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Ruiyao Gao
Dublin Institute of Technology

Aidan O'Dwyer
Dublin Institute of Technology

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Ruiyao Gao, Aidan O'Dwyer

School of Control Systems and Electrical Engineering
Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland
aidan.odwyer@dit.ie

Abstract

This paper presents an approach to stability analysis and control synthesis of affine fuzzy systems. The analysis is based on quadratic Lyapunov functions. The approach considers the nonlinear offset terms in affine fuzzy systems as non-vanishing perturbations added to the corresponding nominal linear blending systems. The affine fuzzy system is bounded by an ultimate limit if the corresponding linear blending system is exponentially stable. A state feedback controller with an extra term is designed with guaranteed global stability. The ultimate bounds are determined for both the open loop system and the compensated system.

Keywords: affine, fuzzy system, stability analysis.

1. Introduction

In recent years, fuzzy control has attracted growing attention in academic and industrial applications. Despite the growing interest in fuzzy systems, the development of systematic methods for analysis and design of fuzzy control systems is still at an early stage. In the literature, although the affine fuzzy system has been widely applied in the modelling of nonlinear systems, reports on linear hybrid systems, and affine hybrid systems, show unevenly distributed interest. Most of the contributions have been devoted to the analysis of linear fuzzy systems (e.g. [1], [2]), although there are several, relatively recent, interesting contributions on the analysis of affine fuzzy systems or piecewise affine systems (e.g. [3]). This paper proposes a new method to investigate the influence of offset terms on the stability of affine fuzzy systems by using quadratic Lyapunov functions. It deals with the offset term as a 'non-vanishing' disturbance of a system, which stabilizes at the origin.
As is known, the linear local model has its equilibrium point centred at the origin $x = 0$. In contrast, the affine local model is inhomogeneous and has a constant offset term, whose equilibrium point is close to, but not at, the origin. Thus, it is more difficult to deal with the stability analysis and controller design for affine fuzzy systems.
The paper is organised as follows. Section 2 briefly introduces affine fuzzy systems. Section 3 determines the stability limits of open loop affine fuzzy systems based on quadratic Lyapunov functions. In section 4, a state feedback controller is employed to compensate the affine fuzzy system and the stability bound is determined for the corresponding closed-loop system. Concluding remarks are provided in section 5.

2. System Description

Consider the nonlinear system

$$\dot{x} = f(x,u) \tag{1}$$

The nonlinear system (1) is approximated by a set of local models, as follows:

$$\dot{x} = \sum_{i=1}^{N} \rho_i(x,u) f_i(x,u) \tag{2},$$

where state vector $x \in \Re^N$, input $u \in \Re^P$, the model $f_i(\cdot,\cdot)$ is one of N vector functions of the states and the input, and is valid in a region defined by the scalar validity function $\rho_i(x,u)$; $\rho_i(x,u)$ are normalised membership functions of the rule satisfying $0 \le \rho_i(x,u) \le 1$ and $\sum_i \rho_i(x,u) = 1$. Typically, the local models f_i are chosen to be of affine form $f_i(x,u) = A_i x + B_i u + d_i$, resulting in constituent dynamic systems $\sum_i$ given by:

$$\dot{x} = A(x,u)x + B(x,u)u + d(x,u) \tag{3},$$

where $A(x,u) = \sum_{i=1}^{N} \rho_i(x,u) A_i$, $B(x,u) = \sum_{i=1}^{N} \rho_i(x,u) B_i$ and $d(x,u) = \sum_{i=1}^{N} \rho_i(x,u) d_i$.

3. Quadratic Stability of the Fuzzy System

3.1 Linear Fuzzy Systems

For a fuzzy system with linear local models, whose offset terms fade to zero, a common sufficient condition for stability is given by analysing the Lyapunov function. Recall that the state space representation of the local model network is given by equation (2). The open loop system corresponding to (3) is

$$\dot{x} = \sum_{i=1}^{N} \rho_i A_i x \tag{4}$$

Each linear component $A_i x(t)$ is called a subsystem. The sufficient conditions for ensuring stability are formulated in Theorem 1 ([1]).

Theorem 1: The equilibrium point of a system (4) is asymptotically stable in the large if there exists a common positive definite matrix *P* such that

$$A_i^T P + PA_i < 0, i = 1,2,\cdots,N \tag{5}$$

i.e., a common *P* has to exist for all subsystems to guarantee stability. In this case, the nominal global system has a uniformly exponentially asymptotically stable equilibrium point at the origin. To check stability i.e. to find a common positive quadratic Lyapunov function *P,* or to show that there is no such common *P* that exists for the system, converts to a problem of solving linear matrix inequality (LMI) functions. Numerically, LMI problems can be solved efficiently by means of some powerful mathematical tools, such as the MATLAB LMI toolbox.

3.2 Affine Fuzzy Systems

For a linear fuzzy system, the global equilibrium point is centered at the origin. In contrast, the affine local model allows its equilibrium point to be close to, but not centered at the origin [4], because the offset term in each affine local model doesn't fade to zero, but is instead a constant. Thus, the origin *x* = 0 may not be the equilibrium point of the affine fuzzy system. Thus, the stability of the system with the origin as an equilibrium point may no longer be studied, nor is there an expectation that the solution of the offset term approachs the origin as $t \to \infty$.

One possibility is to consider the offset term $d(x,u)$ in equation (3) as a non-vanishing perturbation. It is hoped that if the 'perturbation term' $d(x,u)$ is small in some sense, then $x(t)$ will be ultimately bounded by a small bound i.e. $\|x(t)\|$ will be small for sufficiently large *t*.

Consider the open-loop system as in equation (3) and rewrite it as follows:

$$\dot{x} = \sum_{i=1}^{N} \rho_i A_i x + \sum_{i=1}^{N} \rho_i d_i \tag{6}$$

Note that equation (4) is termed a 'nominal system' and equation (6) is termed a 'perturbed system' for convenience. Suppose the nominal system (4) has a uniformly asymptotically stable equilibrium point at the origin, what can be said about the stability behaviour of the perturbed system (6)? A natural approach to address this question is to use the Lyapunov function for the nominal system as a Lyapunov function candidate for the perturbed system. The new element here is that the 'perturbation term' will not vanish at the origin i.e. the origin will not be an equilibrium point of the perturbed system. Therefore, the problem can no longer be studied as a question of the stability of equilibrium points. The best that can be hoped for is that the perturbation term, bounded by a small bound $\|x(t)\|$, will be small for sufficiently large *t*.

From Theorem 1, it is known that if there is a common positive definite *P* existing for all the local linear models, then $V(t,x)=x^T Px$ is a Lyapunov function of the global blending system (4). The conditions to ensure the global stability of system (6) are more complicated, as more analysis needs to be performed. As a special application of the ultimate boundedness theory [5], Lemma 1 is first considered.

Lemma 1: Let $x = 0$ be an equilibrium point for the blended system in equation (4), where A_i is Hurwitz. Let $V(x)$ be a Lyapunov function of the nominal system. Then *V* satisfies the inequalities:

$$c_1\|x\|_2^2 \le V(x) \le c_2\|x\|_2^2 \tag{7}$$

$$\frac{\partial V}{\partial x}\sum_{i=1}^{N}\rho_i A_i x \le -c_3\|x\|_2^2 \tag{8}$$

$$\left\|\frac{\partial V}{\partial x}\right\|_2 \le c_4\|x\|_2 \quad (9),$$

for some positive constants c_1, c_2, c_3 and c_4, where $c_1 = \lambda_{min}(P)$, $c_2 = \lambda_{max}(P)$, $c_3 = \min_{i=1\cdots N}(\lambda_{min}(Q_i))$ and $c_4 = 2\lambda_{max}(P)$.

Proof: Assuming that the Lyapunov function is defined as $V(x)=x^T Px$, *P* being the common positive definite matrix, then

$$\lambda_{min}(P)I \le P \le \lambda_{max}(P)I$$

$$\Rightarrow x^T\lambda_{min}(P)x \le V(x) \le x^T\lambda_{max}(P)x$$

$$\Rightarrow \lambda_{min}(P)\|x\|_2^2 \le V(x) \le \lambda_{max}(P)\|x\|_2^2.$$ This result proves (7).

$$\frac{\partial V}{\partial x}\sum_{i=1}^{N}\rho_i A_i x = x^T\left(P\sum_{i=1}^{N}\rho_i A_i + \sum_{i=1}^{N}\rho_i A_i^T P\right)x = -x^T\sum_{i=1}^{N}\rho_i Q_i x$$

$$\le -\sum_{i=1}^{N}\rho_i\lambda_{min}(Q_i)\|x\|_2^2 \le -\min_{i=1\cdots N}(\lambda_{min}(Q_i)).$$ This result proves (8).

$$\left\|\frac{\partial V}{\partial x}\right\|_2 \le \left\|2x^T P\right\|_2 \le 2\|P\|_2\|x\|_2 \le 2\lambda_{max}(P)\|x\|_2.$$ This result proves (9).

Now some special scalar functions are introduced and Theorem 2 ([5]) is outlined, which will help to characterize and study the stability behaviour of the affine fuzzy systems.

Definition 1: A continuous function $\alpha : [0,a) \to [0,\infty)$ is said to belong to the class *K* function if it is strictly increasing and $\alpha(0)=0$. It is said to belong to the class K_∞ function if $a=\infty$ and $\alpha(r)\to\infty$ as $r\to\infty$.

Definition 2: A continuous function $\beta : [0,a) \to [0,\infty)$ is said to be a class *KL* function if for each fixed *s*, the mapping $\beta(r,s)$ belongs to class *K* with respect to *r*, and for each fixed *r* the mapping $\beta(r,s)$ is decreasing with respect to *s* and $\beta(r,s)\to\infty$ as $r\to\infty$.

Theorem 2: Let $D=\{x\in R^n \mid \|x\| < r\}$ and $f : [0,\infty)\times D \to R^n$ be piecewise continuous in *t* and locally Lipschitz in *x*. Let $V : [0,\infty)\times D \to R$ be a continuous differentiable function such that

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|) \text{ and } \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -\alpha_3(\|x\|), \forall \|x\| \ge \mu > 0 \quad (10),$$

$\forall t \ge 0$, $\forall x \in D$, where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha_3(\cdot)$ are class K functions defined on $[0,r)$ and $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then, there exists a class *KL* function $\beta(\cdot,\cdot)$ and a finite time t_1 (dependent on $x(t_0)$ and μ) such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0),\ \forall t_0 \le t \le t_1 \text{ and } \|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu)),\ \forall t \ge t_1 \quad (11),$$

$\forall \|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$. Moreover, if all the assumptions hold with $r = \infty$, that is, $D = R^n$, and $\alpha_1(\cdot)$ belongs to class K_∞, then the inequalities in (11) hold for any initial state $x(t_0)$. Furthermore, if $\alpha_i(r) = k_i r^c$, for some positive constants k_i and c, then $\beta(r,s) = kr\,exp(-rs)$ with $k = (k_2/k_1)^{1/c}$ and $r = (k_3/k_2 c)$. The inequalities in (11) show that $x(t)$ is uniformly bounded for all $t \ge t_0$. They also show that $x(t)$ is uniformly ultimately bounded with an ultimate bound $\alpha_1^{-1}(\alpha_2(\mu))$. It is significant that the ultimate bound is a class *K* function of μ, because the smaller the value of μ the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero. Based on Lemma 1 and Theorem 2, Theorem 3 is developed for the analysis of the blending of affine local models when the origin of the nominal system is exponentially stable.

Theorem 3: Let $x = 0$ be an exponentially stable equilibrium point of the nominal system. Let *V(x)* be a Lyapunov function of the nominal system, and for some positive $0 < \theta < 1$, the solution of the perturbed system *x(t)* satisfies:

$$\|x(t)\| \le k\,exp[-r(t - t_0)]\|x(t_0)\|,\ \forall t_0 \le t < t_1 \text{ and } \|x(t)\| \le b, \forall t \ge t_1,$$

for some finite time t_1, where $k = \sqrt{c_2/c_1}$, $r = \dfrac{(1-\theta)c_3}{2c_2}$, $b = \dfrac{c_4}{c_3}\sqrt{\dfrac{c_2}{c_1}}\dfrac{\delta}{\theta}$.

Proof: Assume $V(x)$ is a Lyapunov function candidate for the perturbation system (7)-(9). The derivative of $V(x)$ satisfies

$$\begin{aligned}
\dot{V}(x) &= x^T P \dot{x} + \dot{x}^T P x = x^T P\left(\sum_{i=1}^{N} \rho_i A_i x + \sum_{i=1}^{N} \rho_i d_i\right) + \left(\sum_{i=1}^{N} \rho_i A_i x + \sum_{i=1}^{N} \rho_i d_i\right)^T P x \\
&= x^T\left(P\sum_{i=1}^{N} \rho_i A_i + \sum_{i=1}^{N} \rho_i A_i^T P\right)x + 2x^T P \sum_{i=1}^{N} \rho_i d_i \\
&\le -c_3\|x\|_2^2 + c_4\|x\|_2 \max_{i=1\cdots N}(d_i) \quad \text{(from Lemma 1 and } 0 \le \rho_i \le 1,\ \sum_{i=1}^{N}\rho_i = 1) \\
&= -(1-\theta)c_3\|x\|_2^2 - (\theta c_3\|x\|_2 - c_4\delta)\|x\|_2,\ \delta = \max_{i=1\cdots N}(d_i) \\
&\le -(1-\theta)c_3\|x\|_2^2,\ \forall \|x\|_2 \ge c_4\delta/c_3\theta
\end{aligned}$$

The application of Theorem 2 completes the proof.

Theorem 3 shows the effect, from the offset term, of affine models on the properties of fuzzy systems. This result demonstrates that if linear fuzzy system (4) is exponentially stable with respect to the origin, then the corresponding affine fuzzy system (6) is uniformly bounded with ultimate bound *b*. Moreover, note that the ultimate bound *b* is proportional to the upper bound on the perturbation $\delta = \max_{i=1\cdots N}(d_i)$. The ultimate bound can be viewed as a robustness property of nominal systems having exponentially stable equilibria at the origin, because it shows that arbitrarily small (uniformly bounded) perturbations will not result in large steady-state deviations from the origin.

4. Stabilization via Feedback Control

The standard state feedback stabilization problem for the system (1) is the problem of designing a feedback control law, $u = \gamma(x)$, such that the origin $x = 0$ is an asymptotically stable equilibrium point

of the closed-loop system $\dot{x} = f(x, \gamma(x))$. In a typical control problem, there are additional goals for the design, like meeting requirements on the transient response or certain constraints on the control input, but the discussion in this section is concentrated on the basic problem of stabilizing nonlinear systems.
The most practical way to approach the stabilization problem for nonlinear systems is to appeal to the results available in the linear case, that is, via linearization. This approach, however, generally only provides a local result, which has an attractive region of robustness locally. Alternatively, fuzzy systems formulate instant linear (affine) models to approximate the nonlinear systems continuously, and provide the possibility to design a non-local controller, which is conceptually simple and straightforward. Linear feedback control techniques can be utilized to stabilize the system. The following subsections present the stabilization of linear fuzzy systems by state feedback control first; then the results will be applied for the analysis and design of state feedback controllers for affine fuzzy systems.

4.1 Linear Fuzzy Control Systems

When state feedback control is employed to design the controller, the main idea is to design each local state feedback controller so as to compensate each local model to achieve the desired performance. Consequently, the fuzzy control system is formulated by combining these local feedback laws via membership functions.
Assume each pair of (A_i, B_i) is controllable, or at least stabilizable. Design a matrix F_i to assign the eigenvalues of $A_i + B_i F_i$ to desired locations in the open left-half complex plane. Then, the fuzzy control system based on the state feedback control approach can be written as follows:

$$u(t) = -\sum_{i}^{N} \rho_i F_i x(t) \tag{12}$$

Substituting (12) into (3) (with $d(x,u) = 0$ for linear fuzzy systems) gives

$$\dot{x} = \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i \rho_j \left(A_i - B_i F_j\right) x \tag{13}$$

Clearly, the origin is an equilibrium point of the closed-loop system. In a similar manner to Theorem 1, Theorem 4 gives the sufficient conditions to guarantee the stability of system (13).
Theorem 4: The equilibrium point of system (13) is globally asymptotically stable if there exists a common positive definite matrix *P* such that

$$\left(A_i - B_i F_j\right)^T P + P\left(A_i - B_i F_j\right) < 0, \text{ for } \rho_i \cdot \rho_j \neq 0, \forall i, j = 1, \cdots, N \tag{14}$$

The controller design involves determining the local feedback gains F_i, $i = 1, \cdots, N$. The feedback gain F_i assures that $A_i + B_i F_i$ are Hurwitz, and that there is a positive definite matrix *P* satisfying Theorem 4. These requirements, on the one hand, satisfy a desired overall closed-loop system performance requirement; on the other hand, the global stability of the closed-loop system is guaranteed.
Wang *et al.* ([2]) proposed the so-called parallel-distributed compensation (PDC) method as a design framework for the discrete time system and also modified Tanaka and Sugano's stability theorem ([1]) to include a control algorithm. An important observation in Wang's approach is that the stability problem is structured as a standard feasibility problem with several LMI's when the feedback gains are predetermined, and can be solved numerically using an algorithm named the interior-point method. Thus the controller design is simple and natural. Basically, Wang's stability criterion is adopted and tailored here for the analysis of the stability problem in the continuous time domain.
Rewrite (13) as

$$\dot{x} = \sum_{i=1}^{N} \rho_i^2 \left(A_i - B_i F_i\right) x + 2 \sum_{i<j} \rho_i \rho_j G_{ij} x \text{ where } G_{ij} = \frac{\left(A_i - B_i F_j\right) + \left(A_j - B_j F_i\right)}{2}, i < j. \tag{15}$$

Therefore theorem 4 may be reformulated as theorem 5 with relaxed constraints.
Theorem 5: The equilibrium point of system (13) is globally asymptotically stable if there exists a common positive definite *P* matrix such that the following two conditions are satisfied:

$$\left(A_i - B_i F_i\right)^T P + P\left(A_i - B_i F_i\right) < 0,\ \ i = 1,2,\cdots,N \tag{16}$$

$$G_{ij}^T P + P G_{ij} < 0\ ,\ i < j \le N \tag{17}$$

As for the affine fuzzy systems, extra tools and analysis are required to obtain a closed-loop system with guaranteed stability in addition to meeting the conditions in (16) and (17).

4.2 Affine Fuzzy Control Systems

Assuming a set of state feedback gains F_i exist, which satisfy the conditions in Theorem 5, the idea of cancelling the offset term of the affine models is proposed here for controller design. Similar ideas have been previously explored (e.g. [6]). It is clear that cancelling the offset term $d(x,u)$ can be done by directly subtracting the control signal u from equation (12). Therefore, using feedback control converts an affine local state equation into a controllable linear state equation by cancelling the offset term locally. However, because of the interpolation technique involved, the closed-loop system is not generally a simple sum of linear subsystems. A detailed analysis is given below.

Design a set of feedback gains F_i such that each $A_i + B_i F_i$ is Hurwitz. The feedback control signal is

$$u = -\sum_{i=1}^{N} \rho_i\left(\sigma_i d_i + F_i x\right) \tag{18),}$$

where $\sigma_i = [\sigma_1 \quad \cdots \quad \sigma_N]$, N is the system order. Substituting (18) into (3) results in the overall closed-loop system description

$$\dot{x} = \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \rho_j \left(A_i - B_i F_j\right)x + \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \rho_j \left(d_i - B_i \sigma_j d_j\right) \tag{19}$$

There are two parts to (19). The first part is the same as (13) in section 4.1. It plays a vital role in stabilizing the global system; the second part is nonlinear and is parameter dependent (it relies on the states, input or output parameters applied in validity functions), due to the interpolation technique involved, but it is bounded. Rewrite the second part of (19) as follows:

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \rho_i \rho_j \left(d_i - B_i \sigma_j d_j\right) = \sum_{i=1}^{N} \rho_i \cdot \rho_i \left(d_i - B_i \sigma_i d_i\right) + 2\sum_{i<j} \rho_i \rho_j \theta_{ij} \tag{20),}$$

where $\theta_{ij} = \dfrac{(d_i - B_i \sigma_j d_j) + (d_j - B_j \sigma_i d_i)}{2}, i < j$.

Consider the special case that $B_i = B, i = 1,\cdots,N$ first. Thus,

$$\theta_{ij} = \frac{(d_i - B\sigma_j d_j) + (d_j - B\sigma_i d_i)}{2} = \frac{\left(d_i - B\sigma_i d_i\right) + (d_j - B\sigma_j d_j)}{2}.$$

If we set $d_i - B\sigma_i d_i = 0$, $i = 1,\cdots,N$, then $\theta_{ij} = 0$; thus, the closed-loop system is associated only with the first part of (19). Therefore, the cancellation of the offset term leads to a linear system description. Consequently, the stabilization problem for the nonlinear system is reduced to a stabilization problem for a set of controllable linear systems. A stabilizing linear state feedback controller may be subsequently designed to locate the eigenvalues of the closed-loop system in the open left half plane, as presented in section 4.1.

Consider the more general case, i.e. B_i does not have to be equal to B_j, $\forall i \ne j$, then

$$\theta_{ij} = \frac{(d_i - B_i \sigma_j d_j) + (d_j - B_j \sigma_i d_i)}{2} = \frac{\left(d_i - B_i \sigma_i d_i\right) + (d_j - B_j \sigma_j d_j) + (\sigma_i d_i - \sigma_j d_j)(B_i - B_j)}{2}.$$

If we set $d_i - B_i \sigma_i d_i = 0$, $i = 1,\cdots,N$, then

$$\theta_{ij} = \frac{\left(\sigma_i d_i - \sigma_j d_j\right)\left(B_i - B_j\right)}{2} \tag{21}$$

θ_{ij} is proportional to the difference between the model parameters B_i, $i = 1,\cdots,N$ and a difference functionally related to the designed controller offset parameter σ_i, $i = 1,\cdots,N$. Thus the second part of (20) can be written as

$$2\sum_{i<j}\rho_i\rho_j\frac{(\sigma_i d_i-\sigma_j d_j)(B_i-B_j)}{2} \tag{22}$$

Now $0<\rho_i<1$, $i=1,\cdots,N$ is the weighting function of the i^{th} local model, so (22) is bounded by

$$\delta=\frac{N\cdot(N-1)}{8}max\left(\left\|(\sigma_i d_i-\sigma_j d_j)\cdot(B_i-B_j)\right\|\right) \tag{23},$$

for a finite number N of affine fuzzy systems. Two cases are distinguished:

1. Consider the case that $\rho_i=1$ and $\rho_j=0$, $i\neq j,i,j=1,\cdots,N$.
2. Consider that the operating point resides in more than one region $\Re_i$, $i=1,\cdots,N$, so that the local systems are interpolated. This is the general case.

For special case 1, $\theta_{ij}=0$ and the design and analysis issue is simplified as discussed previously, i.e. when the system has a common B. To deal with the general case, the closed-loop system can be written as

$$\dot{x}=\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_i\rho_j(A_i-B_iF_j)x+2\sum_{i<j}\rho_i\rho_j(\sigma_i d_i-\sigma_j d_j)(B_i-B_j) \tag{24}$$

Equation (24) includes an offset term, which is nonlinear parameter dependent. Applying the stability analysis in section 4.2, Lemma 2 is developed.

Lemma 2: The affine fuzzy system (3) is uniformly bounded for all $t\geq t_0$ via state feedback controller (18), if there exits a positive definite matrix P, which satisfies (16) and (17). Let $V=x^TPx$ be a Lyapunov function of the system (3), for some positive $0<\theta<1$, then the solution of the system $x(t)$ satisfies

$$\|x(t)\|\leq k\,exp[-r(t-t_0)]\|x(t_0)\|,\ \forall t_0\leq t<t_1 \text{ and } \|x(t)\|\leq b,\forall t\geq t_1$$

for some finite time t_1, where $k=\sqrt{c_2/c_1}$, $r=\frac{(1-\theta)c_3}{2c_2}$, $b=\frac{2\lambda_{max}(P)}{c_3}\sqrt{\frac{c_2}{c_1}}\frac{\delta}{\theta}$.

Proof: Assume $V=x^TPx$ is the Lyapunov function of the linear fuzzy system (13), i.e. there is a common positive definite matrix P existing for each local system that (16) and (17) are satisfied and

$$c_1\|x\|_2^2\leq V\leq c_2\|x\|_2^2,\ c_1=\lambda_{min}(P), c_2=\lambda_{max}(P).$$

Then, define

$$(A_i-B_iF_i)^TP+P(A_i-B_iF_i)=-Q_{il}, \text{ for } i=1,\cdots,N$$
$$\text{and } G_{ij}^TP+PG_{ij}=-Q_{ij}, \text{ for } i<j,i,j=1,\cdots,N.$$

Based on Lemma 1 and Theorem 3, the differential of the Lyapunov function can be written as

$$\dot{V}=x^T\left(P\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_i\rho_j(A_i-B_iF_j)+\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_i\rho_j(A_i-B_iF_j)^TP\right)x+2x^TP\left(2\sum_{i<j}\rho_i\rho_j\theta_{ij}\right)$$

$$=x^T\left(\sum_{i=1}^{N}\rho_i^2\left[(A_i-B_iF_i)^TP+P(A_i-B_iF_i)\right]\right)x$$

$$+2x^T\sum_{i<j}\rho_i\rho_j(G_{ij}^TP+PG_{ij})x+2x^TP\left(\sum_{i<j}\rho_i\rho_j(\sigma_i d_i-\sigma_j d_j)(B_i-B_j)\right)$$

$$=-x^T\left(\sum_{i=1}^{N}\rho_i^2Q_{il}+2\sum_{i<j}\rho_i\rho_jQ_{ij}\right)x+2x^TP\left(\sum_{i<j}\rho_i\rho_j(\sigma_i d_i-\sigma_j d_j)(B_i-B_j)\right)$$

$$\leq-\left(min(\lambda\, min(Q_{il}))+\frac{N\cdot(N-1)}{2}\cdot min(\lambda_{min}(Q_{ij}))\right)\|x\|_2^2+2\|x\|_2\lambda_{max}(P)\delta$$

(From Lemma 1 and equations (22) and (23))

$$\leq-c_3\|x\|_2^2+2\|x\|_2\lambda_{max}(P)\delta=-(1-\theta)c_3\|x\|_2^2-(\theta c_3-2\lambda_{max}(P)\delta)\|x\|_2$$

$$\leq -(1-\theta)c_3\|x\|_2^2, \ \forall\|x\|_2 \geq \frac{2\lambda_{max}(P)\delta}{\theta c_3}.$$

Apply theorem 2 to complete the proof. Lemma 2 shows the closed loop system is bounded by an ultimate bound *b*, which strongly depends on the interpolation function ρ_i, $i = 1,\cdots,N$, which in turn depends on the states, system input and system output. To stabilize the closed-loop system to the origin, it is expected that δ should satisfy the inequality,

$$\|\delta(x)\| \leq r\|x\|_2 \qquad (25),$$

for some nonnegative constant *r*. This is possible, if (25) is considered as an extra constraint in affine fuzzy control design, although there is no obvious answer. Using $V = x^T Px$ as a Lyapunov function candidate for the closed-loop system, we have

$$\dot{V} = x^T\left(P\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_i\rho_j\left(A_i - B_iF_j\right) + \sum_{i=1}^{N}\sum_{j=1}^{N}\rho_i\rho_j\left(A_i - B_iF_j\right)^T P\right)x + 2x^T P\left(2\sum_{i<j}\rho_i\rho_j\theta_{ij}\right)$$

$$\leq -\left(min(\lambda\, min(Q_{il})) + \frac{N\cdot(N-1)}{2}min\left(\lambda_{min}\left(Q_{ij}\right)\right)\right)\|x\|_2^2 + 2\|x\|_2\lambda_{max}(P)\delta$$

$$\leq -\left(min(\lambda\, min(Q_{il})) + \frac{N\cdot(N-1)}{2}min\left(\lambda_{min}\left(Q_{ij}\right)\right) - 2\lambda_{max}(P)\delta\right)\|x\|_2^2,$$

which will be negative definite if

$$\delta < min(\lambda\, min(Q_{il})) + \frac{N\cdot(N-1)}{2}min\left(\lambda_{min}\left(Q_{ij}\right)\right)/\,2\lambda\, max(P).$$

5. Concluding Remarks

This paper presented a novel approach to the synthesis and analysis of affine fuzzy systems, in which the offset term is nonlinear, parameter dependent and bounded. Under the assumption that the nominal linear fuzzy system is exponentially asymptotically stable i.e. that there is a common positive definite matrix *P* existing for all the local systems, the corresponding affine fuzzy system will be bounded by an ultimate value *b*, which is proportional to the maximum of the offset terms of the local systems. The smaller the bound *b* is, the smaller the deviation of the affine fuzzy systems from the stabilizing origin of the linear fuzzy systems.

A state feedback controller, with an extra term, is proposed to cancel the offset term of the affine fuzzy systems. In some special cases, the overall compensated system is stable at the origin via state feedback control; this happens, for instance, if the local systems have a common *B*, or if the weighting functions of the local models satisfy $\rho_i\rho_j = 0$, for $i \neq j,\ i, j = 1,\cdots,N$. Generally, the overall compensated system is bounded by an ultimate bound *b*.

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