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A note on coslender groups

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A NOTE ON COSLENDER GROUPS

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Abstract. The notion of a coslender group has been introduced previously in [4] by the first author. This work continues the investigation of such groups, defines coslender part of a group, proves embeddability results, gives a characterization of finite rank coslender groups, and proves a result on smooth ascending chains of coslender groups with a conjecture that every countable conslender torsion free group is a smooth ascending union of finite rank coslender pure subgroups.

The notion of a coslender group has been introduced previously in [4] by the first author. This work continues the investigation of such groups, explores some indecomposability properties of the groups and finishes with a conjecture on the structure of coslender groups of at most countable rank.

All groups are additively written Abelian groups and our terminology and notation follow the standard works of Fuchs [5].

Definition 1. An Abelian group A is *coslender* if for every homomorphism $f: A \to Z^N$ and almost all $n \in N$, $p_n f = 0$ (here $p_n (n \in N)$ denote the coordinate projections).

Thus coslenderness is in some sense dual to the notion of slenderness (cf. [5], §94).

The following result is fundamental and easily established (cf. [4], Proposition 1):

PROPOSITON 1. *The following are equivalent*:

(*a*) *A is a coslender group*;

(*b*) $Hom(A, Z) = 0$;

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(*c*) *Hom*$(A/tA, Z)=0$ (*here* tA *is the torsion subgroup of* A);

(*d*) Z *is not a direct summand of* A.

It follows from (*c*) above that it is sufficient to examine torsion-free coslender groups; (*d*) indicates that coslender groups have some indecomposability properties.

Example 1. Algebraically compact, cotorsion and indecomposable groups not isomorphic to Z are conslender, while Z, free groups and Z^I are not.

PROPOSITION 2. (*a*) *If* A_i $(i \in I)$ *are coslender subgroups of a group* A, *then* $\sum_{i \in I} A_i$ *is also coslender*;

(*b*) *Every group* A *contains a maximum coslender subgroup* (*i.e. such that it contains every coslender subgroup of* A).

Proof. (*a*) If $f: \sum_{i \in I} A_i \to Z$ is a homomorphism, then (since each A_i is coslender) $f|\sum_{i \in J} A_i = 0$, $J \subseteq I$ and J finite, giving $f=0$;

(*b*) The sum of all coslender subgroups is the maximum coslender subgroup.

Definition 2. Call the maximum coslender subgroup of a group A, *the coslender part of* A and denote it by $C(A)$.

The next proposition shows that many important algebraic constructions preserve coslenderness.

PROPOSITION 3. *The class of coslender groups is closed under*:

(*a*) *homomorphic images*;
(*b*) *direct summands*;
(*c*) *extensions*;
(*d*) *direct sums*;
(*e*) *direct limits*;
(*f*) *nonmeasurable direct products*.

Proof. See [4].

LEMMA 4. *If* C *is a coslender subgroup of a group* A, *then the pure subgroup* C_* *generated by* C *in* A *is also coslender*.

Proof. By hypothesis C_*/C is torsion, thus by Example 1 – coslender. An application of Proposition 3(*c*) to the exact sequence $0 \to C \to C_* \to C_*/C \to 0$ provides the proof that C_* is coslender.

Proof. This follows immediately from Lemma 4 and Proposition 3 (a).

Our next example shows that we cannot replace the requirement of being a summand in Proposition 3 (b) by purity (even in the finite rank case).

Example 2. A pure subgroup of a coslender group need not be coslender even in the finite rank case.

Proof. Let A denotes the ring of Gaussian integers $Z[i]$ and A its underlying additive group. Using the finite version of Corner's Realization Theorem (see [1], Theorem B and its proof), one obtains a group G of rank at most four such that $A \leq_* G \leq_* \hat{A}$ (here $\hat{A}$ is the natural completion of A) and $End(G) \simeq A$. The last isomorphism implies that G is indecomposable and thus coslender, while its pure subgroup A is free of rank two.

In the case of countable groups we are able to say more with respect to coslenderness.

PROPOSITION 6. *Every countable group A is of the form $A = C \oplus F$, where C is the uniquely determined coslender part of A and F is a free group.*

Proof. This is essentially the result of K. Stein (see [5], Corollary 19.3), where it is shown that $A = C \oplus F$, with a coslender group $C = \bigcap_\eta \operatorname{Ker} \eta$ (η ranges over all homomorphisms $A \to Z$). However if C_1 is any coslender subgroup of A and $\eta\colon A \to Z$ is a homomorphism, then clearly $C_1 \leq \operatorname{Ker} \eta$, whence $C_1 \leq C$. Thus C is the coslender part of A.

Our Example 2 can be considerably strenghtened to show that every group may be embedded as a pure subgroup of a coslender group; this "coslender hull" will in general be different from either the "divisible hull" or the "cotorsion hull".

THEOREM 7. *Every group A may be embedded as a pure subgroup of a coslender group C. Moreover, the embedding may be chosen so that $C/A \cong \bigoplus_I Q$, where $|I| \leq \operatorname{rk} A$ and $\operatorname{rk} C = 1 + \operatorname{rk} A$; if A is of finite rank, then I may be chosen with $|I| = 1$.*

Proof. Assume first that A is of infinite rank $\varkappa$. Thus we have a free resolution of A: $0 \to H \to F \to A \to 0$, where $F = \bigoplus_\varkappa Z$. But by a Result of Corner ([5], §88, Exercise 8) there is an indecomposable (and thus coslender) group C' of rank two with $C'/Z \cong Q$. Let $C'' = \bigoplus_\varkappa C'$. Clearly F may be embedded as a pure subgroup of C''. But then $A \cong F/H \cong C''/H$; set $C = C''/H$ and observe that C is a coslender group of rank $\varkappa$ such that

$C/A \cong C''/F \cong \bigoplus_{\varkappa} Q$. Thus $|I| = \text{rk}\, A = \varkappa$ and rk $C = 1 + \text{rk}\, A$, since $\varkappa$ is infinite.

If A has finite rank n then by Proposition 6 $A = C_0 \oplus F_0$, where F_0 is free of rank at most n. By Corner's result, there is an indecomposable group B such that $F_0 \leq_* B$, with $B/F_0 \cong Q$ and $\text{rk}\, B = 1 + \text{rk}\, F_0$. Set $C = C_0 \oplus B$, a coslender group; a routine check verifies the claims in the above theorem.

An alternative proof of this theorem may be given by the use of Lemma 2 of 6 (see also [5], §88, Exercise 7).

Note that the theorem remains valid if the word "rank" is substituted by "generating system of minimal cardinality".

In the remaining section of this note we shall deal only with torsion-free groups and characterize coslender groups of at most countable rank.

PROPOSITION 8. *A group of rank one is coslender unless it is isomorphic to Z.*

Proof. If rk $A = 1$, *then* $A = Z \oplus B$ *is possible if and only if* $B = 0$.

THEOREM 9. *For a torsion-free group A of finite rank, the following are equivalent*:

(*a*) *A is coslender*;

(*b*) *A is a direct sum of indecomposable groups none of which is isomorphic to Z*;

(*c*) *A is a direct sum of coslender groups of rank one and indecomposable groups of rank at least two.*

Proof. It clearly suffices to prove the equivalence of (*a*) and (*b*). The implication (*b*)⇒(*a*) follows immediately from Proposition 3 and Example 1. The reverse implication follows by induction on $n = \text{rk}\, A$, the case $n = 1$ following from Proposition 8. If A is indecomposable, we are done. If $A = B \oplus C$, where $\text{rk}\, B$, $\text{rk}\, C < \text{rk}\, A$, use the inductive hypothesis and Proposition 3.

Theorem 9 indicates a strong link between direct sums of indecomposable groups and coslender groups. However our next example shows that coslender groups even of countable rank may be very far removed from direct sums of indecomposable groups.

Example 3. There exists a coslender superdecomposable group G of any cardinality $\varkappa$ (i.e. a coslender group G of cardinality $\varkappa$ with the

Proof. Corner [2, Theorem B] has established the existence of a group G of cardinality $\varkappa$ such that there exists a group M with decompositions $M = H_1 \oplus H_2 = G \oplus K$ where H_1 and H_2 are indecomposable of infinite rank and every non-zero summand of G is a direct sum of $\varkappa$ non-zero summands. But this group G is also coslender since it is a summand of M, which is coslender because the groups H_1 and H_2 are coslender (see Proposition 3 (d), (b)).

While it is trivially true that an ascending (pure) union of coslender groups is coslender, the converse presents considerably more difficulties. Not having a complete result in that direction, we state the following.

PROPOSITION 10. *Let $\varkappa$ be a regular cardinal and A a group of rank $\varkappa$; then the following are equivalent:*

(*a*) *A is a smooth ascending union of pure coslender subgroups of rank $<\varkappa$;*

(*b*) *For each $a \in A$ there is a pure coslender subgroup C of A with $a \in C$ and* $\operatorname{rk} C < \varkappa$.

Proof. $(a) \Rightarrow (b)$: If $A = \bigcup_{\alpha} C_\alpha$ and every $\operatorname{rk} C_\alpha < \varkappa$ then for each $a \in A$, there is an α such that $a \in C_\alpha$ whence (b) follows.

$(b) \Rightarrow (a)$: Let $A = \langle a_\lambda, \lambda < \varkappa \rangle$ and set $A_\alpha = \langle a_\lambda, \lambda \leq \alpha \rangle$. Construct inductively a chain of pure coslender subgroups C_α of A such that $A_\alpha \leq C_\alpha$ and $\operatorname{rk} C_\alpha < \varkappa$. If C_α $(\alpha < \beta)$ have been constructed and β is a limit ordinal, set $C_\beta = \bigcup_{\alpha} C_\alpha$. If $\beta = \alpha + 1$ then, by hypothesis, there is a pure coslender subgroup C of rank $<\varkappa$ with $a_{\alpha+1} \in C$. Set $C_\beta = \langle \sum_{\lambda \leq \alpha} C_\lambda + C \rangle_*$. Then C_β is the purification of a coslender subgroup and hence coslender. The regularity of $\varkappa$ ensures that $\operatorname{rk} C_\beta < \varkappa$, and by the construction $A = \bigcup_{\beta \leq \varkappa} C_\beta$. Thus (a) is established.

If $\varkappa$ is uncountable, then not every coslender group will be an ascending union of pure coslender subgroups of smaller rank. This follows (assuming CH) from the existence of $\varkappa_1$ – free indecomposable groups of cardinality $\varkappa_1$, constructed by Corner in [3]. Nevertheless there is a strong evidence that in the countable case the following is true:

Conjecture. A countable (torsion free) group is coslender if and only if it is an ascending union of pure finite rank coslender subgroups.

Further development of the theory of coslenderness is to be found in R. Dimitrić, Some remarks on coslender radical (to appear).

REFERENCES:

[1] *A. L. S. Corner*, Every countable reduced torsion-free ring is an endomorphism ring, Proc. London Math. Soc. **13** (1963), 687 – 710.

[2] ———————, On the existence of very decomposable abelian groups, Proc. Honolulu Conf. Abelian Group Theory, 354 – 357 (LNM 1006, Berlin, Heidelberg, New York, Tokyo, 1983).

[3] ———————, The Existence of $\aleph_1$ – free Indecomposable Abelian Groups, Unpublished manuscript, Oxford 1968.

[4] *R. Dimitrić*, On coslender groups, Glas. Mat. Ser. III **21 (41)** (1986), 327 – 329.

[5] *L. Fuchs*, Infinite abelian groups, Vol. I, II, Academic Press, New York, 1970, 1973,

[6] *L. Fuchs* and *F. Loonstra*, On the cancellation of modules in direct sums over Dedekind domains, Indagationes Math., **33** (1971), 163 – 169).

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PRIMJEDBA O KOUSKIM GRUPAMA

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Sadržaj

Pojam kouske grupe uveden je ranije u radu [4] prvog autora. U ovom radu se nastavlja ispitivanje kouskih grupa i u njemu se definiše kouski deo grupe, dokazuju se rezultati o utapanju, daje se karakterizacija kouske grupe konačnog ranga i dokazuje se rezultat o glatkim rastućim lancima kouskih grupa, sa hipotezom da se svaka prebrojiva kouska grupa predstavlja kao glatka rastuća unija kouskih čistih podgrupa konačnog ranga.