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The aim of this talk is to describe the derivation of shallow water model equations for the *constant vorticity* case and to demonstrate how these equations can be related to two integrable systems: a two component integrable generalization of the Camassa-Holm equation and the Kaup - Boussinesq system.

The motion of inviscid fluid is described by Euler's equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \mathbf{g}, \qquad \nabla\cdot\mathbf{v} = 0,$$

where ρ is a constant density, $\mathbf{v}(x,y,z,t)$ is the velocity of the fluid at the point (x,y,z) at the time t, P is the pressure in the fluid, $\mathbf{g}=(0,0,-g)$ is the constant Earth's gravity acceleration.

We consider a motion of a shallow water over a flat bottom, which is located at $z=0$. We assume that the motion is in the x-direction, and that the physical variables do not depend on y. Let h be the mean level of the water and let $\eta(x,t)$ describes the shape of the water surface, i.e. the deviation from the average level. The pressure is $P = P_A + \rho g(h-z) + p(x,z,t)$, where P_A is the constant atmospheric pressure, and p is a pressure variable, measuring the deviation from the hydrostatic pressure distribution.

On the surface $z = h+\eta$, $P = P_A$ and therefore $p = \eta\rho g$. Taking $\mathbf{v}\equiv(u,0,w)$ we can write the kinematic condition on the surface as (e.g. following [1]) $w = \eta_t + u\eta_x$ on $z = h+\eta$. Finally, there is no horizontal velocity at the bottom, thus $w=0$ on $z=0$.

Let us introduce now dimensionless parameters $\varepsilon = a/h$ and $\delta = h/\lambda$, where a is the typical amplitude of the wave and λ is the typical wavelength of the wave. Now we can introduce dimensionless quantities, according to the magnitude of the physical quantities, see [1, 2] for details: $x\to\lambda x$, $z\to zh$, $t\to\frac{\lambda}{\sqrt{gh}}t$, $\eta\to a\eta$, $u\to\varepsilon\sqrt{gh}u$, $w\to\varepsilon\delta\sqrt{gh}w$, $p\to\varepsilon\rho g h$.

Now let us notice that there is an exact solution of the governing equations of the form $u=\tilde{U}(z)$, $0\le z\le h$, $w\equiv 0$, $p\equiv 0$, $\eta\equiv 0$. This solution represents an arbitrary underlying 'shear' flow. In the presence of a shear flow the horizontal velocity of the fluid will be $\tilde{U}(z)+u$. The scaling for such solution is clearly $u\to\sqrt{gh}(\tilde{U}(z)+\varepsilon u)$, and the scaling for the other variables is as before. The system of equations is (the prime denotes derivative with respect to z):

$$\begin{aligned}
u_t + \tilde{U}u_x + w\tilde{U}' + \varepsilon(uu_x + wu_z) &= -p_x,\\
\delta^2(w_t + \tilde{U}w_x + \varepsilon(uw_x + ww_z)) &= -p_z,\\
u_x + w_z &= 0,\\
w = \eta_t + (\tilde{U}+\varepsilon u)\eta_x, \qquad p=\eta, \qquad \text{on} \qquad z &= 1+\varepsilon\eta,\\
w = 0 \qquad \text{on} \qquad z &= 0.
\end{aligned}$$

The simplest nontrivial case is a linear shear, $\tilde{U}(z) = Az$, where A is a constant. We choose $A > 0$, so that the underlying flow is propagating in the positive direction of the x-coordinate.

The vorticity is $\omega = (U + u)_z - w_x$ or in terms of the rescaled variables, $\omega = A + \varepsilon(u_z - \delta^2 w_x)$. We are looking for a solution with constant vorticity $\omega = A$, and therefore we require that $u_z - \delta^2 w_x = 0$. Together with the equation $u_x + w_z = 0$ it gives

$$u = u_0 - \delta^2 \frac{z^2}{2} u_{0xx} + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2), \quad w = -z u_{0x} + \delta^2 \frac{z^3}{6} u_{0xxx} + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2),$$

where $u_0(x, t)$ is the leading order approximation for u.

With these expressions we obtain the following from the condition on the surface, ignoring terms of order $\mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2)$:

$$\eta_t + A\eta_x + \Big[(1 + \varepsilon\eta)u_0 + \varepsilon\frac{A}{2}\eta^2\Big]_x - \delta^2 \frac{1}{6} u_{0xxx} = 0 \qquad (1)$$

From the second of the Euler's equations and the condition on the surface we have $p = \eta - \delta^2 \Big[\frac{1-z^2}{2} u_{0xt} + \frac{1-z^3}{3} A u_{0xx}\Big]$, then the first of the Euler's equations gives (Note that there is no z-dependence!)

$$\Big(u_0 - \delta^2 \frac{1}{2} u_{0xx}\Big)_t + \varepsilon u_0 u_{0x} + \eta_x - \delta^2 \frac{A}{3} u_{0xxx} = 0. \qquad (2)$$

The linearised equations are

$$u_{0t} + \eta_x = 0, \qquad \eta_t + A\eta_x + u_{0x} = 0, \qquad (3)$$

giving $\eta_{tt} + A\eta_{tx} - \eta_{xx} = 0$. This linear equation has a travelling wave solution $\eta = \eta(x - ct)$ with a velocity c satisfying $c^2 - Ac - 1 = 0$, or

$$c = \frac{1}{2}\Big(A \pm \sqrt{4 + A^2}\Big).$$

If there is no shear ($A = 0$), then $c = \pm 1$. In general, there is one positive and one negative solution, representing left and right running waves. Suppose that we have only one of these waves, then $\eta = cu_0 + \mathcal{O}(\varepsilon, \delta^2)$ - e.g. from (3).

By introduction of a new variable $\rho = 1 + \varepsilon\alpha\eta + \varepsilon^2\beta\eta^2 + \varepsilon\delta^2\gamma u_{0xx}$, where

$$\alpha = \frac{1}{3(1 + c^2)} + \frac{2c^2}{3(1 + c^2)}\Big(1 + \frac{Ac}{2}\Big), \quad \beta = \frac{1 - (3 + c^2)(1 + \frac{Ac}{2})}{3(1 + c^2)}\alpha, \quad \gamma = \frac{\alpha}{6(c - A)},$$

and a change of variables (rescaling) $u_0 \to \frac{1}{\alpha\varepsilon} u_0$, $x \to \frac{\delta}{\sqrt{B}}\, x$, $t \to \frac{\delta}{\sqrt{B}} t$ where $B = \frac{1}{2} + \frac{1}{6(c-A)}\Big(A - \frac{1}{c-A}\Big)$ the equations (1), (2) transform into the system

$$m_t + Am_x - Au_{0x} + 2mu_{0x} + u_0 m_x + \rho\rho_x = 0, \qquad m = u_0 - u_{0xx} \qquad (4)$$

$$\rho_t + A\rho_x + (\rho u_0)_x = 0, \qquad (5)$$

Before the rescaling we had $\alpha\varepsilon\eta = \rho - 1 - \varepsilon^2\beta c^2 u_0^2 - \varepsilon\delta^2\gamma u_{0xx}$. Since in the leading order $\eta = cu_0$ the rescaling of η is $\eta \to \frac{1}{\alpha\varepsilon}\eta$. Thus in terms of the rescaled variables $\eta = \rho - 1 - \frac{\beta c^2}{\alpha^2} u_0^2 - B\frac{\gamma}{\alpha} u_{0xx}$.

The system (4), (5) is an integrable 2-component Camassa-Holm system that appears in [3], generalizing the famous Camassa-Holm equation [4]. The Lax representation for this system is (ζ is a spectral parameter)

$$\begin{aligned} \Psi_{xx} &= \Big(-\zeta^2\rho^2 + \zeta(m - \frac{A}{2}) + \frac{1}{4}\Big)\Psi, \\ \Psi_t &= \Big(\frac{1}{2\zeta} - u_0 - A\Big)\Psi_x + \frac{1}{2}u_{0x}\Psi. \end{aligned}$$

An alternative derivation for the case of zero vorticity, based on the Green-Naghdi equations is reported in [5].

Another integrable system matching the water waves asymptotic equations to the first order of the small parameters ε, δ is the Kaup - Boussinesq system. We describe briefly its derivation. Introducing $V = u - \delta^2(\frac{1}{2} - \frac{A}{3c})u_{xx}$ the equation (2) can be written as $V_t + \varepsilon V V_x + \eta_x = 0$. Equation (1) in the first order in ε, δ is

$$\eta_t + \Big[A\eta + (1+\varepsilon\eta)u_0 + \varepsilon\frac{A}{2}\eta^2\Big]_x - \delta^2\frac{1}{6}u_{0xxx} = 0$$

and with a shift $\eta \to \eta - \frac{1}{\varepsilon}$ it becomes

$$\eta_t + \varepsilon(1 + \frac{Ac}{2})(\eta u_0)_x - \delta^2\frac{1}{6}u_{0xxx} = 0 \quad \text{or} \quad \eta_t + \varepsilon\frac{1+c^2}{2}(\eta V)_x - \delta^2\frac{1}{6}V_{xxx} = 0.$$

Further rescaling leads to the Kaup - Boussinesq system

$$V_t + VV_x + \eta_x = 0, \qquad \eta_t - \frac{1}{4}V_{xxx} + \frac{1+c^2}{2}(\eta V)_x = 0,$$

which is integrable iff $A = 0$ ($c^2 = 1$) with a Lax pair [6]

$$\Psi_{xx} = -\Big((\zeta - \frac{1}{2}V)^2 - \eta\Big)\Psi, \qquad \Psi_t = -(\zeta + \frac{1}{2}V)\Psi_x + \frac{1}{4}V_x\Psi.$$

It is interesting to investigate further which specific properties of the original governing equations are preserved in the 'integrable' approximate models. For example the 2-component Camassa-Holm system for certain initial data admits breaking waves solutions [5].

References

[1] R.S. Johnson, *A modern introduction to the mathematical theory of water waves*, Cambridge: Cambridge University Press (1997).

[2] R.S. Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid. Mech. **457** (2002) 63–82.

[3] P. Olver and P. Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support*, Phys. Rev. E **53** (1996), 1900.

[4] R. Camassa and D.D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993), 1661–1664.

[5] A. Constantin and R. Ivanov, *On an integrable two-component Camassa-Holm shallow water system*, Phys. Lett. A **372** (2008), 7129–7132.

[6] D.J. Kaup, *A higher-order water-wave equation and the method for solving it*, Progr. Theoret. Phys. **54** (1975), 396–408.