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Aidan O'Dwyer

Dublin Institute of Technology, aidan.odwyer@dit.ie

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Development of the Analytical Design Method for Single Stage Phase Lead and Phase Lag Compensators

Aidan O'Dwyer

*School of Control Systems and Electrical Engineering,
Dublin Institute of Technology,
Kevin St., Dublin 8, Ireland
aidan.odwyer@dit.ie*

Abstract – The design of phase lead and phase lag compensators in the frequency domain, using the Bode diagram, is a commonly used method of designing control systems and is taught in many undergraduate and postgraduate courses in control engineering. The graphical trial and error procedure used may be replaced by an analytical procedure. However, the analytical approach tends to be formulated as a “black-box” approach in textbooks, without engineering insight. This paper presents the full development of the analytical design procedure. The instructional approach has been successfully used by the author over the past two academic years.

Keywords – cascade compensators, frequency domain, analytical

I INTRODUCTION

The use of frequency domain controller design procedures is a staple part of the undergraduate experience in control engineering. There is some evidence that such procedures are used in many industrial design problems [1]. In particular, the design of series or cascade compensators in the frequency domain is popular, partly because the procedure may be used if the process to be controlled is unknown and only experimental data is available [1]. Considering single stage continuous-time phase lead or phase lag compensators in particular, the conventional design approach is a trial and error one, based most commonly on a Bode diagram.

The first analytical design method for continuous-time phase lead controllers was proposed over thirty years ago by Wakeland [2]; subsequently, Mitchell [3] modified the method to allow design of continuous-time phase lag controllers. Other analytical design procedures have been detailed in the continuous-time domain [1], [4], [5] and in the discrete-time domain [6]. However, analytical design methods have not received much attention in textbooks; in a survey of twenty-nine textbooks [7-35], the author has found just three textbooks which outline continuous-time analytical methods [12], [25], [30], with Dorf and Bishop [12] outlining the methods of Wakeland [2] and Mitchell [3], and Phillips and Harbor [25] and Shahian and Hassul [30] outlining the method of Phillips [4].

This paper will consider the analytical design procedure outlined by Wakeland [2] and Mitchell [3], and recently detailed by Wang [1]. In Dorf and Bishop [12], the method is presented as a recipe for controller design, based on the user specification of

the desired gain crossover frequency, ω_c and the desired phase margin, ϕ_m . In teaching the method from the book, it is the author's experience that students dislike the method, due to the lack of transparency of the design procedure. This issue has been partly addressed by Wang [1], who presents an outline development of the unique analytical solution to the continuous time, single stage phase lead and phase lag controller design approach. However, Wang [1] uses a different notation to that used by Dorf and Bishop [12].

This paper details the step-by-step full development of the analytical method, in a student-friendly manner, in an extension of the information provided by Wang [1] and Dorf and Bishop [12]. The author has found that teaching the analytical method by firstly introducing this full development of the analytical procedure, followed by the demonstration of the advantages of the analytical approach over the more commonly used trial and error graphical approach, through simulation examples, has led to wider student acceptance of the approach.

II DESIGN PROCEDURE RECIPE

A single stage compensator may be represented as

$$G_c(s) = \frac{1 + \alpha Ts}{1 + Ts} \quad (1)$$

with $\alpha > 1$ being a lead compensator and $\alpha < 1$ being a lag compensator. The phase contribution of the compensator at the desired gain crossover (i.e. 0 dB) frequency, ω_c , may be straightforwardly calculated (Dorf and Bishop, [12]) as

$$p = \tan \phi = \frac{\alpha \omega_c T - \omega_c T}{1 + \alpha (\omega_c T)^2} \quad (2)$$

Similarly, the following equation may be easily deduced, from which the magnitude of the compensator at ω_c , $|G_c(j\omega_c)|$, may be determined:

$$c = |G_c(j\omega_c)|^2 = \frac{1 + (\alpha\omega_c T)^2}{1 + (\omega_c T)^2} \quad (3)$$

Equations (2) and (3) may be used, by eliminating $\omega_c T$, to obtain a quadratic equation from which α may be calculated:

$$(p^2 - c + 1)\alpha^2 + (2p^2 c)\alpha + (p^2 c^2 + c^2 - c) = 0 \quad (4)$$

This equation is presented [2], [3], [12], but not developed; subsequently, it is declared [2], [12] that a phase lead controller will result from this equation (i.e. $\alpha > 1$), provided

$$c > p^2 + 1 \quad (5)$$

Similarly, it is declared [3], [12] that a phase lag controller will result from this equation (i.e. $\alpha < 1$), provided

$$c < (p^2 + 1)^{-1} \quad (6)$$

Neither of these statements is developed. Finally, having determined α , it is stated [2], [3], [12] that

$$T = \frac{1}{\omega_c} \sqrt{\frac{1-c}{c-\alpha^2}} \quad (7)$$

The analytical design procedure has been summarised by the following recipe [12]:

- Select desired crossover frequency, ω_c
- Determine phase margin desired, and therefore the required phase from the compensator, ϕ . Then $p = \tan \phi$.
- Determine the magnitude of the compensator required, M, in dB. Calculate $c = 10^{0.1M}$.
- Check if $c > p^2 + 1$ (lead compensator) or if $c < (p^2 + 1)^{-1}$ (lag compensator).
- Determine α from the solution of equation (4):

$$(p^2 - c + 1)\alpha^2 + (2p^2 c)\alpha + (p^2 c^2 + c^2 - c) = 0$$
- Determine T from equation (7): $T = \frac{1}{\omega_c} \sqrt{\frac{1-c}{c-\alpha^2}}$.

III DEVELOPMENT OF EQUATIONS (5) AND (6), AND EQUATIONS FOR α , T

a) Development of equations (5) and (6)

In this section of the paper, the analytical procedure outlined by Wang [1] is modified and simplified. An explicit equation for α may be developed by eliminating $\omega_c T$ from equations (2) and (3). Following the development of Wang [1], label $\alpha\omega_c T = \sigma$. Then, equations (2) and (3) become

$$p = \frac{\sigma - \alpha}{1 + \frac{\sigma^2}{\alpha}} \quad (8) \text{ and } c = \frac{1 + \sigma^2}{1 + \left(\frac{\sigma}{\alpha}\right)^2} \quad (9).$$

It is straightforward to deduce, from equation (8), that

$$\frac{\sigma}{\alpha} = \frac{\sigma - p}{\sigma p + 1} \quad (10)$$

Substituting equation (10) into equation (9) allows equation (11) to be directly developed:

$$(1 + \sigma^2)(\sigma p + 1)^2 = c[(\sigma p + 1)^2 + (\sigma - p)^2] \quad (11)$$

It is easy to show that the right hand side of equation (11) may be written as $c(1 + \sigma^2)(1 + p^2)$ i.e. equation (11) becomes

$$(1 + \sigma^2)[(\sigma p + 1)^2 - c(p^2 + 1)] = 0 \quad (12)$$

Now, since $\sigma = \alpha\omega_c T > 0$, σ may be deduced from the solution of the quadratic equation

$$(\sigma p + 1)^2 - c(p^2 + 1) = 0 \quad (13)$$

Using the quadratic formula,

$$\sigma = \frac{\sqrt{c(p^2 + 1)} - 1}{p} \quad (14)$$

Note that only the positive solution of equation (13) is relevant (as $\sigma = \alpha\omega_c T > 0$).

For a phase lead controller, $c = |G_c(j\omega_c)|^2 > 1$; therefore, from equation (14), $c > p^2 + 1$ for $\sigma > 0$, thus proving equation (5). Interestingly, this condition implies, from equation (14), that $\sigma > p$; of course, the minimum value of $p = \tan \phi = 0$.

For a phase lag controller, $c = |G_c(j\omega_c)|^2 < 1$ and $p < 0$. Thus, from equation (14), $\sigma > 0$ if $\sqrt{c(p^2 + 1)} < 1$ i.e. if $c < (p^2 + 1)^{-1}$, proving equation (6).

b) Development of equation for α

From equations (10) and (14), α may subsequently be determined. Re-writing equation (10),

$$\alpha = \frac{\sigma(\sigma p + 1)}{\sigma - p} \quad (15)$$

Substituting equation (14) into equation (15) gives, after simple step-by-step development,

$$\alpha = \frac{\sqrt{c} \left[\sqrt{c(p^2 + 1)} - 1 \right]}{\sqrt{c} - \sqrt{p^2 + 1}} \quad (16)$$

An alternative to equation (16) for α may be determined by explicitly solving equation (4) (assuming this equation is valid). The development below is again a modified and simplified version of that of Wang [1].

$$\alpha = \frac{-2p^2 c \pm \sqrt{4p^4 c^2 - 4(p^2 - c + 1)(p^2 c^2 + c^2 - c)}}{2(p^2 - c + 1)} \quad (19)$$

This may be simplified in a step-by-step manner to

$$\alpha = \frac{-p^2 c \pm |c - 1| \sqrt{c(p^2 + 1)}}{p^2 - c + 1} \quad (20)$$

In turn, for a phase lead controller (i.e. when $c > p^2 + 1$), a positive solution for α from equation (20) may be determined straightforwardly to be

$$\alpha = \frac{-p^2c - (c-1)\sqrt{c(p^2+1)}}{p^2 - c + 1} \quad (21)$$

For a phase lag controller, (i.e. when $c < (p^2 + 1)^{-1}$), a positive solution for α from equation (20) may be determined easily to be

$$\alpha = \frac{-p^2c + (1-c)\sqrt{c(p^2+1)}}{p^2 - c + 1} \quad (22)$$

It is straightforward to show that equation (21) or (22) is equivalent to equation (16). This indirectly demonstrates the validity of equation (4). The most convenient equation from a hand calculation or programming point of view has been found to be equation (16).

c) Development of equation for T

It is straightforward to develop the equation for T, equation (7), from equation (3). An alternative to equation (7) may be deduced from equation (10), bearing in mind that $T = \sigma/\omega_c\alpha$. In a further simplified and modified version of the development of Wang [1], it is straightforward to deduce that

$$T = \frac{\sqrt{c} - \sqrt{p^2 + 1}}{p\omega_c\sqrt{c}} \quad (23)$$

It is easy to verify that $T > 0$, from equations (7) or (23), for both the phase lead and phase lag controllers, using equations (5) and (6), as appropriate. A further alternative equation for T may be easily determined from equation (2) as the positive version of

$$T = \frac{\omega_c(\alpha - 1) \pm \sqrt{\omega_c^2(\alpha - 1)^2 - 4\alpha\omega_c^2p^2}}{2\alpha\omega_c^2p} \quad (24)$$

The most convenient equation from a hand calculation or programming point of view has been found to be equation (23). In particular, equation (23) is better than equation (7) for phase lag controller design, when $c - \alpha^2$ can be close to zero, giving rise to the possibility of round-off errors.

IV NEW SUMMARY OF THE ANALYTICAL DESIGN PROCEDURE

- Select desired crossover frequency, ω_c .
- Determine phase margin desired, and therefore the required phase from the compensator, ϕ . Then $p = \tan \phi$.
- Determine the magnitude of the compensator required, M, in dB. Calculate $c = 10^{0.1M}$.
- Check if $c > p^2 + 1$ (lead compensator) or if $c < (p^2 + 1)^{-1}$ (lag compensator).

- Calculate $\alpha = \frac{\sqrt{c} \left[\sqrt{c(p^2+1)} - 1 \right]}{\sqrt{c} - \sqrt{p^2+1}}$
- Calculate $T = \frac{\sqrt{c} - \sqrt{p^2+1}}{p\omega_c\sqrt{c}}$.

V PEDAGOGICAL ISSUES

At DIT Kevin St, compensator design methods in the frequency domain are covered, as part of a suite of controller design approaches, in the Control Systems Major subject of the BE in Electrical/Electronic Engineering and the Advanced Control Systems subject of the taught ME in Advanced Engineering. Firstly, the equations and plots summarising the relationship between time domain metrics (such as overshoot and rise time) to the damping factor, ξ , and subsequently to frequency domain metrics (such as phase margin and gain crossover frequency), which are exact for closed loop systems in second order form, are introduced. Subsequently, the trial and error graphical methods for phase lead and phase lag compensator design are used to design compensators for a simple process, $G_p(s) = 40/s(s+3)$, to achieve realistic time domain metrics (maximum overshoot and maximum settling time to a servo step input). These metrics translate to a desired compensated phase margin of 45° , and a desired $\omega_c = 7.4$ rads/s (phase lead compensation) or a desired $\omega_c = 2.4$ rads/s (phase lag compensation), following a standard design procedure [12]. Such a process, when compensated, will be reasonably represented in second order form. The simplicity of the problem means that compensator designs to achieve the specifications may be determined on the first iteration; the compensators determined are:

$$\frac{1+0.189s}{1+0.096s} \text{ (lead) and } \frac{1+4.17s}{1+23.42s} \text{ (lag).}$$

In subsequent checks, the phase margins of the phase lead and phase lag compensated systems are determined to be, respectively, 41.1° and 46.6° .

Then, the step-by-step development of the analytical procedure is detailed and the method is used in the example for both phase lead and phase lag controller design. Even for this simple problem, the calculation (using a hand-held calculator) of the compensators using the equations is more straightforward than the graphical approach. The compensators determined are:

$$\frac{1+0.193s}{1+0.085s} \text{ (lead) and } \frac{1+2.91s}{1+12.77s} \text{ (lag).}$$

In subsequent checks, the phase margins of the phase lead and phase lag compensated systems are determined to be, respectively, 44.9° and 45.1° . Clearly, the analytical procedure is more accurate, as no graphical asymptotic approximations are required. It is true that the production of the Bode diagram can

be automated using MATLAB (which also eliminated the asymptotic approximations) but, of course, it is also straightforward to automate the analytical procedure by the programming of the equations.

Though the analytical procedure is more direct (particularly for more involved problems), the author has found that, in a learning environment, as previously mentioned, it is important not to treat the method as a recipe, as engineering insight and understanding is consequently reduced. Thus, it is important to introduce the step-by-step full development of the analytical procedure prior to its use. The combination of the development and application of the procedure allows a similar level of insight to be obtained with the analytical method as with the trial and error graphical approach.

Of course, the achievement of the time domain metrics used in the design examples provide a somewhat one-dimensional view of closed loop system performance. Subsequently, the design is evaluated by considering disturbance rejection, robustness to process variations and model uncertainty and sensitivity to measurement noise, in addition to set-point tracking performance. Such evaluation is done using SIMULINK or MATLAB (the latter for determining how, for example, closed loop sensitivity varies with frequency). Thus, the teaching method does not exclude the sensible use of CAD tools. However, these tools are used after the design is completed, as the author's experience is that the teaching method allows increased student understanding, than an approach in which CAD tools are integrated further into the design procedure. The background of the students on one of the courses (ME in Advanced Engineering) is a factor. This course takes students from a wide variety of engineering backgrounds, domestically and internationally. Students from a domestic electrical/electronic engineering background tend to have some familiarity with MATLAB, for example. However, this is not true, in the author's experience, of students from a mechanical engineering background, or international students who typically come from China or India. An over-emphasis on CAD tools is thus unwise.

VI CONCLUSIONS

This paper has reported on the teaching of the analytical design procedure for continuous time, single stage phase lead and phase lag controllers. The paper details the step-by-step full development of the analytical method, in a more transparent and student-friendly manner than heretofore. It has been found that the method of instruction detailed in the paper, followed by the demonstration of the advantages of the analytical approach over the more commonly used trial and error graphical approach, through simulation examples, has led to increased student

understanding. Formal measurement of such increased understanding has not been carried out to date. In further work, the author will consider the use of the universal design chart [36], which allows a direct evaluation of which single-stage controller is appropriate to achieve a given closed loop system specification, and whether it is possible to further improve the quality of the closed loop response with the chosen controller structure.

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