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AN ESSENTIALLY SEMI-RIGID CLASS OF MODULES

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Introduction

A class $\mathscr{C}$ of abelian groups is said to be rigid if $\operatorname{Hom}(A, B) = 0$ for all pairs of different groups $A, B \in \mathscr{C}$. The class $\mathscr{C}$ is said to be semi-rigid if $\operatorname{Hom}(A, B) \neq 0$ implies $\operatorname{Hom}(B, A) = 0$ for all pairs of different groups $A, B \in \mathscr{C}$. (This is a slight modification of the concept used by Fuchs in [4] where in addition it is required that for each $A \in \mathscr{C}$, $\operatorname{Hom}(A, A)$ should be a subgroup of the group of rationals $\mathbb{Q}$.) It is known from model theory that the existence of a proper (that is, not a set) rigid class of groups is not provable in ZFC (see for example [**9**, §17, 196–203]. In a recent paper [**5**], Göbel and Shelah have established the existence of a semi-rigid class of cotorsion-free abelian groups. In this note we wish to investigate the analogous problem for reduced torsion-free modules over a complete discrete valuation ring R. Since all such modules are necessarily separable, there is no possibility of finding a rigid or semi-rigid class. However, exploiting the concept of inessential homomorphism used previously in [**1**, **2**, **6**, **7**], it is natural to raise the question of the existence of essentially rigid and semi-rigid classes. We show in this note that the existence of an essentially semi-rigid class is easily established.

2. *Notation and some preliminaries*

Throughout we shall suppose that R is a complete discrete valuation ring of cardinality ν having unique prime ideal p. Our notation will follow the standard works of Fuchs [**3**, **4**]; set-theoretic concepts may be found in Jech [**8**].

For an infinite cardinal λ, let S_λ denote a free R-module of rank λ; note that $|S_\lambda| = \lambda\nu$. Let $\hat{S}_\lambda$ denote the completion of S_λ in the p-adic topology. (All topological references will be to this topology.) It follows then that $|\hat{S}_\lambda| = |S_\lambda|^{\aleph_0} = \lambda^{\aleph_0}\nu^{\aleph_0}$. Define the class Γ by

$$\Gamma = \{\lambda \mid \lambda \text{ is an infinite cardinal with } \lambda^{\aleph_0} = 2^\lambda \geqslant \nu\}.$$

LEMMA 1. (i) *Γ is a proper class.*

(ii) *If $\lambda \in \Gamma$ then $|\hat{S}_\lambda| = 2^\lambda$.*

Proof. (ii) is trivial. To show that (i) holds note that if α is any cardinal $\geqslant \nu$, then $\kappa = \sup\{\alpha, 2^\alpha, 2^{2^\alpha}, \ldots\}$ belongs to Γ. (In fact κ is a strong limit of cofinality ω.) The following definition was introduced in [**7**].

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Definition. An R-module G is said to be a *maximal pure submodule* of the complete R-module $\hat{S}$ if G is a pure submodule of $\hat{S}$ containing S and $\hat{S}/G \cong Q$, the field of fractions of R.

For each $\lambda \in \Gamma$ let $\{G_{\lambda i}\}$ $(i \in I_\lambda)$ denote the family of maximal pure submodules of $\hat{S}_\lambda$.

Lemma 2. *For each* $\lambda \in \Gamma$, $|I_\lambda| = 2^{2^\lambda}$.

Proof. For $\lambda \in \Gamma$, $\hat{S}_\lambda/S_\lambda$ is a Q-vector space of dimension 2^λ. By a well-known result on vector spaces there are precisely $|Q|^{2^\lambda}$ subspaces of $\hat{S}_\lambda/S_\lambda$ of codimension one. Since a maximal pure submodule is just the preimage of a subspace of codimension one, there are 2^{2^λ} maximal pure submodules of $\hat{S}_\lambda$, that is, $|I_\lambda| = 2^{2^\lambda}$.

Lemma 3. *If* $\alpha, \beta \in \Gamma$ *and* $\alpha \geqslant \beta$, *then* $|\mathrm{Hom}(\hat{S}_\alpha, \hat{S}_\beta)| \leqslant 2^\alpha$.

Proof. Every homomorphism from $\hat{S}_\alpha$ to $\hat{S}_\beta$ is determined by its effect on the basic submodule S_α, of rank α. Hence there are at most $|\hat{S}_\beta|^\alpha$ such homomorphisms. If $\alpha, \beta \in \Gamma$ and $\alpha \geqslant \beta$ then this upper bound becomes $2^{\beta\alpha} = 2^\alpha$.

3. *An essentially semi-rigid class*

The concept of an inessential homomorphism is a modification of a concept introduced by Corner at the Montpellier Symposium 1967; it has been used previously for working with modules over a complete discrete valuation ring by Goldsmith [7] and Dugas, Göbel and Goldsmith [1]; in a wider context it has also been exploited by Dugas and Göbel [2].

Definition. A homomorphism $\phi : G \to H$, where G, H are reduced torsion-free R-modules, is said to be *inessential* if the unique extension $\hat{\phi}$ of ϕ from $\hat{G} \to \hat{H}$ is such that $\hat{G}\hat{\phi} \subseteq H$. The collection of inessential homomorphisms from G to H is denoted by $\mathrm{Ines}(G, H)$.

Definition. A class $\mathscr{C}$ of R-modules is said to be *essentially semi-rigid* if $\mathrm{Hom}(A, B) \neq \mathrm{Ines}(A, B)$ implies $\mathrm{Hom}(B, A) = \mathrm{Ines}(B, A)$ for all pairs of different modules $A, B \in \mathscr{C}$. An *essentially rigid* class is defined similarly.

Note. If A is complete then every homomorphism $A \to B$ is automatically inessential and so when we raise the problem of the existence of essentially rigid and semi-rigid classes we are, of course, looking for non-trivial examples, that is, examples in which the modules in the class are not complete.

Theorem. *If R is a complete discrete valuation ring then there exists a non-trivial essentially semi-rigid class of R-modules.*

Proof. The essentially semi-rigid class will be constructed by choosing inductively one module from each of the sets $\{G_{\lambda i}\}$ $(i \in I_\lambda, \lambda \in \Gamma)$. Thus suppose that the collection $\{G_\lambda\}$ $(\lambda < \zeta)$, where $\lambda, \zeta \in \Gamma$, has been constructed with the property that

$$\mathrm{Hom}(G_\lambda, G_\alpha) = \mathrm{Ines}(G_\lambda, G_\alpha) \qquad \text{for } \alpha < \lambda < \zeta.$$

Fix any $\alpha < \zeta$ and consider the collection $\{\mathrm{Hom}(G_{\zeta i}, G_\alpha)\}$ $(i \in I_\zeta)$. Define the subset J_α of I_ζ by

$$J_\alpha = \{i \in I_\zeta \mid \mathrm{Hom}(G_{\zeta i}, G_\alpha) \neq \mathrm{Ines}(G_{\zeta i}, G_\alpha)\}\,.$$

We claim that $|J_\alpha| \leqslant 2^\zeta$. For suppose that $|J_\alpha| > 2^\zeta$ and pick a set of homomorphisms $\{\phi_i : G_{\zeta i} \to G_\alpha\}$ $(i \in J_\alpha)$ such that each ϕ_i is not inessential. Since each homomorphism ϕ_i extends uniquely to a homomorphism $\hat{\phi}_i : \hat{S}_\zeta \to \hat{S}_\alpha$, we deduce from Lemma 3 that $\hat{\phi}_i = \hat{\phi}_j$ for some $i \neq j \in J_\alpha$. However, it follows easily from the property of being maximal pure that $\hat{S}_\zeta = G_{\zeta i} + G_{\zeta j}$ and hence $\hat{S}_\zeta \hat{\phi}_i \subseteq G_{\zeta i}\hat{\phi}_i + G_{\zeta j}\hat{\phi}_j \subseteq G_\alpha$, contrary to the choice of ϕ_i. Thus $|J_\alpha| \leqslant 2^\zeta$. Since $|I_\zeta| = 2^{2^\zeta}$ we may pick $i_0 \in I_\zeta \setminus \bigcup_{\alpha < \zeta} J_\alpha$ and clearly $\mathrm{Hom}(G_{\zeta i_0}, G_\alpha) = \mathrm{Ines}(G_{\zeta i_0}, G_\alpha)$ for all $\alpha < \zeta$. Set $G_\zeta = G_{\zeta i_0}$. The construction is completed by transfinite induction. Since the class of R-modules $\{G_\lambda\}$ $(\lambda \in \Gamma)$ has the property $\mathrm{Hom}(G_\lambda, G_\alpha) = \mathrm{Ines}(G_\lambda, G_\alpha)$ for $\alpha < \lambda$, and each G_λ is a maximal pure submodule of $\hat{S}_\lambda$ (and hence is not complete), it follows that $\{G_\lambda\}$ $(\lambda \in \Gamma)$ is the desired non-trivial essentially semi-rigid class.

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