad (5)$$

where the validity function $1 \geq \rho_i \geq 0$ and $\sum_{i=1}^{Nm} \rho_i = 1$. Each linear component $A_i x(t)$ is called a subsystem. The sufficient conditions for ensuring stability of equation (5) are usually formulated in Theorem 1 (Tanaka and Sugeno, 1992):

Theorem 1: The equilibrium point of a system (5) is asymptotically stable in the large if there exists a common positive definite matrix P such that

$$A_i^T P + P A_i < 0, \; i = 1,2,\cdots,Nm \qquad (6)$$

i.e., a common P has to exist for all subsystems to guarantee the overall stability. In this case, the nominal global system has a uniformly exponentially asymptotically stable equilibrium point at the origin. This theorem reduces to the Lyapunov stability theorem for linear systems when N_m =1.

The stability condition of Theorem 1 is derived using a quadratic function $V(x) = x^T P x$. If there exists a P>0 such that the quadratic function proves the stability of system (5), system (5) is also said to be quadratically stable and the V is called a quadratic Lyapunov function. Theorem 1 thus presents a sufficient condition for the quadratic stability of system (5).

Checking the stability of system (5) has long been recognized to be difficult for there is a lack of a systematic procedure to find a common positive definite matrix P. Solving this problem requires two questions to be answered: Is there a common quadratic Lyapunov function that exists? How can a common quadratic Lyapunov function be determined?

Deriving sufficient conditions under which exponential stability will be assured has been investigated by a number of authors. Narendra and Balakrishnan ([13]) introduce 'commutativity' to assure the existence of a common Lyapunov function; however, the converse of the approach does not hold in general, i.e. if there is no such commuting Lyapunov function found for the overall system, it doesn't mean that the overall global system is not stable, so the utilization of other approaches to look for the common P matrix, if it exists, is needed. Shorten and Narendra ([14]) presented the necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for two stable second order linear systems. Subsequently, Shorten and Narendra ([15]) extended the approach to check the existence of a common quadratic Lyapunov function for a finite number of stable second order linear systems. Recently, Shorten made a further step and generalized the above results in ([16]).

To determine the common quadratic Lyapunov function, most of the time a trial-and-error procedure has been used ([4]). In the literature, since the middle of the 1990s, there is a rapidly growing interest in finding out the common Lyapunov function P by solving a convex optimization problem using linear matrix inequality (LMI) approach ([5], [7]). A very important property of this approach is that the stability condition of theorem 1 is expressed in LMI form. To check stability, which means to find a common positive quadratic Lyapunov function P, or

to show that there is no such common P that exists for the system, converts to a problem of solving LMI functions. Numerically, the LMI problems can be solved efficiently by means of some powerful tools available in the mathematical programming literature, like the Matlab LMI toolbox.

IV AFFINE BLENDING SYSTEMS

For the blending of linear local models, each of which has a stable node with an equilibrium point centered at the origin, the global system has its equilibrium point centered at the origin x=0. In contrast, the affine local model allows its equilibrium point to be close to, but not centered at the origin ([2]), because the offset term in each affine local model doesn't fade to zero, but is instead a constant. Thus, the origin x=0 may not be equilibrium of the blended system. We can no longer study stability of the origin as an equilibrium point, nor should we expect the solution of the offset term to approach the origin as $t \to \infty$.

One possibility is to consider the offset term $d(x,u,t)$ in equation (2) as a non-vanishing perturbation. It is hoped that if the 'perturbation term' $d(x,u,t)$ is small in some sense, then $x(t)$ will be ultimately bounded by a small bound; that is, $\|x(t)\|$ will be small for sufficiently large t.

Consider the open-loop system in equation (2) and rewrite it as follows:

$$\dot{x} = \sum_{i=1}^{Nm} \rho_i A_i x + \sum_{i=1}^{Nm} \rho_i d_i \tag{7}$$

Note that equation (5) is termed a nominal system and equation (7) is termed a perturbed system for convenience.

Suppose the nominal system (5) has a uniformly asymptotically stable equilibrium point at the origin, what can we say about the stability behavior of the perturbed system (7)? A natural approach to address this question is to use a Lyapunov function for the nominal system as a Lyapunov function candidate for the perturbed system. The new element here is that the 'perturbation term' will not vanish at the origin, i.e. the origin will not be an equilibrium point of the perturbed system. Therefore, the problem can no longer be studied as a question of the stability of equilibria. The best that can be hoped for is that if the perturbation term, bounded by a small bound $\|x(t)\|$, will be small for sufficiently large t.

From Theorem 1, it is known that if there is a common positive definite P existing for all the local linear models, then $V(t,x) = x^T P x$ is a Lyapunov function of the global blending system (5). The conditions to ensure the global stability of system (7) are more complicated, as more analysis needs to be performed. Lemma 1 is first developed for the linear blending systems.

Lemma 1: Let x=0 be an equilibrium point for the blending system as equation (5), where A_i is Hurwitz. Let $V(x)$ be a Lyapunov function of the nominal system. Then V satisfies the inequalities:

$$c_1 \|x\|_2^2 \le V(x) \le c_2 \|x\|_2^2 \tag{8}$$

$$\frac{\partial V}{\partial x} \sum_{i=1}^{Nm} \rho_i A_i x \le -c_3 \|x\|_2^2 \tag{9}$$

$$\left\| \frac{\partial V}{\partial x} \right\|_2 \le c_4 \|x\|_2 \tag{10}$$

for some positive constants c_1, c_2, c_3 and c_4, where $c_1 = \lambda_{\min}(P)$, $c_2 = \lambda_{\max}(P)$, $c_3 = \min_{i=1\cdots Nm} (\lambda_{\min}(Q_i))$, $c_4 = 2\lambda_{\max}(P)$.

Proof: Assuming that the Lyapunov function is defined as $V(x) = x^T P x$, P being the common positive definite matrix, then

$$\lambda_{\min}(P) I \le P \le \lambda_{\max}(P) I$$

$$\Rightarrow x^T \lambda_{\min}(P) x \le V(x) \le x^T \lambda_{\max}(P) x$$

$$\Rightarrow \lambda_{\min}(P) \|x\|_2^2 \le V(x) \le \lambda_{\max}(P) \|x\|_2^2$$

$$\frac{\partial V}{\partial x} \sum_{i=1}^{Nm} \rho_i A_i x$$

$$= x^T \left(P \sum_{i=1}^{Nm} \rho_i A_i + \sum_{i=1}^{Nm} \rho_i A_i^T P \right) x$$

$$= -x^T \sum_{i=1}^{Nm} \rho_i Q_i x$$

$$\le -\sum_{i=1}^{Nm} \rho_i \lambda_{\min}(Q_i) \|x\|_2^2$$

$$\le -\min_{i=1\cdots Nm} (\lambda_{\min}(Q_i))$$

$$\left\| \frac{\partial V}{\partial x} \right\|_2 \le \left\| 2x^T P \right\|_2 \le 2\|P\|_2 \|x\|_2 \le 2\lambda_{\max}(P) \|x\|_2$$

Now we introduce some special scalar functions and Theorem 2 ([17]) that will help to characterize and study the stability behavior of the blending affine local model systems.

Definition 1: A continuous function $\alpha:[0,a)\to[0,\infty)$ is said to belong to the class K function if it is strictly increasing and $\alpha(0)=0$. It is said to belong to the class K_∞ function if $a=\infty$ and $\alpha(r)\to\infty$ as $r\to\infty$.

Definition 2: A continuous function $\beta:[0,a)\to[0,\infty)$ is said to be a class *KL* function if for each fixed *s*, the mapping $\beta(r,s)$ belongs to class *K* with respect to *r*, and for each fixed *r* the mapping $\beta(r,s)$ is decreasing with respect to *s* and $\beta(r,s)\to\infty$ as $r\to\infty$.

Theorem 2: Let $D=\{x\in R^n \mid \|x\|<r\}$ and $f:[0,\infty)\times D\to R^n$ be piecewise continuous in t and locally Lipschitz in x. Let $V:[0,\infty)\times D\to R$ be a continuous differentiable function such that

$$\alpha_1(\|x\|)\le V(t,x)\le\alpha_2(\|x\|) \quad (11)$$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\le-\alpha_3(\|x\|),\forall\|x\|\ge\mu>0 \quad (12)$$

$\forall t\ge 0,\ \forall x\in D$, where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, and $\alpha_3(\cdot)$ are class *K* function defined on $[0,r)$ and $\mu<\alpha_2^{-1}(\alpha_1(r))$. Then, there exists a class *KL* function $\beta(\cdot,\cdot)$ and a finite time t_1 (dependent on $x(t_0)$ and μ) such that

$$\|x(t)\|\le\beta(\|x(t_0)\|,t-t_0),\ \forall t_0\le t\le t_1 \quad (13)$$

$$\|x(t)\|\le\alpha_1^{-1}(\alpha_2(\mu)),\ \forall t\ge t_1 \quad (14)$$

$\forall\|x(t_0)\|<\alpha_2^{-1}(\alpha_1(r))$. Moreover, if all the assumptions hold with $r=\infty$, that is, $D=R^n$, and $\alpha_1(\cdot)$ belongs to class K_∞, then inequalities (13)-(14) hold for any initial state $x(t_0)$. Furthermore, if $\alpha_i(r)=k_i r^c$, for some positive constants k_i and c, then $\beta(r,s)=kr\exp(-rs)$ with $k=(k_2/k_1)^{1/c}$ and $r=(k_3/k_2c)$.

Inequalities (13)-(14) show that $x(t)$ is uniformly bounded for all $t\ge t_0$. They also show that $x(t)$ is uniformly ultimately bounded with an ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$. It is significant that the ultimate bound is a class *K* function of μ, because the smaller the value of μ the smaller the ultimate bound. As $\mu\to 0$, the ultimate bound approaches zero.

Based on Theorem 2, a Lemma (Lemma 2) is dveloped for the analysis of the blending of affine local models when the origin of the nominal system is exponentially stable.

Lemma 2: Let x=0 be an exponentially stable equilibrium point of the nominal system. Let V(x) be a Lyapunov function of the nominal system and for some positive $0<\theta<1$, the solution of the perturbed system x(t) satisfies:

$$\|x(t)\|\le k\exp[-r(t-t_0)]\|x(t_0)\|,\ \forall t_0\le t<t_1$$

and $\|x(t)\|\le b,\forall t\ge t_1$

for some finite time t_1, where

$$k=\sqrt{c_2/c_1},\ r=\frac{(1-\theta)c_3}{2c_2},\ b=\frac{c_4}{c_3}\sqrt{\frac{c_2}{c_1}}\frac{\delta}{\theta}$$

Proof: Assume $V(x)$ is a Lyapunov function candidate for the perturbation system (7). The derivative of $V(x)$ along the trajectories of (7) satisfies

$$\dot V(x)=x^TP\dot x+\dot x^TPx$$

$$=x^TP\left(\sum_{i=1}^{Nm}\rho_iA_ix+\sum_{i=1}^{Nm}\rho_id_i\right)+\left(\sum_{i=1}^{Nm}\rho_iA_ix+\sum_{i=1}^{Nm}\rho_id_i\right)^TPx$$

$$=x^T\left(P\sum_{i=1}^{Nm}\rho_iA_i+\sum_{i=1}^{Nm}\rho_iA_i^TP\right)x+2x^TP\sum_{i=1}^{Nm}\rho_id_i$$

$$\le-c_3\|x\|_2^2+c_4\|x\|_2\max_{i=1\cdots Nm}(d_i)$$

$$=-(1-\theta)c_3\|x\|_2^2-(\theta c_3\|x\|_2-c_4\delta)\|x\|_2,\ \delta=\max_{i=1\cdots Nm}(d_i)$$

$$\le-(1-\theta)c_3\|x\|_2^2\quad\forall\|x\|_2\ge c_4\delta/c_3\theta$$

Application of Theorem 2 completes the proof.

Lemma 2 shows the effect, from the offset term, of affine models on the property of blended systems. This result demonstrates that if linear blending system (5) is exponentially stable with respect to the origin, then the corresponding affine blending system (7) is uniformly bounded with ultimate bound *b*. Moreover, note that the ultimate bound *b* is proportional to the upper bound on the perturbation $\delta=\max_{i=1\cdots Nm}(d_i)$. The ultimate bound can be viewed as a robustness property of nominal systems having exponentially stable equilibria at the origin, because it shows that arbitrarily small (uniformly bounded) perturbations will not result in large steady-state deviations from the origin.

V CONCLUDING REMARKS

This paper investigated the stability of affine blending systems, in which the offset term is nonlinear, parameter dependent and bounded. Assuming the nominal linear blending system is exponentially asymptotically stable, the corresponding affine blending system is bounded by an ultimate value *b*, which is proportional to the maximum of the offset terms of local models. The smaller the bound *b* is, the smaller the deviation of the affine blending systems from the stabilizing origin of the linear blending systems. Further work will focus on the systematic analysis and controller design of affine blending systems to ensure the closed-loop compensated system has guaranteed stability.

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