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On Endomorphisms and Automorphisms of Some Torsion-Free Modules

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Abstract The relationship between the endomorphism ring and the Automorphism group of an Abelian p-group has been extensively investigated. It is known that the automorphism group need not generate the full endomorphism ring. In this paper we investigate the analagous problem for torsion-free modules over a complete discrete valuation ring obtaining similar results.

Some twenty of more years ago, L. Fuchs [4] raised the question of the relationship between the automorphism group and the endomorphism ring of an Abelian p-group; in particular does the automorphism group generate the endomorphism ring? Fairly comprehensive answers were given in [1], [3], [7] and [9]. The object of this present work is to investigate the analogous problem for reduced torsion-free modules over a complete discrete valuation ring. We show in Theorem 7, using a recent realization result on such modules [2], that the answer is, in general, in the negative.

Throughout the work R will denote a complete discrete valuation ring i.e. a commutative principal ideal ring with exactly one prime element p (up to unit factors) which is complete in the topology having the ideals p^nR as a base of neighbourhoods of 0. We shall always assume that 2 is a unit in R. If G is a reduced torsion-free R-module, then we shall denote the endomorphism ring and automorphism group of G by E(G) and Aut G respectively. $E_0(G)$ will denote the subring of E(G) consisting of endomorphism of finite rank. Unexplained terms may be found in the standard works of Fuchs, [5], [6]; the main exception being that maps are written on the right.

We begin establishing that in some cases E(G) is actually generated by Aut G; indeed the stronger result that in some of these cases every endomorphism is a sum of two automorphisms is established.

Proposition 1

If M is a free R-module of finite rank, then every endomorphism of M is a sum of two automorphisms of M.

Proof

By induction on the rank of M. The result is clearly true if M has rank 1. Suppose M = R + G where rkG = rkM -1. If $\phi \in E(M)$ then ϕ has a matrix representation of the form $\phi = \begin{bmatrix} \mu & \alpha \\ \beta & \psi \end{bmatrix}$ where $\mu \in E(R)$, $\psi \in E(G)$ and α, β are appropriate homorphisms. But by induction $\mu = \mu_1 + \mu_2$ and $\psi = \psi_1 + \psi_2$ where the μ_i, ψ_i are units. But then $\phi = \theta_1 + \theta_2$ where

$$\theta_1 = \begin{bmatrix} \mu_1 & \alpha \\ 0 & \psi_1 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} \mu_2 & 0 \\ \beta & \psi_2 \end{bmatrix}$$

and it is easy to check that θ_1, θ_2 are automorphisms.

Proposition 2

If M is a countably generated reduced torsion-free module, then every endomorphism of M is a sum of automorphisms of M.

Proof

Since M is countably generated it is free and we may write $M = \bigoplus_{i=1}^{\infty} \langle e_i \rangle$. Let $\phi \in E(M)$. Then for each i we have $e_i \phi = \Sigma a_{ij} e_j$, where $a_{ij} \in R$ for all i,j. Since 2 is a unit in R, each element of R can be expressed as a sum of two units and so, for each i, we may write $a_{ii} = b_{ii} + d_{ii}$ where b_{ii}, d_{ii} are units in R. Thus we may write

$$e_i \phi = (\sum_{j<i} a_{ij} e_j + b_{ii} e_i) + (d_{ii} e_i) + (\sum_{j>i} a_{ij} e_j).$$

Hence $e_i \phi = e_i \phi_1 + e_i \phi_2 + e_i \phi_3$ where the endomorphisms ϕ_i are defined by the successive terms in the above expression. A simple check shows that ϕ_1 and ϕ_2 are automorphisms. The endomorphism ϕ_3 given by $e_i \phi_3 = \sum_{j>i} a_{ij} e_j$ is an α-endomorphism in the terminology of Freedman [3] and so can be written as a sum of two locally nilpotent endomorphisms [3, Lemma 2]. Thus $\phi_3 = \eta_1 + \eta_2$ where each η_i is locally nilpotent. But then $\phi_3 = (\eta_1 - 1) + (1 + \eta_2)$ where 1 is the identity endomorphism. However the local nilpotence of the η_i ensure that $(\eta_1 - 1)$ and $(1 + \eta_2)$ are units in E(M). Thus ϕ is a sum of (four) automorphisms of M.

Corollary 3

If M is a free R-module, then every endomorphism of M is a sum of automorphisms of M.

Proof

This follows from Proposition 2 by using an analogous set - theoretic construction to that given by Castagna [1, Theorem 2.2].

Lemma 4:

If G is a reduced torsion-free R-module of infinite rank which is complete in its p-adic topology, then the Jacobson radical of G, J(E(G)) = pE(G).

Proof

See e.g. [8, Theorem 2.3].

Lemma 5:

Let A be a ring with Jacobson radical J(A). If each element of A/J(A) is a sum of n units, then each element of A is a sum of n units.

Proof

See e.g. [9, Lemma 2].

Proposition 6

If G is a reduced torsion-free R-module of infinite rank which is complete in its p-adic topology, then every endomorphism of G can be written as a sum of two automorphisms.

Proof

Since G is complete, J(E(G)) = pE(G) and thus E(G)/J(E(G)) is a vector space over the field of p elements, Z_p. But the results of Hill[7] clearly hold for such vector spaces and so each element of E(G)/J(E(G)) is a sum of two units. The result follows from Lemma 5.

The results obtained in Corollory 3 and Proposition 6 could be loosely interpreted as saying that at both extremities (in the sense that any reduced torsion-free R-module can be regarded as lying between a free R-module and a complete R-module) endomorphism rings are generated by the automorphism group in a strong fashion. Despite this, our next result shows that this does not always hold.

Theorem 7

There exists a reduced torsion-free R-module G such that the endomorphism ring of G is not generated by the automorphism group of G.

Proof

Let $A = R[x]$, the polynomial ring in a single variable x. It follows from a result of Dugas, Goldsmith and Gobel [2, Theorem 4.1] that there exists a reduced torsion-free R-module G with $E(G) = A \oplus E_0(G)$. Define $\gamma : E(G) \to A \to A/pA = Z_p[x]$. The only units in $Z_p[x]$ are non-zero constant polynomials and so the non-constant polynomials are not sums of units. Hence their pre-images under γ are not sums of units.

Remark

It is not necessary to involve the full strength of theorem 4.1 in [2]. The simpler split realization result Theorem 0.1 in [2] will suffice but this was avoided here since the concept of inessential used there has no place in the current context.

Our final result shows that despite theorem 7, the automorphism group generates a large part of the endomorphism ring.

Lemma 8:

If α is a finite rank endomorphism of the reduced torsion-free R-module G, then $\mathrm{Ker}\alpha$ is a direct summand of G. Moreover there exists a direct decomposition $G = F \oplus K$ where F has finite rank, $G\alpha \subseteq F$ and $K \subseteq \mathrm{Ker}\alpha$.

Proof

This is a well known property of finite rank endomorphisms of torsion-free modules over a complete discrete valuation ring; see e.g. [8, Lemma 3.2].

Proposition 9

If G is a reduced torsion-free R-module and $\phi \in E_0(G)$ then ϕ is a sum of two automorphisms.

Proof

From Lemma 8 we may write $G = F \oplus K$ where $G\phi \subseteq F$, $K \subseteq \mathrm{Ker}\,\phi$. Then $\phi|F$ is an endomorphism of F which has finite rank and so, by Proposition 1, $\phi|F = u + v$ where u, v are automorphisms of F. Now define endomorphisms μ,ν of G by:-

$$f\mu = fu,\ k\mu = k \text{ for } f \in F,\ k \in K,$$
$$f\nu = fv,\ k\nu = -k \text{ for } f \in F,\ k \in K.$$

Clearly μ,ν are units in E(G) and moreover

$$f(\mu+\nu) = fu + fv = f(u + v) = f\phi$$
$$k(\mu+\nu) = k + (-k) = 0 = k\phi.$$

Thus $\phi = \mu + \nu$ is a sum of two automorphisms.

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